GROUPS AND RINGS (MA22017)

SEMESTER 2 MATHEMATICS: PROBLEM SHEET 4

Homework questions, marked **H**, *should be handed in according to the directions given by your tutor. Other questions are marked* **W** *for warmup or work or* **E** *for extra or enthusiast. A copy of this sheet is on Moodle and at*

`http://people.bath.ac.uk/masgks/MA22017/sheet4.pdf`

1 H For each of the following commutative rings, say whether it is an integral domain, a field, or neither. Give brief reasons. What are the units in each case?

(a) The set of *Gaussian integers* $\mathbb{Z}[i] = \big\{a + ib \in \mathbb{C} \mid a, b \in \mathbb{Z}\big\}$, (where $i = \sqrt{-1}$) with the usual operations of complex numbers.

(b) $\mathbb{Z}/9$, with the usual operations.

(c) $\mathbb{C}[t]$.

(d) $\mathbb{Q}[i] = \big\{a + ib \in \mathbb{C} \mid a, b \in \mathbb{Q}\big\}$, with the usual operations of complex numbers.

2 E Is it true that if a finite G acts on a set X and the orbits $\mathrm{orb}_G(x)$ and $\mathrm{orb}_G(y)$ are the same size, then $\mathrm{Stab}_G(x) \cong \mathrm{Stab}_G(y)$? Give a proof or a counterexample.

3 W Decide whether each of the following is a subring, an ideal, or neither; prove your assertions.

(a) $S_1 = \{-1, 0, 1\} \subset \mathbb{Z}$;

(b) $S_2 = \{a_0 + a_2t^2 + a_4t^4 + \cdots . \mid a_i \in \mathbb{Q}\} \subset \mathbb{Q}[t]$;

(c) $S_3 = \{a_2t^2 + a_3t^3 + a_4t^4 + \cdots . \mid a_i \in \mathbb{Q}\} \subset \mathbb{Q}[t]$;

(d) $S_4 = \{\text{polynomials of degree } \leq 2\} \subseteq \mathbb{Q}[t]$;

(e) $S_5 = \{p \in \mathbb{Q}[t] \mid p(1) = 0\} \subset \mathbb{Q}[t]$.

4 H Show that if R is an integral domain, a, b, $c \in R$, and $ab = ac$, and $a \neq 0$, then $b = c$: that is, one may cancel. Is this the same as the statement "multiplication by a is injective"?

GKS, 28/2/25