GROUPS AND RINGS (MA22017)

SOLUTIONS TO PROBLEM SHEET 6

1 H. In this question R is a commutative ring and I and J are ideals in R. Say whether each of the following statements is true or not: give a proof or a counterexample.

(a) If I and J are both prime ideals then $I \cap J$ is a prime ideal.

(b) If I and J are both prime ideals then IJ is a prime ideal.

(c) If I and J are both prime ideals then $I + J$ is a prime ideal.

(d) If I and J are both maximal ideals then $I \cap J$ is a maximal ideal.

(e) If I and J are both maximal ideals then IJ is a maximal ideal.

(f) If I and J are both maximal ideals then $I + J$ is a maximal ideal.

Solution:

(a) This false in $\mathbb{Z}$, because $\langle p\rangle \cap \langle q\rangle = \langle \mathrm{lcm}(p, q)\rangle$ (Example V.39), which is never prime if p and q are different primes: for example $\langle 2\rangle \cap \langle 3\rangle = \langle 6\rangle$. The point is that if $ab \in I \cap J$ then a or b belongs to I and a or b belongs to J, but we don't have a way to exclude the possibility that $a \in I$ and $b \in J$ but neither is in $I \cap J$.

(b) Again this fails in $\mathbb{Z}$ by Example V.39, because $\langle p\rangle\langle q\rangle = \langle pq\rangle$ so the same counterexample as in (a) works.

(c) This fails in $\mathbb{Z}$ for the trivial reason that if p and q are different primes then they are coprime so $\langle p\rangle + \langle q\rangle = \mathbb{Z}$ which is explicitly excluded from being a prime ideal by the definition.

The last three may be done by simply pointing out that in $\mathbb{Z}$, all non-zero prime ideals are maximal, so all the counterexamples above still work. There is plenty of room for discussion (can we ever make any of these be prime or maximal? can we make $I + J$ be a proper non-prime ideal?) if anybody is in the mood for that.

2 W Recall that $\xi \in \mathbb{C}$ is said to be algebraic if the ideal K_ξ, which is defined to be the kernel of $\mathrm{ev}_\xi \colon \mathbb{Q}[t] \to \mathbb{C}$, is not zero. Show that the image $\mathbb{Q}[\xi]$ of ev_ξ is a field if and only if ξ is algebraic. (Hint: Notice that $\mathbb{Q}[\xi]$ is the image of ev_ξ: use the first isomorphism theorem and Theorem VI.24.)
Solution: *If ξ is transcendental then ev_ξ is injective so by the first isomorphism theorem, $\mathbb{Q}[\xi] \cong \mathbb{Q}[t]$, and $\mathbb{Q}[t]$ is not a field. If ξ is algebraic, then*

again by the first isomorphism theorem, $\mathbb{Q}[\xi] \cong \mathbb{Q}[t]/K_\xi$. But $\mathbb{Q}[t]$ is a Euclidean domain, hence a PID: set $K_\xi = \langle f \rangle$ (so f is the minimal polynomial of ξ). According to Theorem VI.24, it is therefore sufficient to show that f is irreducible. But if $f = gh$ with g and h nonunits (so both of positive degree) then $0 = \mathrm{ev}_\xi(f) = f(\xi) = g(\xi)h(\xi)$, so wlog $g(\xi) = 0$. Therefore $g \in K_\xi$, so $f|g$, but we also have $g|f$, so h is a unit, which is a contradiction.

3 W Justify the assertion made in lectures that 2, 3 $1 \pm \sqrt{-5}$ are all irreducible in $\mathbb{Z}[\sqrt{-5}]$. Put $N(x) = x\bar{x}$ for any $x \in \mathbb{Z}[\sqrt{-5}]$, where $\bar{x}$ denotes the complex conjugate.

(a) Show that $N(xy) = N(x)N(y)$.

(b) Show that $N(x)$ for $x \neq 0$ is a positive integer of the form $a^2 + 5b^2$.

(c) Show that if $N(x) = 1$ then $x \in \mathbb{Z}[\sqrt{-5}]$ is a unit.

(d) Show that if x is reducible then $N(x)$ is the product of two integers greater than 1 and of the form $a^2 + 5b^2$.

(e) Compute $N(x)$ for each of the four elements above.

(f) Hence deduce that these elements are irreducible.

Solution:

(a) Simply $N(xy) = (xy)(\overline{xy}) = xy\bar{x}\bar{y} = (x\bar{x})(y\bar{y}) = N(x)N(y)$.

(b) If $x = a + b\sqrt{-5}$ then $N(x) = (a + b\sqrt{-5})(a - b\sqrt{-5}) = a^2 + 5b^2$.

(c) If $N(x) = 1$ then $\bar{x} = 1/x \in \mathbb{C}$ but $\bar{x} \in \mathbb{Z}[\sqrt{-5}]$ so x has an inverse, namely $\bar{x}$.

(d) If x is reducible then $x = yz$ with y and z nonzero nonunits, so $N(y) \geq 2$ and $N(z) \geq 2$, but also $N(x) = N(y)N(z)$.

(e) $N(2) = 4$, $N(3) = 9$, $N(1 \pm \sqrt{-5}) = 6$.

(f) The factorisation implied by (d) would involve 2 or 3 in each case, but neither 2 nor 3 is of the form $a^2 + 5b^2$. (They are simply too small, but note also that $a^2 + 5b^2$ is a square mod 5 and the squares mod 5 are 0, 1 and 4.)

4 H Suppose that R is a principal ideal domain. For each of the following rings, say whether it is necessarily a PID too: give a proof, or a counterexample.

(a) A nontrivial subring S of R that contains 1_R.

(b) A quotient ring $A = R/I$ where I is a prime ideal (so A is an integral domain).

(c) The ring $R[t]$.

(d) A quotient $R[t]/J$ where J is a prime ideal.

Solution:

(a) No: for example $\mathbb{Q}[t]$, which is a PID, contains $\mathbb{Z}[t]$, which is not.

(b) Yes: in fact A is a field, by Theorem VI.24.

(c) No: $\mathbb{Z}$ is a PID but $\mathbb{Z}[t]$ is not.

(d) No: $\mathbb{Z}[t]/\langle t^2+5\rangle$ is $\mathbb{Z}[\sqrt{-5}]$ which is not a PID. We shall see later that any PID is in fact a UFD, and we already know that $\mathbb{Z}[\sqrt{-5}]$ is not a UFD, but in fact $\langle 2, 1+\sqrt{-5}\rangle$ is not a principal ideal because both elements are irreducible, neither divides the other, and the ideal does not contain 1. To see this, suppose that $2(a+b\sqrt{-5})+(1+\sqrt{-5})(c+d\sqrt{-5}) = 1$; then $2a+c-5d = 1$ and $2b+c+d = 0$ but subtracting these gives $2a-2b-4d = 1$ which is impossible because the left-hand side is odd and the right-hand side is even.

GKS, 14/3/25