GROUPS AND RINGS (MA22017)

SOLUTIONS TO PROBLEM SHEET 11

1 W Suppose M, N and W are R-modules.

(a) If $\varphi\colon M \to W$ and $\psi\colon N \to W$ are linear maps, is $a\colon M \times N \to W$ given by $a(m,n) = \varphi(m) + \varphi(n)$ a bilinear map?

(b) If W is an R-algebra (see Definition IV.17 and replace K with R) and $\varphi\colon M \to W$ and $\psi\colon N \to W$ are linear maps, is $b\colon M \times N \to W$ given by $b(m,n) = \varphi(m)\varphi(n)$ a bilinear map?

Solution:

(a) No: indeed $a(rm,n) = r\varphi(m) + \psi(n)$ rather than $r\varphi(m) + r\psi(n)$ as would be required if a were bilinear. To reconcile them one would have to have $(r-1)\psi(n) = 0$ for all r and n, so we can even take $r = 0$ and that shows that we must have $\psi = 0$. Similarly we would require $\varphi = 0$, to this is never true except in trivial cases.

(b) Yes: for instance $b(rm + sm', n) = \varphi(rm + sm')\psi(n) = (r\varphi(m) + s\varphi(m'))\psi(n) = r\varphi(m)\psi(n) + s\varphi(m')\psi(n) = rb(m,n) + sb(m',n)$.

2 H Compute

(a) $\mathbb{Z}/6\mathbb{Z} \otimes \mathbb{Z}/4\mathbb{Z}$

(b) $\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/3\mathbb{Z} \otimes \mathbb{Z}/4\mathbb{Z}$

(the tensor products are over $\mathbb{Z}$).
Solution:

$\mathbb{Z}$ when I'm writing answers. This is exactly like the computation of $\mathbb{Z}/2 \otimes \mathbb{Z}/2$ that we saw in Example 8.20(iv) (which itself was the same as the computation in 8.20(iii).) The tensor product $A \otimes B$ tells you what bilinear maps from $A \times B$ there are. If A is generated as an R-module by elements $a_1, \ldots, a_m$, so that any element of A is of the form $r_1a_1 + \cdots + r_ma_m$, and B is generated by $b_1, \ldots, b_n$, and $\mathbf{f}$ is a bilinear map from $A \times B$ to somewhere, then if I know what $\mathbf{f}(a_i, b_j)$ for all pairs (i,j), then I certainly know what $\mathbf{f}(a,b)$ is. So we can try possible values for $\mathbf{f}(a_i, b_j)$ and see which ones actually work. In this case, $A = \mathbb{Z}/6$ and $B = \mathbb{Z}/4$: these are both generated by one element, $a = 1 \in \mathbb{Z}/6$ and $b = 1 \in \mathbb{Z}/6$, because every integer mod 4 is a multiple of 1. So all we need to know is what the value of $\mathbf{f}(1,1)$ is: call it x (it is an element of some module, but I haven't decided which one yet). On top of that, we know that $\mathbf{f}(4,4) = 0$ because in the first slot, $4 = 0$,

so $4 = 0 = 0 \times 1$ *so* $4x = \mathbf{f}(4,1) = \mathbf{f}(0,1) = 0 \times x = 0$ *(by bilinearity). Similarly* $6x = \mathbf{f}(1,6) = \mathbf{f}(1,0) = 0$. *So* $6x = 4x = 0$, *so* $2x = 0$. *But we can't conclude that* $x = 0$, *because the module that* x *is in might be* $\mathbb{Z}/2$; *and indeed there is a perfectly good bilinear map* $\mathbb{Z}/4 \times \mathbb{Z}/6$ *that sends* $(1,1)$ *to* $1 \in \mathbb{Z}/2$. *So we've found all the bilinear maps: we just need to know the value of* x, *and that's either* 0 *or* $1 \in \mathbb{Z}/2$, *so* $\mathbb{Z}/4 \otimes_{\mathbb{Z}} \mathbb{Z}/6 = \mathbb{Z}/2$. *More generally, if I replace* 4 *and* 6 *with* c *and* d, *we get* $\mathbb{Z}/c \otimes_{\mathbb{Z}} \mathbb{Z}/d = \mathbb{Z}/e$, *where* $e = \operatorname{hcf}(c,d)$, *by exactly the same argument.*

(a) *We saw that there are no bilinear maps from* $\mathbb{Z}/2 \times \mathbb{Z}/3$ *so this is zero. Indeed, in the above, we have* $c = 2$ *and* $d = 3$ *so* $e = 1$, *and* $\mathbb{Z}/1 = \{0\}$

3 E If V is an n-dimensional vector space over a field k, the *symmetric algebra* is the ring

$$S^*V = \bigoplus_{d=0}^{\infty} \operatorname{Sym}^d V$$

with the convention that $\operatorname{Sym}^0 V = k$ and the product given by

$$(v_1 \dots v_d)(w_1 \dots w_e) = v_1 \dots v_d w_1 \dots w_e$$

– recall that $v_1 \dots v_d = \sum_{\sigma \in S_d} v_{\sigma(1)} \dots v_{\sigma(d)}$. Show that this makes S^*V into a k-algebra, and that it is isomorphic to $k[x_1, \dots, x_n]$.
Solution: *This is for enthusiasts to think about.*

GKS, 28/4/25