GROUPS AND RINGS (MA22017)

SEMESTER 2 MATHEMATICS: PROBLEM SHEET 6

Homework questions, marked **H**, *should be handed in according to the directions given by your tutor. Other questions are marked* **W** *for warmup or work or* **E** *for extra or enthusiast. A copy of this sheet is on Moodle and at*

`http://people.bath.ac.uk/masgks/MA22017/sheet6.pdf`

1 H. In this question R is a commutative ring and I and J are ideals in R. Say whether each of the following statements is true or not: give a proof or a counterexample.

(a) If I and J are both prime ideals then $I \cap J$ is a prime ideal.

(b) If I and J are both prime ideals then IJ is a prime ideal.

(c) If I and J are both prime ideals then $I + J$ is a prime ideal.

(d) If I and J are both maximal ideals then $I \cap J$ is a maximal ideal.

(e) If I and J are both maximal ideals then IJ is a maximal ideal.

(f) If I and J are both maximal ideals then $I + J$ is a maximal ideal.

2 W Recall that $\xi \in \mathbb{C}$ is said to be algebraic if the ideal K_ξ, which is defined to be the kernel of $\mathrm{ev}_\xi \colon \mathbb{Q}[t] \to \mathbb{C}$, is not zero. Show that the image $\mathbb{Q}[\xi]$ of ev_ξ is a field if and only if ξ is algebraic. (Hint: Notice that $\mathbb{Q}[\xi]$ is the image of ev_ξ: use the first isomorphism theorem and Theorem VI.24.)

3 W Justify the assertion made in lectures that 2, $3\,1 \pm \sqrt{-5}$ are all irreducible in $\mathbb{Z}[\sqrt{-5}]$. Put $N(x) = x\bar{x}$ for any $x \in \mathbb{Z}[\sqrt{-5}]$, where $\bar{x}$ denotes the complex conjugate.

(a) Show that $N(xy) = N(x)N(y)$.

(b) Show that $N(x)$ for $x \neq 0$ is a positive integer of the form $a^2 + 5b^2$.

(c) Show that if $N(x) = 1$ then $x \in \mathbb{Z}[\sqrt{-5}]$ is a unit.

(d) Show that if x is reducible then $N(x)$ is the product of two integers greater than 1 and of the form $a^2 + 5b^2$.

(e) Compute $N(x)$ for each of the four elements above.

(f) Hence deduce that these elements are irreducible.

4 H Suppose that R is a principal ideal domain. For each of the following rings, say whether it is necessarily a PID too: give a proof, or a counterexample.

(a) A nontrivial subring S of R that contains 1_R.

(b) A quotient ring $A = R/I$ where I is a prime ideal (so A is an integral domain).

(c) The ring $R[t]$.

(d) A quotient $R[t]/J$ where J is a prime ideal.

GKS, 14/3/25