GROUPS AND RINGS (MA22017)

SEMESTER 2 MATHEMATICS: PROBLEM SHEET 2

Homework questions, marked H, should be handed in according to the directions given by your tutor. A copy of this sheet is on Moodle and at

`http://people.bath.ac.uk/masgks/MA22017/sheet2.pdf`

1 W In each of the following cases say what the order $|G|$ of G is.

(a) $G = S_5$, the symmetric group on 5 letters.

(b) The alternating group A_5.

(c) $\mathbb{Z}/n$.

(d) The subgroup $n\mathbb{Z}$ of $\mathbb{Z}$

2 H In each of the following cases say what the order $o(g)$ of g is. Verify that $o(g)$ divides $|G|$.

(a) $G = S_5$ and $g = (13)(245)$.

(b) $G = \mathrm{SL}(2, \mathbb{F}_2)$ and $g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. (By $\mathbb{F}_2$ we mean $\mathbb{Z}/2$ considered as a field, i.e. with both addition and multiplication mod 2.)

3 W Suppose that G is a cyclic group of finite order n and that g generates G (in this case, we say that g is a generator). Is g^2 a generator? Which other elements of G are generators? How many of them are there?

4 W By considering left and right cosets, show that if $H < G$ and $|G : H| = 2$ then $H \lhd G$. Give an example of a group G and a non-normal subgroup of index 3.

5 H Suppose that G is a group and H is a subgroup.

(a) Verify that there is no such thing as "left index" and "right index": the number of left cosets of H in G is equal to the number of right cosets. You may wish to consider the map $gH \mapsto Hg^{-1}$.

(b) Suppose that every non-identity element of G has order 2. Show that G is abelian.

(c) Show that the alternating group A_4, which is of order 12, does not have a subgroup of order 6: hence the converse of Lagrange's Theorem is false in general.

6 E Verify the assertion made in 2.20(iv): the group $\mathrm{Aff}(\mathbb{R}^2)$ of affine linear transformations of a plane has a normal subgroup consisting of the translations. The quotient is isomorphic to $\mathrm{GL}(2,\mathbb{R})$, but although $\mathrm{GL}(2,\mathbb{R})$ is a subgroup of $\mathrm{Aff}(\mathbb{R}^2)$, it is not a normal subgroup.

GKS, 13/2/25