GROUPS AND RINGS (MA22017)

SEMESTER 2 MATHEMATICS: PROBLEM SHEET 3

Homework questions, marked **H**, *should be handed in according to the directions given by your tutor. Other questions are marked* **W** *for warmup or work or* **E** *for extra or enthusiast. A copy of this sheet is on Moodle and at*

`http://people.bath.ac.uk/masgks/MA22017/sheet3.pdf`

1 W Consider the map $\varphi\colon \mathbb{R} \to \mathbb{C}^*$ given by $\varphi(x) = e^{2\pi i x}$. (Remember what the group operations on $\mathbb{R}$ and $\mathbb{C}^*$ are.) Verify that φ is a group homomorphism. What is its kernel? Describe the three maps π, $\bar{\varphi}$ and ι from the factorisation in Corollary II.26.

2 H,E In each of the following cases say what the kernel and image of the group homomorphism φ are and describe π, $\bar{\varphi}$ and ι briefly.

(a) **H** $\varphi\colon S_n \to \mathbb{Z}/2$ where $\varphi(\sigma)$ is the signature of σ.

(b) **E** Suppose p is a prime number, and remember the notation $\mathbb{F}_p$, which is $\mathbb{Z}/p$ but as a filed, i.e. with multiplication mod p as well as addition mod p. Take $\varphi\colon \mathrm{SL}(2,\mathbb{Z}) \to \mathrm{SL}(2,\mathbb{Z}/p)$ to be the reduction mod p map: that is, $\varphi(M)$ is M mod p. *[The hard part is to determine the image of φ: you may want to use the Chinese Remainder Theorem.]*

3 W In I.40 we mentioned "the smallest subgroup that contains S" (a subset of G) as another way to describing $\langle S\rangle$. Let G be a group, suppose $S \subset G$ and let H be the intersection of all (not necessarily proper) subgroups of G that contain S. Show that H is a subgroup, and that any subgroup that contains S also contains H. Deduce that $H = \langle S\rangle$.

4 W,E

(a) **W** Let G be a group and suppose $S \subseteq G$ is a subset. Is there a smallest normal subgroup of G that contains S? If so, can you describe what the elements look like?

(b) **E** If $H < G$, define the normaliser $N_G(H)$ to be the largest subgroup of G such that H is normal in $N_G(H)$. Make this definition precise, and show that $N_G(H)$ is a subgroup of G. Is $N_G(H)$ a normal subgroup of G?

5 E Prove the assertions in III.17(v) in the notes: that in the action of $\mathrm{SL}(2,\mathbb{Z})$ on the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}\, z > 0\}$, the stabiliser of most $z \in \mathbb{H}$ is $\pm I$, but the stabiliser of $i \in \mathbb{H}$ is a group of order 4 generated by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and the stabiliser of $\omega = e^{2\pi i/3}$ is of order 6, generated by $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$.

6 H Prove the assertion in the proof of Proposition III.18, that left multiplication by G on $X = \{gH \mid g \in G\}$ defines a group action and that the stabiliser of $1_G H$ is H.

GKS, 19/2/25