GROUPS AND RINGS (MA22017)

SEMESTER 2 MATHEMATICS: REVISION SHEET

This is a revision sheet on the Sections 1–6. A copy of this sheet is on Moodle and at

`http://people.bath.ac.uk/masgks/MA22017/sheet.revision.pdf`

1 Recall (Example III.7(viii)) that any group G acts on itself by conjugation: $a(g,h) = ghg^{-1}$. The orbits are called *conjugacy classes*.

(a) Show that for this action, the map $a_g \colon G \to G$ is in fact a group homomorphism.

(b) Show that any normal subgroup of G is a union of conjugacy classes.

(c) Let $\mathcal{W}$ denote the set of all subgroups of G. Show that G acts on $\mathcal{W}$ by conjugation.

(d) Suppose $H \leq G$, so $H \in \mathcal{W}$. Show that $H \lhd G$ if and only if $\mathrm{Stab}_G(H) = G$ (under the conjugation action of G on W), and more generally that $H \lhd \mathrm{Stab}_G(H)$.

(e) Deduce that $\mathrm{Stab}_G(H) = N_G(H)$, the *normaliser* of H in G, which is by definition the largest subgroup $N < G$ such that $H \lhd N$.

2 Compute the following products of permutations:

(a) $(134)(125)(453)$

(b) $(12)(13)(12)$

(c) $(134)^{-1}(12)(34)(134)$

(d) $(134)^{-1}(12)(24)(134)$

3 Show that the dihedral group D_{2n} (the symmetries of an n-gon) is generated by two elements of order 2 by showing the following things:

(a) If $n = 2m-1$ is odd, then D_{2n} is generated by the rotation $a = (123\ldots n)$ and the reflection $b = (2\ n)(3\ n{-}1)\ldots(m\ m{+}1)$; if $n = 2m$ is even then instead $b = (1\ n)(2\ n-1)\ldots(m\ m+1)$.

(b) a has order n and b has order 2.

(c) bab^{-1} also has order n.

(d) $c = ba$ has order 2. Thus D_{2n} is generated by b and c, with relation $b^2 = c^2 = (bc)^n = 1$.

Note about notation: We may write the statement in (d) as

$$D_{2n} = \langle b, c \mid b^2 = c^2 = (bc)^n = 1 \rangle,$$

meaning that D_{2n} is generated by b and c and they satisfy the relations $b^2 =$, $c^2 = 1$ and $(bc)^n = 1$, and no others apart from the ones that follow from those. It is also true that

$$D_{2n} = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1} \rangle.$$

Either of these may be taken as the definition of D_{2n}, if you want a definition purely in terms of elements, not referring to the polygon that D_{2n} acts on.
4 For each of the following polynomials in $\mathbb{Q}[t]$, say whether it is irreducible or not.

(a) $t^5 + 132t^4 - 99t^3 - 143t^2 + 121t + 11$. *[Eisenstein.]*

(b) $t^5 + 132t^4 - 99t^3 - 143t^2 + 121t + 34$. *[Look for a linear factor.]*

(c) $t^4 + 4t^3 - 3t^2 - 14t + 8$. *[Subtract $(t^2 + 2t - 3)^2$.]*

5 What is the characteristic of each of these rings?

(a) $\mathbb{F}_{25}$

(b) $\mathbb{F}_{25}[t]$

(c) $\mathbb{F}_{25}[t]/\langle t^2 \rangle$

(d) $\mathbb{Z}/25\mathbb{Z}$

(e) $R/3R$, where $R = \mathbb{Z}/15\mathbb{Z}$

(f) $\mathbb{Z}[t]/\langle t^5 \rangle$

6 Suppose G is a group and V is a finite-dimensional vector space (over some field K). We say that G acts linearly on V if G acts on V and $g(\lambda v + \mu w) = \lambda g(v) + \mu g(w)$, for all $g \in G$, v, $w \in V$, λ, $\mu \in K$. Recall that if G acts on X then $\alpha(g)$ is the element of $\mathrm{Sym}(X)$ given by $\alpha(g)(x) = g(x)$.

(a) Show that G acts linearly on V if and only if the image of $\alpha\colon G \to \mathrm{Sym}\, V$ is a subgroup of $\mathrm{GL}(V)$. (Remember that $\mathrm{Sym}\, V$ is the group consisting of all bijections from V to V, which may totally disregard the vector space structure: $\mathrm{GL}(V)$ is the group of bijective linear maps from V to V, sometimes called $L(V)$ or $\mathrm{Aut}(V)$.)

(b) Show that if G acts linearly on V then G also acts on the set (called $\mathbb{P}V$) of all lines through the origin in V, To do this, note that to specify $\ell \in \mathbb{P}V$ we only need to specify a non-zero point $v \in \ell$, because then $\ell = \{\lambda v \mid \lambda \in K\}$ and that μv and v determine the same ℓ if $\mu \neq 0$.

(c) If G acts on a set X, then X^G denotes the set of invariants of G: that is, $X^G = \{x \in X \mid \mathrm{Stab}_G(x) = G\}$. Show that if G acts linearly on V then V^G is the set of vectors that are eigenvectors of g with eigenvalue 1 for every $g \in G$.

(d) Show that $\ell = \{\lambda v \mid \lambda \in K\}$ is in $(\mathbb{P}V)^G$ if and only if v is an eigenvector of g for every $g \in G$.

7 Find all abelian groups of order 1400.

8

(a) $\mathbb{Z}/2$ and $\mathbb{Z}/4$ are both $\mathbb{Z}$-modules. Compute $\mathbb{Z}/2 \otimes \mathbb{Z}/4$.

(b) There are six $\mathbb{Z}$-modules (abelian groups) G of order 1400: you computed them in Question 7. For each of them, say what $\mathbb{Z}/2 \otimes G$ is. What about $\mathbb{Z}/5 \otimes G$ and $\mathbb{Z}/7 \otimes G$?

(c) Define a map $\mathbb{Z}/2 \times \mathbb{Z}/4 \to \mathbb{Z}/4$ by calling the elements of $\mathbb{Z}/2$ by the names 0_2 and 1_2, and those of $\mathbb{Z}/4$ by $0_4, 1_4, 2_4, 3_4$, and setting $a_2 b_4 = ab_4$, where ab denotes usual integer arithmetic. Does this make $\mathbb{Z}/4$ into a $\mathbb{Z}/2$-module?

9 Find the minimum polynomial of $\sqrt{2} + \sqrt[3]{2}$ over $\mathbb{Z}$ (you may use a computer).

GKS, 1/5/26