GROUPS AND RINGS (MA22017)

SOLUTIONS TO PROBLEM SHEET 7

1 W What is the content of each of the following polynomials?

(a) $3x^3 - 12x^2 - 9 \in \mathbb{Z}[x]$.

(b) $3x^3 - 12x^2 - 9 \in \mathbb{Q}[x]$.

(c) $3wx^3 - 12wx^2 - 9w \in R[x]$, where $R = \mathbb{Z}[w]$.

(d) $3wx^3 - 12wx^2 - 9w \in S[w]$, where $S = \mathbb{Z}[x]$.

(e) $3wx^3 + 3x^3 - 12w^2x^2 - 12wx^2 - 9w - 9 \in R[x]$, where $R = \mathbb{Z}[w]$.

Solution:

(a) 3.

(b) 1 *(although* 3 *is not wrong, as it is a unit too).*

(c) $3w$.

(d) 3.

(e) $3(w + 1)$.

2 W Say whether each of the following polynomials is reducible or irreducible, giving reasons.

(a) $3x^3 - 12x^2 - 9 \in \mathbb{Z}[x]$.

(b) $3x^3 - 12x^2 - 9 \in \mathbb{Q}[x]$.

(c) $x^2 + 5x - 3 \in \mathbb{F}_{11}[x]$.

(d) $x^2 + 5x - 3 \in \mathbb{F}_{13}[x]$.

(e) $x^2 + 5x - 3 \in \mathbb{F}_{37}[x]$.

(f) $x^3 + 5x - 3 \in \mathbb{F}_{13}[x]$.

(g) $x^3 + 5x - 3 \in \mathbb{F}_{11}[x]$.

Solution:

(a) $3x^3 - 12x^2 - 9 \in \mathbb{Z}[x]$ *is reducible because it has a factor of* 3.

(b) $3x^3 - 12x^2 - 9 \in \mathbb{Q}[x]$ *is irreducible because* 3 *is now a unit so we are interested in factorising* $x^3 - 4x^2 - 3$*: we know that if we can do that in* $\mathbb{Q}[x]$ *then we can do it in* $\mathbb{Z}[x]$ *and then we would have to have a monic linear factor, i.e. an integer root. There aren't any of those: they would have to be odd,* ± 1 *don't work and neither does* ± 3 *(no need to compute actual values, they won't be divisible by* 9*) and* 5 *is too big.*

(c) $x^2 + 5x - 3 \in \mathbb{F}_{11}[x]$ *has discriminant* $25 + 12 = 37 = 4$ *which is a square so this is reducible.*

(d) $x^2 + 5x - 3 \in \mathbb{F}_{13}[x]$ *has discriminant* $37 = -2$ *which is not a square: you just try the possibilities. In fact there is a way to calculate quickly whether* a *is a square in* $\mathbb{F}_p$ *for and* a *and any* p*, and in this case the answer is no and the reason is that* 13 *is* 5 *mod* 8*.*

(e) $x^2 + 5x - 3 \in \mathbb{F}_{37}[x]$ *has discriminant* $37 = 0$ *so this is in fact a square,* $(x + 21)^2$ *to be precise.*

(f) $x^3 + 5x - 3 \in \mathbb{F}_{13}[x]$ *has the root* 3 *so has a factor of* $x - 3$ *so it is reducible.*

(g) $x^3 + 5x - 3 \in \mathbb{F}_{11}[x]$ *has no root: you just try all eleven possibilities. So it is irreducible.*

3 H Say whether each of the following polynomials is reducible or irreducible in $\mathbb{Q}[x]$, giving reasons. You may want to look at Question 2 sometimes.

(a) $x^4 - 10x^3 - 15 \in \mathbb{Z}[x]$.

(b) $x^4 - 10x^3 - 15 \in \mathbb{Q}[x]$.

(c) $x^4 - x^3 - 10x^2 + 7x + 3 \in \mathbb{Q}[x]$.

(d) $x^4 - 14x^3 + 36x^2 - 34x - 4 \in \mathbb{Q}[x]$.

(e) $x^3 + 5x - 3 \in \mathbb{Q}[x]$.

Solution:

(a) $x^4 - 10x^3 - 15 \in \mathbb{Z}[x]$ *is irreducible by Eisenstein's criterion with* $p = 5$*.*

(b) $x^4 - 10x^3 - 15 \in \mathbb{Q}[x]$ *is irreducible for the same reason.*

(c) $x^4 - x^3 - 10x^2 + 7x + 3 \in \mathbb{Q}[x]$ *is reducible: it has a factor of* $x - 1$*. More generally, if a quartic doesn't have linear factors and doesn't yield to Eisenstein's criterion quickly, then it may well have two quadratic factors and those can often be guessed by factorising the constant term and trying to choose the coefficients of* x *so as to make everything else work out.*

(d) $x^4 - 14x^3 + 36x^2 - 34x - 4 \in \mathbb{Q}[x]$ *is irreducible. It is not Eisenstein for* $p = 2$ *because of the constant term* 4*, but putting* $x = y + 1$ *gives* $y^4 - 10y^3 - 15$ *which is Eisenstein with* $p = 5$*. Generally, with small integer coefficients, if Eisenstein doesn't do the job immediately, it is worth trying* $x = y \pm 1$ *but if those don't work either it's time to start looking for factors.*

(e) $x^3 + 5x - 3 \in \mathbb{Q}[x]$ *is irreducible because we have already seen that it is irreducible mod* 11 *(unfortunately it is reducible mod* 5 *because* $2^3 = 3$*).*

GKS, 21/3/25