GROUPS AND RINGS (MA22017)

SEMESTER 2 MATHEMATICS: PROBLEM SHEET 0

These questions are not for handing in. A copy of this sheet is at

`http://people.bath.ac.uk/masgks/MA22017/sheet0.pdf`

The notation $\mathbb{Z}$ means the group of integers where the operation is addition. (Why not multiplication?) $\mathbb{Z}/r$ means the integers mod r: you may have called this $\mathbb{Z}_r$ or $\mathbb{Z}/r\mathbb{Z}$ or C_r, which are common alternatives.

0. Find your notes from the algebra course you did in Semester 1 of your *first* year (this may be the hard part) and read them again, paying special attention to Subsections 1.3 (Prime factorisation), 1.4 (Modular arithmetic) and 2.6 (Permutations), and to Section 3 (Polynomials and unique factorisation).

1. Compute, by hand (or even in your head):

(a) The product of permutations (125)(26)(4563)(23)

(b) The highest common factor d of the integers 1547 and 7531.

(c) Integers λ and μ such that $1547\lambda + 7531\mu = d$

(d) Polynomials $q(t)$ and $r(t)$, with rational coefficients, such that

$$39t^4 - 21t^3 + 6t^2 + t - 11 = q(t)(3t^3 + t^2 + 2t - 8) + r(t)$$

and $\deg r < 3$.

2. Which of the following are groups and which are not? If not, what is wrong?

(a) $\mathbb{N}$ with addition.

(b) $\mathbb{N}$ with multiplication.

(c) $\mathbb{R}^3$ with vector cross product.

(d) $\mathbb{R}^3$ with vector dot product.

(e) For a fixed non-empty set S, the set of functions $f\colon S \to \mathbb{Z}/2$, with addition: that is, we define $(f_1 + f_2)(x) := f_1(x) + f_2(x)$, where the $+$ on the right-hand side is addition in $\mathbb{Z}/2$.

(f) $H \subset S_5$, where $\sigma \in H$ if $\sigma = \rho$ or $\sigma = (45)\rho$ for some $\rho \in S_3$.

(g) For a fixed non-empty set S, the power set of S (the set of all subsets of S) with the operation of symmetric difference: $A \cdot B = (A \cup B) \smallsetminus (A \cap B)$.

3. Here is a list of maps between groups. Say which ones are group homomorphisms. Don't guess: try to find a proof.

(a) $\phi\colon \mathbb{Z} \to \mathbb{Z}$ given by $\phi(x) = 0$.

(b) $\phi\colon \mathbb{Z} \to \mathbb{Z}$ given by $\phi(x) = 1$.

(c) $\phi\colon \mathbb{Z} \to \mathbb{Z}$ given by $\phi(x) = -x$.

(d) $\phi\colon \mathbb{Z} \to \mathbb{Z}$ given by $\phi(x) = x^2$.

(e) $\varepsilon\colon S_n \to \mathbb{Z}/2$, where ε is the sign of σ, i.e. 0 if σ is an even permutation and 1 if it is odd.

(f) $\eta\colon S_n \to \mathbb{Z}/2$ where $\eta(\sigma) = 0$ if σ is the product of an even number of disjoint cycles (possibly of length 1, i.e. fixed points) and 1 if that number is odd.

GKS, 4/2/25