GROUPS AND RINGS (MA22017)

SEMESTER 2 MATHEMATICS: PROBLEM SHEET 8

Homework questions, marked **H***, should be handed in according to the directions given by your tutor. Other questions are marked* **W** *for warmup or work or* **E** *for extra or enthusiast. A copy of this sheet is on Moodle and at*

`http://people.bath.ac.uk/masgks/MA22017/sheet8.pdf`

1 W Suppose that R is a nontrivial ring (i.e. $0_R \neq 1_R$). Prove the assertion made in lectures that the map $i\colon X \to F_X$ in the definition of a free module is automatically injective.

(a) Suppose that i is not injective, so that there exist $x, x' \in X$ with $x \neq x'$ but $i(x_0) = i(x_1)$. Construct an R-module M (you could take $M = R$) and a map of sets $f\colon X \to M$ such that $f(x) \neq f(x')$.

(b) Show that no linear map $\varphi\colon F_X \to M$, as required by the definition of free module, can exist.

2 H Choose an integral domain R (your choice) and give an example of a free R-module, an example of a torsion R-module and an example of an R-module that is neither free nor torsion.

3 H Explain why $2\mathbb{Z}$ cannot be a direct summand of $\mathbb{Z}$ as a $\mathbb{Z}$-module.

4 E Show that if X and Y have the same cardinality then any free module on X is isomorphic to any free module on Y, as follows.

Let $a\colon X \to Y$ be a bijection and let $b\colon Y \to X$ be its inverse. Suppose that F_X is free on X with respect to the map $i\colon X \to F_X$ and F_Y is free on Y with respect to the map $j\colon Y \to F_Y$.

(a) In the diagram that arises from the fact that F_X is free, put $M = F_Y$ and hence construct a linear map $\alpha\colon F_X \to F_Y$.

(b) Similarly construct a linear map $\beta\colon F_Y \to F_X$.

(c) In the diagram that arises from the fact that F_X is free, put $M = F_X$ and hence deduce that there is a *unique* linear map $\varphi\colon F_X \to F_X$ with a certain property. Show that both id_{F_X} and $\beta\alpha$ have this property.

(d) Do the same thing with F_Y, and deduce that α and β are mutually inverse maps and hence (by definition of isomorphism) they are isomorphisms.

GKS, 28/3/25