GROUPS AND RINGS (MA22017)

SOLUTIONS TO PROBLEM SHEET 1

1 W. In each of the following cases say whether H is a subgroup of G. If it is, say whether it is a normal subgroup.

(a) $G = \mathbb{Z}$ and $H = \{2^k \mid k \in \mathbb{N}\}$

(b) $G = \mathbb{Q}^*$ (with multiplication) and $H = \{2^k \mid k \in \mathbb{Z}\}$.

(c) $G = \mathrm{SL}(2, \mathbb{R})$ and $H = \mathrm{SL}(2, \mathbb{Z})$.

Solution:

(a) This not a subgroup: for example, 2 has no inverse. This is because $-2 \notin H$, not because $\frac{1}{2} \notin H$: the alleged group operation is addition, the same as in $\mathbb{Z}$, not multiplication.

(b) This is a subgroup because $H \neq \varnothing$ and $2^k(2^l)^{-1} = 2^{k-l} \in H$, which verifies the condition for H to be a subgroup in Lemma I.14. Notice that here the group operation is multiplication: that's what it has to be to make $\mathbb{Q}^*$ into a group, so we don't have to specify it explicitly. H is a normal subgroup, because G is abelian.

(c) This is a subgroup, because the condition of being in SL *is that the determinant is 1 so the inverse of an integral matrix is again integral, and of course the product of integral matrices is integral. But it is not normal: e.g.*

$$\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{5}{4} \\ 1 & -\frac{1}{2} \end{pmatrix}$$

so we have found an element $g \in \mathrm{SL}(2, \mathbb{R})$ and an element $h \in \mathrm{SL}(2, \mathbb{Z})$ such that $g^{-1}hg \notin \mathrm{SL}(2, \mathbb{Z})$.

2 H Prove or disprove the following statements.

(a) If G is a group and H and K are subgroups of G, then $H \cap K$ is always a subgroup of G.

(b) If G is a group and H and K are subgroups of G, then $H \cup K$ is always a subgroup of G.

(c) If G is a group and H and K are subgroups of G, and $K \lhd G$, then $H \cap K \lhd G$.

(d) If G is a group and H and K are subgroups of G, and $K \lhd G$, then $H \cap K \lhd H$.

Solution:

(a) Yes. It is enough to check that $g_1 g_2^{-1} \in H \cap K$ if $g_1, g_2 \in H \cap K$; but in that case, $g_1 g_2^{-1} \in H$ because $g_1, g_2 \in H$ and similarly for K.

(b) No. For example, take $G = \mathbb{Z}$, and $H = 2\mathbb{Z}$ and $K = 3\mathbb{Z}$. Then $H \cup K \not\ni 5 = 2 + 3$ so $H \cup K$ is not closed under the group operation (which is + here). Or you could look at the example from vector spaces: the $x - y$ plane and the z-axis are both subgroups of $\mathbb{R}^3$ (with addition, of course) but their union is not.

(c) No. But it is not enough to say "it is not necessarily the case that $ghg^{-1} \in H \cap K$": you must give a counterexample, not just a failed attempt at a proof. For example, we know that $A_n \lhd S_n$ (check it!), so let us take $G = S_4$ and $K = A_4$. Then we could take H to be the cyclic subgroup of order 3 generated by (123): that is contained in A_3 so $K \cap H = H$, but $(14)(123)(14) = (423) \notin H$ so H is not a normal subgroup of $G = S_4$.

(d) Yes. If $h \in H$ and $k \in H \cap K$ then $h^{-1}kh \in K$ because $K \lhd G$ and $h \in G$, and $h^{-1}kh \in H$ because $h, k \in H$. With (iii), this gives an example of a subgroup that is normal in H but not in $G > H$; a simpler example is to take any non-normal subgroup of H and observe that $H \lhd H$.

3 W Discuss the question in 1.17: is $\mathbb{Z}/2$ a subgroup of $\mathbb{Z}/6$? Is $\mathbb{Z}/3$ a subgroup of $\mathbb{Z}/6$? Also: is $\mathbb{Z}/2$ a subgroup of $\mathbb{Z}/2 \times \mathbb{Z}/2$?

Solution: *The first at least of these is genuinely debatable. At one level, no: residue classes mod 2 are not residue classes mod 6. On the other hand, there is a unique subgroup of $\mathbb{Z}/6$, uniquely isomorphic to $\mathbb{Z}/2$, generated by 3. What about $\mathbb{Z}/3$? Again there is a unique subgroup isomorphic to $\mathbb{Z}/3$ but is there really a preferred isomorphism? Do you care if there isn't? The third is slightly different again: this time there is a subgroup isomorphic to $\mathbb{Z}/2$, but it isn't unique.*

4 W/H Prove:

(a) **(W)** Lemma 1.27 (the image of a group homomorphism is a group);

(b) **(H)** Lemma 1.29 (a group homomorphism is injective if and only if its kernel is trivial);

(c) **(W)** 1.34 (isomorphism is an equivalence relation on groups: first formulate a precise statement of this).

Solution:

(a) *If* $\varphi\colon G \to H$ *is a group homomorphism and* $h_1,\, h_2 \in H_0 := \operatorname{Im}\varphi$ *then* $h_1 = \varphi(g_1)$ *and* $h_2 = \varphi(g_2)$ *for some* $g_1,\, g_2 \in G$ *then*

$$h_1 h_2^{-1} = \varphi(g_1)\varphi(g_2)^{-1} = \varphi(g_1)\varphi(g_2^{-1}) \in \operatorname{Im}\varphi = H_0$$

So H_0 *is a subgroup as long as it is not empty: but* $H_0 \ni \varphi(1_G) = 1_H$.

(b) *If* $\operatorname{Ker}\varphi$ *(notation as above) is not trivial then it contains at least two elements (*1_G *and something else) both of which are mapped to* 1_H *by* φ*, so* φ *is not injective. Conversely, suppose that* $\operatorname{Ker}\varphi = \{1_G\}$ *and suppose that* $\varphi(g_1) = \varphi(g_2)$*, for* $g_1,\, g_2 \in G$*. Then* $1_H = \varphi(g_1)\varphi(g_2)^{-1} = \varphi(g_1 g_2^{-1})$*, so* $g_1 g_2^{-1} \in \operatorname{Ker}\varphi$ *so* $g_1 g_2^{-1} = 1_H$*. Therefore* $g_1 = g_2$*, i.e.* φ *is injective.*

(c) *The precise statement is this: suppose* G*,* H*, and* K *are groups. Then* $G \cong G$*; if* $G \cong H$ *then* $H \cong G$*; and if* $G \cong H$ *and* $H \cong K$ *then* $G \cong K$*. You have to say it this way: you can't talk about "the set of all groups".*

For the first, observe that $\mathrm{id}\colon G \to G$ *is an isomorphism, with inverse again* id*.*

For the second, suppose that $\varphi\colon G \to H$ *is an isomorphism, with inverse* $\psi\colon H \to G$*. Then* ψ *is an isomorphism, with inverse* φ*: there is nothing to check.*

For the third, suppose that $\varphi\colon G \to H$ *is an isomorphism with inverse* ψ *and* $\varphi'\colon H \to K$ *is an isomorphism with inverse* ψ'*. Then clearly* $\varphi'\varphi$ *is an isomorphism with inverse* $\psi\psi'$*, provided those maps are homomorphisms. But the composite of two homomorphisms is a homomorphism. This has been implicit but not stated directly so we should check it: if* $\theta\colon G \to H$ *and* $\theta'\colon H \to K$ *are group homomorphisms and* $a,\, b \in G$ *then*

$$\begin{aligned}(\theta'\theta)(ab) &= \theta'(\theta(ab)) = \theta'(\theta(a)\theta(b)) \\ &= \theta'(\theta(a))\theta'(\theta(b)) = (\theta'\theta(a))((\theta'\theta)(b))\end{aligned}$$

so $\theta'\theta$ *is indeed always a homomorphism.*

GKS, 4/2/26