GROUPS AND RINGS (MA22017)

MOCK EXAM 2025

This is a mock exam for MA22017, because there aren't any real ones. The format is right, but this is probably a bit longer and harder than a real exam. A copy of this mock exam is on Moodle and at

`http://people.bath.ac.uk/masgks/MA22017/mockexam.pdf`

Section A

1.

(a) Say what it means for a map $\alpha\colon G \times X \to X$ to be an *action* of a group G on a set X. [2]

(b) Show that an action of G on X determines a homomorphism $G \to \mathrm{Sym}(X)$, where $\mathrm{Sym}(X)$ is the group of permutations of X. [2]

(c) Define what it means for an action to be *free*, to be *faithful*, and to be *transitive*. [3]

(d) For each of the conditions in (c), either give an example of an action satisfying the other two but not that one, or show that this is impossible. [3]

2 In this question, R is a commutative ring.

(a) Define what it means for an ideal I in R to be a *prime ideal*. [1]

(b) Define what it means for an ideal I in R to be a *maximal ideal*. [1]

(c) Prove that any maximal ideal is prime. [3]

(d) Define what it means for an ideal to be *finitely generated*. [1]

(e) Suppose that R is Noetherian; that is, every ideal in R is finitely generated. Show that if I_j is an ideals in R, for every $j \in \mathbb{N}$, and $I_j \subseteq I_{j+1}$, then there exists $N \in \mathbb{N}$ such that $I_j = I_N$ for every $j \geq N$. [4]

3 In this question R is a commutative ring.

(a) Define what it means for an R-module M to be *free*. [2]

(b) If M is an R-module and N is a submodule of M, define what it means for N to be *direct summand* of M. [2]

(c) Give, with justification, an example of a module M and a submodule N that is not a direct summand. [2]

(d) State and prove a sufficient condition for N to be a direct summand of a finitely generated R-module M. [2]

(e) Show by giving an example that the condition in (d) is not a necessary condition. [2]

4 Let G be a group, which may be infinite, and let H be a subgroup of G.

(a) Define what is meant by a *left coset* of H in G. [1]

(b) Show that if $g \in G$ then there is a unique left coset of H containing g. [2]

(c) Define what it means for H to be a *normal subgroup* of G. [1]

(d) Show that if $|G : H| = 2$ then H is a normal subgroup of G. Remember that G may be infinite. [2]

(e) By considering the group $G = D_8$ (the symmetries of a square), or otherwise, give an example of a group G with subgroups H_1 and H_2 such that $|G : H_1| = |G : H_2|$ but H_1 is a normal subgroup of G and H_2 is not. [4]

Section B

5 In this question R is an integral domain.

(a) Define what it means for an element of R to be *prime*, and what it means for an element of R to be *irreducible*. [2]

(b) Show that if $p \in R$ is prime then p is irreducible. [2]

(c) Show that if R is a UFD and $p \in R$ is irreducible then p is prime. [3]

(d) Suppose that R is a UFD, that $f \in R[t]$ is primitive, that $\deg f > 0$, and that $p \in R$ is prime. Put $S = R/pR$. Denote by red_p the quotient map $R[t] \to R[t]/\langle p \rangle = S[t]$. Suppose that $\deg \mathrm{red}_p(f) = \deg f$, and that $\mathrm{red}_p(f) \in S[t]$ is irreducible. Show that f is irreducible. [3]

(e) Suppose that $f \in R[t]$, and that there exists $g \in R[t]$ with $\deg g < \deg f$ such that $f + g^2 = h^2$ for some $h \in R[t]$. Show that f is reducible. [2]

(f) Are the following polynomials in $\mathbb{Z}[t]$ irreducible or not? [8]

(i) $t^3 - 14t^2 + 21t + 24$ *[Use (d)]*

(ii) $t^4 + 3t^2 + 10t - 21$ *[Use (e) with $g = t - 5$]*

(iii) $t^4 + 3t^2 + 9t - 21$

(iv) $t^4 + 4t^3 + 11t^2 + 4t + 26$ *[Put $t = s - 1$].*

6 In this question R is a commutative ring and K is a field.

(a) Define what is meant by the dual $M^\vee$ of M. [2]

(b) Show that if V is a finite-dimensional K-vector space then $V^\vee$ is isomorphic to V. [3]

(c) Give, with justification, an example of a ring R and a finitely generated R-module M such that $M^\vee$ is not isomorphic to M. [3]

(d) Let M be any R-module. Exhibit, with justification, a linear map $M \to M^{\vee\vee}$ from M to its double dual, which is injective in the case where $R = K$ and M is a finite-dimensional K-vector space. [3]

(e) Give an example to show that the map in (d) need not be injective in general. [3]

(f) Let X be the $\mathbb{Z}$-module consisting of all finite sequences of integers: that is, $X = \{f : \mathbb{Z} \to \mathbb{Z} \mid f(n) = 0 \text{for all but finitely many} n\}$.

(i) By considering the map $f \mapsto \sum_i f(i)$, show that the map $\mathrm{ev} : \mathbb{Z} \to X^\vee$ given by $\mathrm{ev}(r)(f) = f(r)$ is not surjective. [4]

(ii) Show that if $a : \mathbb{Z} \to \mathbb{Z}$ is a map of sets, there is a map $\hat{a} \in X^\vee$ given by $\hat{a}(\delta_i) = a(i)$, where $\delta_i \in X$ is the sequence with $\delta_i(j) = \delta_{ij}$. Deduce that in fact $X^\vee$ is uncountable, so cannot be isomorphic to X. [5]

7 In this question G is a group. Let G' be the group generated by the elements $[a, b] = aba^{-1}b^{-1}$ for $a,\ b \in G$.

(a) Show that $G' = 1$ if and only if G is abelian. [3]

(b) Show that G' is a normal subgroup of G. Deduce that if G is a non-abelian simple group then $G' = G$. [5]

(c) A group is called *solvable* if the sequence

$$G^{(1)} = G',\ G^{(2)} = (G')' \ldots$$

reaches $G^{(r)} = 1$ for some $r \in \mathbb{N}$. Show that the dihedral group D_{2n} is solvable. [6]

(d) Show that if $n > 4$ then the symmetric group S_n is not solvable. You may use the fact that A_n is simple for $n > 4$. [6]

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