GROUPS AND RINGS (MA22017)

SEMESTER 2 MATHEMATICS: PROBLEM SHEET 10

This is a revision sheet on the Sections 1–6. A copy of this sheet is on Moodle and at

`http://people.bath.ac.uk/masgks/MA22017/sheet10.pdf`

1 Recall (Example III.7(viii)) that any group G acts on itself by conjugation: $a(g,h) = ghg^{-1}$. The orbits are called *conjugacy classes*.

(a) Show that for this action, the map $a_g \colon G \to G$ is in fact a group homomorphism.

(b) Show that any normal subgroup of G is a union of conjugacy classes.

(c) Let $\mathcal{W}$ denote the set of all subgroups of G. Show that G acts on $\mathcal{W}$ by conjugation.

(d) Suppose $H \leq G$, so $H \in \mathcal{W}$. Show that $H \lhd G$ if and only if $\mathrm{Stab}_G(H) = G$ (under the conjugation action of G on W), and more generally that $H \lhd \mathrm{Stab}_G(H)$.

(e) Deduce that $\mathrm{Stab}_G(H) = N_G(H)$, the *normaliser* of H in G, which is by definition the largest subgroup $N < G$ such that $H \lhd N$.

2 Compute the following products of permutations:

(a) $(134)(125)(453)$

(b) $(12)(13)(12)$

(c) $(134)^{-1}(12)(34)(134)$

(d) $(134)^{-1}(12)(24)(134)$

3 Show that the dihedral group D_{2n} (the symmetries of an n-gon) is generated by two elements of order 2 by showing the following things:

(a) If $n = 2m-1$ is odd, then D_{2n} is generated by the rotation $a = (123\dots n)$ and the reflection $b = (2\ n)(3\ n{-}1)\dots(m\ m{+}1)$; if $n = 2m$ is even then instead $b = (1\ n)(2\ n-1)\dots(m\ m+1)$.

(b) a has order n and b has order 2.

(c) bab^{-1} also has order n.

(d) $c = ba$ has order 2. Thus D_{2n} is generated by b and c, with relation $b^2 = c^2 = (bc)^n = 1$.

4 For each of the following polynomials in $\mathbb{Q}[t]$, say whether it is irreducible or not.

(a) $t^5 + 132t^4 - 99t^3 - 143t^2 + 121t + 11$. *[Eisenstein.]*

(b) $t^5 + 132t^4 - 99t^3 - 143t^2 + 121t + 34$. *[Look for a linear factor.]*

(c) $t^4 + 4t^3 - 3t^2 - 14t + 8$. *[Subtract* $(t^2 + 2t - 3)^2$*.]*

5 What is the characteristic of each of these rings?

(a) $\mathbb{F}_{25}$

(b) $\mathbb{F}_{25}[t]$

(c) $\mathbb{F}_{25}[t]/\langle t^2 \rangle$

(d) $\mathbb{Z}/25\mathbb{Z}$

(e) $R/3R$, where $R = \mathbb{Z}/15\mathbb{Z}$

(f) $\mathbb{Z}[t]/\langle t^5 \rangle$

(g) $\mathrm{Hom}(R, S)$, the set of ring homomorphisms $\varphi\colon R \to S$ where R and S are rings, with addition and multiplication defined by $(\varphi + \psi)(r) = \varphi(r) + \psi(r)$ and $(\varphi\psi)(r) = \varphi(r)\psi(r)$.

GKS, 27/4/25