GROUPS AND RINGS (MA22017)

SOLUTIONS TO SHEET 10

1 Recall (Example III.7(viii)) that any group G acts on itself by conjugation: $a(g,h) = ghg^{-1}$. The orbits are called *conjugacy classes*.

(a) Show that for this action, the map $a_g\colon G \to G$ is in fact a group homomorphism.

(b) Show that any normal subgroup of G is a union of conjugacy classes.

(c) Let $\mathcal{W}$ denote the set of all subgroups of G. Show that G acts on $\mathcal{W}$ by conjugation.

(d) Suppose $H \leq G$, so $H \in \mathcal{W}$. Show that $H \lhd G$ if and only if $\mathrm{Stab}_G(H) = G$ (under the conjugation action of G on W), and more generally that $H \lhd \mathrm{Stab}_G(H)$.

(e) Deduce that $\mathrm{Stab}_G(H) = N_G(H)$, the *normaliser* of H in G, which is by definition the largest subgroup $N < G$ such that $H \lhd N$.

Solution:

(a) $a_g(h_1h_2) = gh_1h_2g^{-1} = (gh_1g^{-1})(gh_2g^{-1}) = a_g(h_1h_2)$.

(b) If $h \in H$ and $h' \in [h]$, the conjugacy class of h, then $h' = ghg^{-1} \in gHg^{-1} = H$ so $[h] \subset H$. So $H = \bigcup_{h\in H}[h]$.

(c) We need to show that $gHg^{-1} < G$ if $H < G$. But $gHg^{-1} \neq \varnothing$ because it contains $1 = g1g^{-1}$, and if $a = ghg^{-1}$, $b = gh'g^{-1} \in gHg^{-1}$, then $ab^{-1} = (ghg^{-1})(gh'g^{-1})^{-1} = (ghg^{-1})(gh'^{-1}g^{-1}) = ghh'^{-1}g^{-1} \in gHg^{-1}$. And we also need to show that that this is a group action, i.e. that $a(g_1g_2, H) = a(g_1, a(g_2, H))$, but $a(g_1, a(g_2, H)) = g_1(g_2Hg_2^{-1})g_1^{-1} = (g_1g_2)H(g_1g_2)^{-1} = a(g_1g_2, H)$.

(d) If $H \lhd G$ then for any $g \in G$ we have $gH = Hg$ so $gHg^{-1} = H$; that is, g stabilises $H \in \mathcal{W}$ under conjugation. So $\mathrm{Stab}_G(H) = G$. Conversely, if $\mathrm{Stab}_G(H) = G$ then $gHg^{-1} = H$ for all g so $gH = Hg$ for all g. More generally, even if $\mathrm{Stab}_G(H) \neq G$, if $g \in \mathrm{Stab}_G(H)$ then $gHg^{-1} = H$ so $gH = Hg$, so $H \lhd \mathrm{Stab}_G(H)$.

(e) We need to show that if $K < G$ and $H \lhd K$ then $K \subseteq \mathrm{Stab}_G(H)$, as then any subgroup of G in which H is normal is contained in $\mathrm{Stab}_G(H)$, which is what it means for that to be the biggest such subgroup. But if not, then there exists $k \in K$ such that $kH \neq Hk$; say $kh \notin Hk$ (or $hk \notin kH$, which is similar). But then $khk^{-1} \notin H$ so $kHk^{-1} \neq H$ so H is not a normal subgroup of K.

2 Compute the following products of permutations:

(a) $(134)(125)(453)$

(b) $(12)(13)(12)$

(c) $(134)^{-1}(12)(34)(134)$

(d) $(134)^{-1}(12)(24)(134)$

Solution:

(a) (12543)

(b) (23)

(c) $(13)(42)$

(d) $(42)(23)$, *or* (234) *which is the same thing.*

3 Show that the dihedral group D_{2n} (the symmetries of an n-gon) is generated by two elements of order 2 by showing the following things:

(a) If $n = 2m - 1$ is odd, then D_{2n} is generated by the rotation $a = (123 \ldots n)$ and the reflection $b = (2\ n)(3\ n-1) \ldots (m\ m+1)$; if $n = 2m$ is even then instead $b = (1\ n)(2\ n-1) \ldots (m\ m+1)$.

(b) a has order n and b has order 2.

(c) bab^{-1} also has order n.

(d) $c = ba$ has order 2. Thus D_{2n} is generated by b and c, with relation $b^2 = c^2 = (bc)^n = 1$.

Solution:

(a) a is the rotation by $2\pi/n$ and b is the reflection in the vertical axis, if we put vertex 1 at the bottom (odd case) or bottom right (even case) and number the vertices anticlockwise.

(b) Obvious from the description above or from the permutations.

(c) In fact $o(ghg^{-1}) = o(h)$ in all cases, because $(ghg^{-1})^r = gh^r g^{-1}$ which is 1 if and only if $h^r = 1$.

(d) $bab = bab^{-1}$ is the opposite rotation: the multiplication gives $bab = (1\ n\ n-1\ \ldots\ 2)$ (odd case) or $bab = (n\ n-1\ \ldots\ 1)$ (even case) which is a^{-1} in both cases, so $c^2 = baba = (bab)a = a^{-1}a = 1$. These two generate D_{2n} becuse $bc = bba = a$ so we recover a and b, and $(bc)^n = a^n = 1$.

4 For each of the following polynomials in $\mathbb{Q}[t]$, say whether it is irreducible or not.

(a) $t^5 + 132t^4 - 99t^3 - 143t^2 + 121t + 11$. *[Eisenstein.]*

(b) $t^5 + 132t^4 - 99t^3 - 143t^2 + 121t + 34$. *[Look for a linear factor.]*

(c) $t^4 + 4t^3 - 3t^2 - 14t + 8$. *[Subtract $(t^2 + 2t - 3)^2$.]*

Solution:

(a) $t^5 + 132t^4 - 99t^3 - 143t^2 + 121t + 11$ *is Eisenstein with* $p = 11$*, hence irreducible.*

(b) $t^5 + 132t^4 - 99t^3 - 143t^2 + 121t + 34$ *has a factor of* $t + 1$ *since putting* $t = -1$ *gives* $-1 + 132 + 99 - 143 - 121 + 34 = -1 + 11 \times (12 + 9 - 13 - 11 + 3) + 1 = 0$.

(c) $t^4+4t^3-3t^2-14t+8-(t^2+2t-3)^2 = -(t+1)^2$ *so* $t^4+4t^3-3t^2-14t+8 = (t^2 + 2t - 3)^2 - (t + 1)^2$ *which factorises because it is the difference of two squares,* $t^4 + 4t^3 - 3t^2 - 14t + 8 = ((t^2 + 2t - 3) + (t + 1))((t^2 + 2t - 3) - (t + 1))$*, so it is reducible.*

5 What is the characteristic of each of these rings?

(a) $\mathbb{F}_{25}$

(b) $\mathbb{F}_{25}[t]$

(c) $\mathbb{F}_{25}[t]/\langle t^2 \rangle$

(d) $\mathbb{Z}/25\mathbb{Z}$

(e) $R/3R$, where $R = \mathbb{Z}/15\mathbb{Z}$

(f) $\mathbb{Z}[t]/\langle t^5 \rangle$

(g) $\mathrm{Hom}(R, S)$, the set of ring homomorphisms $\varphi\colon R \to S$ where R and S are rings, with addition and multiplication defined by $(\varphi + \psi)(r) = \varphi(r) + \psi(r)$ and $(\varphi\psi)(r) = \varphi(r)\psi(r)$.

Solution: *5, 5, 5, 25, 3, 0, the characteristic of* S*.*

GKS, 27/4/25