GROUPS AND RINGS (MA22017)

SEMESTER 2 MATHEMATICS: PROBLEM SHEET 5

Homework questions, marked **H**, *should be handed in according to the directions given by your tutor. Other questions are marked* **W** *for warmup or work or* **E** *for extra or enthusiast. A copy of this sheet is on Moodle and at*

`http://people.bath.ac.uk/masgks/MA22017/sheet5.pdf`

1 H Let R be a commutative ring, and let $a \in R$. Show that if R is an integral domain then the equation $x^2 = a$ has at most two solutions in R. Find a commutative ring R and an element $a \in R$ such that $x^2 = a$ has more than two solutions.

2 W Consider the evaluation homomorphism $\varphi\colon \mathbb{R}[t] \to \mathbb{C}$ defined by setting $\phi(f) = f(i)$. Identify $\mathrm{Ker}(\phi)$: using the division algorithm, prove carefully that your answer is correct.
What does the First Isomorphism Theorem tell us in this case?

3 W Prove that if I and J are ideals in a ring R, then $I + J$, IJ and $I \cap J$ are ideals in R and $IJ \subseteq I \cap J \subseteq I + J$.

4 H Let R be a finite ring, i.e. the number $|R|$ of elements of R is finite. Show that $|R|$ is divisible by $\operatorname{char} R$. Deduce that if $|R| = p$ is prime, then $R \cong \mathbb{Z}/p\mathbb{Z}$.
By considering the map $m_a\colon R \to R$ given by $m_a(b) = ab$, or otherwise, show that a finite integral domain is a field.

GKS, 7/3/25