

# MA22017 Groups and Rings, February-May 2025

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## What this course is about

This course introduces abstract algebra, starting with group theory and then moving on to rings. Later we will study modules, which are for rings what vector spaces are for fields, and finally come back to some linear algebra; but this is very definitely not a linear algebra course.

Here is a non-exhaustive list of conventions for these notes:

- $\mathbb{N}$  denotes the set of non-negative integers, i.e. including 0.
- Unless otherwise stated, group multiplication will be denoted by juxtaposition, so  $gh$  means  $g$  multiplied by  $h$  (on the right). If  $\phi$  and  $\psi$  are maps,  $\phi\psi$  will mean the composition  $\psi$  followed by  $\phi$ . Sometimes both these conventions will apply at once, i.e. the group multiplication will actually be composition.
- The integers mod  $n$  are denoted  $\mathbb{Z}/n$  (whether we think of them as an additive group or as a ring). The notation  $\mathbb{Z}_n$  will not be used.

## I Basic theory of groups

In this section we introduce the basic definitions and axioms of groups, and see some examples and consequences.

### Groups and subgroups

**I.1** One way to look at a group – not the only way, and not always the best way – is to say that it is a set where you are allowed to multiply elements together. There is more to it than that, but that's a starting point.

The corresponding basic informal idea of a ring is that in a ring you are allowed to multiply and you are allowed to add. Again, there's more to it than that.

A field is a special kind of ring: multiplication is commutative and you are allowed to divide by anything that isn't actually zero. Examples of fields include  $\mathbb{Q}$ ,  $\mathbb{R}$ ,  $\mathbb{C}$  and  $\mathbb{Z}/p$  (also called  $\mathbb{F}_p$ ) for  $p$  a prime. In a few places we shall mention fields before we have defined them precisely: these examples will suffice.

Groups are more basic than rings (it is convenient to be able to use the word "group" when defining a ring) so we begin by looking at groups first.

**I.2** Groups are there to capture in a mathematical way the idea of symmetry. So we are going to use this to guess what the axioms of a group should be.

**I.3** The general idea of a group is this (a more formal definition will follow). We have a set  $G$  and a way of "multiplying" together two elements  $a$  and  $b$  of  $G$  to get an element  $ab \in G$ . (If we imagine that  $a$  and  $b$  are symmetries of something, this says "do  $b$  and then do  $a$ ".) We want  $abc$  to make sense, without having to write brackets, but quite possibly  $ab \neq ba$ : again, that is what we expect from thinking about symmetry. There is also a symmetry that says "leave it alone", and so we want a neutral element  $e$  with the property that  $ae = ea = a$  for every  $a \in G$ . And finally, if we have done the symmetry  $a$  and then we wish we hadn't, we can undo it with another symmetry (or just possibly the same one).

**Definition I.4** A *group* is a set  $G$  equipped with a multiplication such that  $ab \in G$  if  $a, b \in G$  and such that

- (a) multiplication is associative:  $(ab)c = a(bc)$  for all  $a, b, c \in G$ ;
- (b) there is an identity element  $e \in G$  such that  $ae = ea = a$  for all  $a \in G$ ;
- (c) for any  $a \in G$  there exists an element  $a^{-1} \in G$  such that  $aa^{-1} = a^{-1}a = e$ .

**I.5** There is nothing wrong with Definition I.4 but it uses the word "multiplication" without explaining it. Also, we want to think about symmetries but you don't talk about multiplying symmetries. Without forgetting or abandoning Definition I.4, let us look at it in a slightly more precise way.

**Definition I.6** A *binary operation* on a set  $S$  is a function

$$*: S \times S \rightarrow S.$$

Binary operations that we are interested in tend to be called addition, denoted by  $+$ , or multiplication, denoted by either  $\cdot$  or nothing at all. In this context, by ancient tradition one writes  $a * b$  rather than  $*(a, b)$  (Polish notation): thus  $7 + 6$  rather than  $+(7, 6)$  and  $a \cdot b$  or  $ab$  rather than  $\cdot(a, b)$ .

**Definition I.7** A *group* is a pair  $(G, *)$ , where  $G$  is a set,  $*$  is a binary operation on  $G$  and the following axioms hold:

- (a) **(associative)**  $(a * b) * c = a * (b * c)$  for all  $a, b, c \in G$ .
- (b) **(identity)** There is an element  $e \in G$  with the property that  $e * a = a = a * e$  for all  $a \in G$ .

(c) (**inverse**) For each  $a \in G$  there exists  $b \in G$  such that  $a * b = b * a = e$ .

**I.8** Often, in fact usually, we know what the group operation  $*$  is, and then we don't need to mention it and we just refer to the group as  $G$  rather as  $(G, *)$ .

Notice that Definition I.7 says exactly the same thing as Definition I.4, but without saying "multiplication".

**I.9** In a group, both the identity element and inverses are unique.

- (i) if  $e, e' \in G$  are two elements satisfying the identity property from I.4, then  $e * e' = e'$  because  $e$  is an identity and  $e * e' = e$  because  $e'$  is. So  $e = e'$ .
- (ii) Given  $a \in G$ , if  $b, b' \in G$  are both elements satisfying the inverse property for  $a$  in I.4, then

$$b = b * e = b * (a * b') = (b * a) * b' = e * b' = b'.$$

This unique element  $b$  from (ii) is called the *inverse* of  $a$ . It is often denoted by  $a^{-1}$ , especially if  $*$  has been denoted by  $\cdot$  or by nothing, in which case the identity element  $e$  is often denoted by 1, or by  $1_G$  if we need to distinguish it from the integer 1.

**I.10** A neat way to say this is that a group has three operations: there is a binary operation (usually called *the* group operation),  $*$ , which takes two elements of  $G$  and gives you a third one; there is a unary operation,  $^{-1}$ , which takes one element of  $G$  and gives you a second one; and there is a nullary operation,  $e$ , which takes no element of  $G$  and gives you a first one. Then there are the group axioms, which tells you how these operations interact.

**Definition I.11** A group  $(G, *)$  is *abelian* (or *commutative*) if  $a * b = b * a$  for all  $a, b \in G$ .

The binary operation in an abelian group is often written as  $+$ , in which case the identity element is denoted 0 and the inverse of an element  $a \in G$  is denoted  $-a$ .

**Example I.12** We already know lots of examples of groups, and of structures that are not groups.

- (i)  $(\mathbb{Z}, +)$ , the ordinary integers with addition, is a group (in fact an abelian group).
- (ii)  $(\mathbb{Z}, \cdot)$ , the ordinary integers with multiplication, is *not* a group because inverses are missing. Most obviously,  $0^{-1}$  is missing, but so is  $2^{-1}$ : you can undo multiplication by 2 by multiplying by  $\frac{1}{2}$ , but  $\frac{1}{2} \notin \mathbb{Z}$ .
- (iii)  $(\mathbb{C}, +)$  is a group: so are  $(\mathbb{R}, +)$  and  $(\mathbb{Q}, +)$ .
- (iv)  $(\mathbb{C}^*, \cdot)$ , the non-zero complex numbers with multiplication, is a group. So are  $(\mathbb{R}^*, \cdot)$  and  $(\mathbb{Q}^*, \cdot)$ , but not  $(\mathbb{C}, \cdot)$  etc., because of 0 not having an inverse.

- (v)  $(\mathbb{Z}/n, +)$ , the integers modulo  $n$  with addition (modulo  $n$ ) is a group for any  $n > 0$ . This is sometimes called the cyclic group  $C_n$ .
- (vi)  $(\mathbb{Z}/n, \cdot)$  is not a group, even if  $n$  is prime, but if  $p$  is prime and  $(\mathbb{Z}/p)^*$  denotes the non-zero elements of  $\mathbb{Z}/p$  then  $((\mathbb{Z}/p)^*, \cdot)$  is a group. If  $n$  is not prime we can still get a group but we have to take  $(\mathbb{Z}/n)^*$  to be the set of elements that are coprime to  $n$  – we should be a bit careful about what this means.
- (vii)  $(\text{GL}(2, \mathbb{C}), \cdot)$ , the set of  $2 \times 2$  matrices with complex entries and non-zero determinant, and matrix multiplication, is a group. It is the first non-abelian group we have seen.
- (viii)  $(M_{2 \times 2}(\mathbb{C}), +)$ , the set of  $2 \times 2$  matrices with complex entries, and matrix addition, is a group.
- (ix)  $D_{2n}$ , the group of symmetries of an  $n$ -gon, is a group with  $2n$  elements (the group operation is composition:  $ab$  means do symmetry  $b$  followed by symmetry  $a$ ). It is non-abelian.
- (x)  $S_n$ , the symmetric (permutation) group on  $n$  symbols (i.e. the symmetries of a set of size  $n$ , is a group. (We could allow  $n$  to be an infinite cardinal here but we won't.) It is non-abelian.
- (xi)  $A_n$ , the even permutations of  $n$  symbols, form a group, also non-abelian.
- (xii) The set of continuous bijections from the circle to itself is a group, called the group of homeomorphisms of  $S^1$ . This is a very big group: too big for use really.
- (xiii) If  $X$  is any set then the set of bijective maps  $f: X \rightarrow X$  forms a group called the symmetry group or automorphism group or permutation group  $\text{Sym}(X)$  of  $X$ . This is also uncomfortably big unless  $X$  is finite, in which case it is just  $S_n$  again.

**Definition I.13** A subset  $H$  of a group  $G$  is called a *subgroup* of  $G$  if and only if it is closed under all the group operations. That means that  $1 \in H$  (so, in particular,  $H \neq \emptyset$ ); that  $a^{-1} \in H$  if  $a \in H$ ; and that  $ab \in H$  if  $a, b \in H$ .

In other words,  $H$  is a subgroup if it is a subset of  $G$  that is also a group (with the same binary operation).

In this case, we write  $H < G$ . If  $H < G$  and  $H \neq G$  we say that  $H$  is a *proper subgroup* of  $G$ .

**Lemma I.14** Suppose that  $H$  is non-empty subset of  $G$ . Then  $H$  is a subgroup of  $G$  if and only if

$$\text{for all } a, b \in H, \text{ we have } ab^{-1} \in H.$$

*Proof:* One implication is immediate from Definition I.13: if  $H$  is a subgroup and  $a, b \in H$  then  $b^{-1} \in H$  and hence  $ab^{-1} \in H$ , by I.4.

Conversely, suppose we have this condition. First choose  $a \in H$  (so we do need to know that  $H \neq \emptyset$ ) and take  $b = a$ . That tells us that  $1 \in H$ , so now we can start again and take  $a = 1$ . That tells us that if  $b \in H$  then  $b^{-1} \in H$ . Finally, if we want to show that  $ab \in H$  we take  $c = b^{-1}$  and then the criterion tells us that  $H \ni ac^{-1} = ab$ .

**Example I.15** We also know many examples of subgroups, and of things that are not subgroups.

- (i)  $A_n < S_n$ : the product and inverse of even permutations are even.
- (ii)  $2\mathbb{Z} < \mathbb{Z}$ : even integers are integers; there are some; and the difference of even integers is even (we are using Lemma I.14 here).
- (iii)  $\mathbb{R}_{>0}$ , the positive real numbers with multiplication, form a subgroup of  $\mathbb{R}^*$ .
- (iv)  $S^1$ , the set of complex numbers of modulus 1, is a subgroup of  $\mathbb{C}^*$ . (This group goes by many different names.)
- (v)  $SL(2, \mathbb{C}) < GL(2, \mathbb{C})$ : it is by definition the set of  $2 \times 2$  matrices of determinant 1.
- (vi)  $\mathbb{Z}/n$  is *not* a subgroup of  $\mathbb{Z}$ : this is important. It is not even a subset of  $\mathbb{Z}$ , in fact, but more than that: there isn't even a subgroup of  $\mathbb{Z}$  that "looks like"  $\mathbb{Z}/n$  (except in the trivial case  $n = 1$ ). We shall make this more precise later: see Definition I.32.
- (vii)  $\mathbb{Z}/2$  a subgroup of  $\mathbb{Z}/6$ ?
- (viii) Indisputably,  $6\mathbb{Z} < 2\mathbb{Z}$ : numbers divisible by 6 are even.
- (ix) There is a subgroup  $R$  of  $D_6$  that consists of 1 and a reflection in a (single, fixed) line of symmetry.

**Definition I.16** A subset  $H$  of a group  $G$  is called a *normal subgroup* of  $G$  if it is a subgroup and

$$g^{-1}hg \in H \text{ whenever } h \in H \text{ and } g \in G.$$

In this case, we write  $H \triangleleft G$ .

**Example I.17** Again we know many examples.

- (i) If  $G$  is an abelian group and  $H < G$  then  $H \triangleleft G$ .
- (ii)  $SL(2, \mathbb{C}) \triangleleft GL(2, \mathbb{C})$ , because if  $\det A = 1$  (so that  $A \in SL(2, \mathbb{C})$ ) and  $B \in GL(2, \mathbb{C})$ , then  $\det(B^{-1}AB) = \det(B^{-1}) \det A \det B = \det(B^{-1}) \det B = 1$ , and therefore  $B^{-1}AB \in SL(2, \mathbb{C})$ .
- (iii) The subgroup of  $D_6$  consisting of rotations of the triangle in the plane (three of them, one trivial) is a normal subgroup of  $D_6$ . However, the subgroup  $R$  of  $D_6$  from Example I.15(ix) is not a normal subgroup.

**I.18** It is not immediately obvious why Definition I.16 is important, but it is. In fact, non-normal subgroups are very common. It is not normal for a subgroup to be normal.

## Maps between groups

**I.19** We do not only want to talk about one group at a time. We want to be able to compare them, and that means having maps that go from one group to another. These maps could just be maps of sets, of course, but then you lose all the group information. We want to keep that.

**Definition I.20** If  $G$  and  $H$  are groups then a map  $\varphi: G \rightarrow H$  is called a *group homomorphism* if

- (a)  $\varphi(1_G) = 1_H$ ;
- (b) if  $g \in G$  then  $\varphi(g^{-1}) = (\varphi(g))^{-1}$ ;
- (c) if  $g_1, g_2 \in G$  then  $\varphi(g_1g_2) = \varphi(g_1)\varphi(g_2)$ .

**I.21** What Definition I.20 says is that  $\varphi$  is a group homomorphism if it respects all the group operations. In effect, what this means is that the result of doing a calculation in  $G$  and sending the answer to  $H$  by using  $\varphi$  is the same as the result of sending all the components to  $H$  by  $\varphi$  and doing the corresponding computation in  $H$ .

**I.22** A very similar definition is the definition of linear map for vector spaces. The only difference is that in groups you can multiply and take inverses, and a group homomorphism respects those operations; in a vector space you are allowed to add and to multiply by scalars, and a linear map respects those operations. A linear map is just a vector space homomorphism.

**Proposition I.23** Let  $G$  and  $H$  be groups and  $\varphi: G \rightarrow H$  a map. Then  $\varphi$  is a group homomorphism if and only if  $\varphi(ab) = \varphi(a)\varphi(b)$  for all  $a, b \in G$ , i.e. if and only if Definition I.20(iii) holds.

*Proof:* Suppose we know Definition I.20(iii). Let us first prove that Definition I.20(i) holds. We have

$$\varphi(a) = \varphi(1_G a) = \varphi(1_G)\varphi(a)$$

and if we multiply on the right by  $\varphi(a)^{-1}$  we get

$$1_H = \varphi(a)\varphi(a)^{-1} = \varphi(1_G)\varphi(a)\varphi(a)^{-1} = \varphi(1_G)$$

as required. Now we can prove Definition I.20(ii) because

$$1_H = \varphi(1_G) = \varphi(aa^{-1}) = \varphi(a)\varphi(a^{-1})$$

and now multiplying on the left by  $\varphi(a)^{-1}$  gives us what we want.

**I.24** Proposition I.23 is very convenient and sometimes it is made the definition. If we know we are talking about groups, we often just say “homomorphism”.

**Definition I.25** If  $\varphi: G \rightarrow H$  is a group homomorphism then

- (a) the *image* of  $\varphi$  is  $\{\varphi(g) \mid g \in G\}$ , a subset of  $H$ ;
- (b) the *kernel* of  $\varphi$  is  $\{g \in G \mid \varphi(g) = 1_H\}$ , a subset of  $G$ .

**Lemma I.26** If  $\varphi: G \rightarrow H$  is a group homomorphism then the image  $\text{Im } \varphi = \varphi(G)$  is a subgroup of  $H$ .

**Lemma I.27** If  $\varphi: G \rightarrow H$  is a group homomorphism then the kernel  $\text{Ker } \varphi$  is a normal subgroup of  $G$ .

*Proof:* Let us write  $K = \text{Ker } \varphi$  and suppose that  $g \in G$  and  $k, k_1, k_2 \in K$ .

First, according to Lemma I.14, we can check that  $K$  is a subgroup by checking that  $k_1 k_2^{-1} \in K$ . But for this it is enough to compute

$$\varphi(k_1 k_2^{-1}) = \varphi(k_1) \varphi(k_2^{-1}) = \varphi(k_1) \varphi(k_2)^{-1} = 1_H.$$

Then we want to check that  $K$  is a normal subgroup. But  $\varphi(g^{-1}) = \varphi(g)^{-1}$  and  $\varphi(k) = 1_H$ , so

$$\varphi(g^{-1} k g) = \varphi(g^{-1}) \varphi(k) \varphi(g) = \varphi(g)^{-1} \cdot 1_H \cdot \varphi(g) = \varphi(g)^{-1} \varphi(g) = 1_H$$

so  $g^{-1} k g \in K$  if  $k \in K$ , as required.

**Lemma I.28** A group homomorphism  $\varphi: G \rightarrow H$  is injective if and only if  $\text{Ker } \varphi = \{1_G\}$ .

**Definition I.29** Let  $G$  and  $G'$  be groups. A homomorphism  $\varphi: G \rightarrow G'$  is called an *isomorphism* if there is a homomorphism  $\psi: G' \rightarrow G$  such that  $\psi\varphi: G \rightarrow G$  is the identity on  $G$  and  $\varphi\psi: G' \rightarrow G'$  is the identity on  $G'$ .

**I.30** Notice that this definition is symmetric:  $\psi: G' \rightarrow G$  is also an isomorphism.

The important part of the definition is the requirement that  $\psi$  is a homomorphism. If we drop that and just allow  $\psi$  to be any map, then the rest of the definition just says that  $\varphi$  is bijective and  $\psi$  is its two-sided inverse map. So if we are given  $\varphi$  we have no choice about what map  $\psi$  is: either  $\varphi$  is not bijective, in which case it cannot be an isomorphism because  $\psi$  doesn't exist at all, or it is, in which case  $\psi$  is its inverse. But the definition also insists that  $\psi$  should be a homomorphism.

In fact this is no extra requirement in this case. But beware: that is true about groups, but this definition generalises, and in some of the other cases we cannot be sure that the inverse map preserves all the structure that we want to preserve.

**Proposition I.31** Let  $\varphi: G \rightarrow G'$  be a bijective group homomorphism. Then  $\varphi$  is an isomorphism.

*Proof:* The only choice for  $\psi$  is the inverse map to  $\varphi$ : if  $\varphi(a) = b$ , then  $\psi(b) = a$ . This is defined for every  $b$  because  $\varphi$  is surjective, and  $a$  is unique because  $\varphi$  is injective. It satisfies  $\varphi\psi = \text{id}_{G'}$  and  $\psi\varphi = \text{id}_G$ .

The task, then, is to show that  $\psi$ , thus defined, is a group homomorphism. According to Proposition I.23, it is enough to show that if  $b_1, b_2 \in G'$  then  $\psi(b_1b_2) = \psi(b_1)\psi(b_2)$ .

But  $\varphi\psi(b_1b_2) = b_1b_2$ , and  $\varphi(\psi(b_1)\psi(b_2)) = \varphi\psi(b_1)\varphi\psi(b_2) = b_1b_2$ , where the first equality holds because  $\varphi$  is a homomorphism. So  $\psi(b_1b_2)$  and  $\psi(b_1)\psi(b_2)$  are mapped to the same element by  $\varphi$ , so they must be equal because  $\varphi$  is injective.

**Definition I.32** Two groups  $G$  and  $H$  are said to be *isomorphic* if there exists an isomorphism  $\varphi: G \rightarrow H$ : we write  $G \cong H$  in this case.

**I.33** Isomorphism – that is, the relation of being isomorphic – is an equivalence relation among groups. Sometimes one thinks of isomorphic groups as being “the same”, but sometimes that is the wrong thing to do. The difficulty is that the definition of “isomorphic” only promises that an isomorphism exists: it doesn’t supply it, the isomorphism may not be unique (in particular, an isomorphism from  $G$  to  $G$  does not have to be the identity) and even if you know  $\varphi$  you may not know  $\psi$ .

Much of cryptography rests on this last possibility. Alice knows both halves of an isomorphism between isomorphic groups  $G$  and  $H$  but she publishes only  $G$ ,  $H$  and  $\varphi$ . She tells Bob to encode his message as an element of  $G$ , and (perhaps after some computations) apply  $\varphi$  and send her the result. Eve knows that to find the message all she needs to do is apply  $\psi$ , but she doesn’t know  $\psi$ : that’s Alice’s secret, and it is computationally hard to work out  $\psi$  from  $\varphi$  even though in principle  $\varphi$  does determine  $\psi$ .

## Order, generators and cyclic groups

**Definition I.34** The *order of a group* is its cardinality: that is, the order of  $G$  is  $|G|$ .

**I.35** We will usually only use this if  $G$  is finite: otherwise we shall say that  $G$  is infinite or that  $|G| = \infty$ . We could also distinguish different cardinalities among infinite groups. Thus we shall say that  $|\mathbb{Z}| = \infty$  and  $|\mathbb{R}| = \infty$ , simply meaning that neither is finite, although in fact  $|\mathbb{Z}| \neq |\mathbb{R}|$ .

**Definition I.36** The *order of an element*  $g \in G$  is the smallest natural number  $n$  such that  $g^n = 1_G$ , or  $\infty$  if there is no such number. We write  $o(g)$  for the order of  $g$ .

**I.37** It is important to note that Definition I.34 and Definition I.36 are entirely different: notice the different notation for order of an element and order of a group. In particular, the order of a group is ascertained by counting: it is a set-theoretic property. On the other hand, to determine the order of an element



one must know how to multiply, so it is a group-theoretic property. The use of the same word is not unintentional, though: they are related, as we shall see in Proposition I.46.

**Lemma I.38** Let  $G$  be a group and  $g \in G$ . Then  $o(g) = 1$  if and only if  $g = 1_G$ .

This is a trivial observation, but a very useful one.

**Definition I.39** Suppose that  $G$  is a group and  $S \subset G$  (note that this means a subset, not a subgroup). We say that  $S$  *generates*  $G$  if no proper subgroup of  $G$  contains  $S$ .

**I.40** There are many ways to rephrase this. One is to say that  $G$  is the smallest subgroup of  $G$  that contains  $S$ . This is all right, but then we need to ask what is meant by “the smallest”. Another way is via the following proposition, which as well as being practically useful also gives a different way of thinking about what Definition I.39 means.

**Proposition I.41** Suppose that  $G$  is a group and  $S \subset G$ . Then  $S$  generates  $G$  if and only if every  $g \in G$  can be written in the form

$$g = s_1 \dots s_k$$

for some  $k \in \mathbb{N}$ , where  $s_i \in S$  or  $s_i^{-1} \in S$  for every  $i$ .

*Proof:* Consider the set  $\langle S \rangle = \{s_1 \dots s_k \mid k \in \mathbb{N}, s_i \in S \text{ or } s_i^{-1} \in S\}$ . This clearly includes  $S$ : we claim that it is a group. We use Lemma I.14, noticing that  $\langle S \rangle$  is not empty, even if  $S$  is empty, because if we take  $k = 0$  we get the empty product, which is 1 by definition. (If  $S = \emptyset$  there is nothing to check anyway.) So we just need to check that if  $g, h \in \langle S \rangle$  then  $gh^{-1} \in \langle S \rangle$ . But this is immediate: if  $g = s_1 \dots s_k$  and  $h = r_1 \dots r_l$  with  $s_i$  or  $s_i^{-1}$  in  $S$  and also  $r_j$  or  $r_j^{-1}$  in  $S$  for each  $i$  and  $j$ , then

$$gh^{-1} = s_1 \dots s_k r_l^{-1} \dots r_1^{-1}$$

which obviously satisfies the same condition.

So the statement that every  $g \in G$  can be written in this form is the same as saying that  $\langle S \rangle = G$ . So we want to show that no proper subgroup of  $G$  contains  $S$  if and only if  $G = \langle S \rangle$ .

If  $G \neq \langle S \rangle$  then  $\langle S \rangle$  is a proper subgroup of  $G$  containing  $S$ , so  $S$  does not generate  $G$ .

Conversely, if  $S$  does not generate  $G$ , then there is a proper subgroup  $H$  of  $G$  containing  $S$ , but then  $\langle S \rangle < H$  because  $H$  is closed under the group operations, and so  $\langle S \rangle \neq G$ .

**Definition I.42** The subgroup  $\langle S \rangle$  in the proof of Proposition I.41 is called the *subgroup generated by  $S$* . We shall use this notation, and if  $S = \{g\}$  consists of a single element we shall usually write  $\langle g \rangle$  instead of  $\langle \{g\} \rangle$ . Notice that  $\langle g \rangle = \{g^i \mid i \in \mathbb{Z}\}$ .

**Example I.43** We already know many examples of this.

- (i) If  $G$  is a group,  $\emptyset$  generates the trivial subgroup  $1$ . So does  $S = \{1_G\}$ .
- (ii) If  $G = \mathbb{Z}$  (with addition, of course) then  $\{1\}$ , which is not the identity element, generates  $\mathbb{Z}$ .
- (iii) If  $G = \mathbb{Z}/p$  with  $p$  a prime then any non-identity element of  $G$  generates  $G$ .
- (iv) If  $G = \mathbb{Z}/6$  then  $\{1\}$  generates  $G$  but  $\{2\}$  does not.
- (v) If  $G = D_6$  or  $S_3$  (in fact  $D_6 \cong S_3$ ) then there is no element  $g \in G$  such that  $\{g\}$  generates  $G$ .

**Definition I.44** We say that a group  $G$  is *cyclic* if  $G = \langle g \rangle$  for some  $g \in G$ .

**I.45** Cyclic groups are abelian. We know all the cyclic groups up to isomorphism: they are  $\mathbb{Z}/n$  for  $n > 0$ , and  $\mathbb{Z}$ .

**Proposition I.46** If  $g \in G$  then  $|\langle g \rangle| = o(g)$ : that is, the order of an element is the same as the order of the (cyclic) group that it generates.

*Proof:* Suppose that  $o(g) = r \in \mathbb{N}$  (if the order is infinite there is nothing to prove) and consider the set  $R := \{g^i \mid 0 < i \leq r\}$ . Recall that  $\langle g \rangle = \{g^i \mid i \in \mathbb{Z}\}$ . Obviously  $R \subseteq \langle g \rangle$ . We claim that  $R = \langle g \rangle$  and that  $|R| = r$ .

For the first, it is enough to note that if  $i \in \mathbb{Z}$  and  $i \equiv j \pmod{r}$  then  $g^i = g^j$ , because we have  $i = j + kr$  for some  $k \in \mathbb{Z}$  so  $g^i = g^{j+kr} = g^j g^{kr} = g^j g^{rk} = g^j 1^k = g^j$ .

For the second we know by Definition I.36 that  $g^r = 1$  and  $g^s \neq 1$  if  $1 < s < r$  (if  $r = 1$  there is nothing to prove). If  $1 \leq s < s' \leq r$  and  $g^s = g^{s'}$  then  $1 = g^{s'}(g^s)^{-1} = g^{s'-s}$  which is impossible as  $1 < s' - s < r$  so the  $r$  elements  $g, g^2, \dots, g^r = 1$  are all different so  $|R| = r$ .

## II Structure of groups

In this section we shall prove two major theorems about the structure of groups: Lagrange's theorem, which is about the orders of finite groups, and the (first) isomorphism theorem, which describes group homomorphisms.

### Cosets and Lagrange's theorem

**Definition II.1** Suppose that  $G$  is a group and  $H$  is a subgroup of  $G$ . A *left coset* of  $H$  in  $G$  is a subset of  $G$  of the form

$$gH := \{gh \mid h \in H\}$$

where  $g \in G$  is fixed. A *right coset*  $Hg$  is defined similarly.

**II.2** In general, cosets are not subgroups: for one thing, if  $g \notin H$  then  $1_G \notin g^{-1}H$ . They are analogous to affine linear subspaces, e.g. planes in  $\mathbb{R}^3$  that don't go through the origin. Those aren't vector subspaces but we are still allowed to think about them.

Notice that  $gH$  can be equal to  $g'H$  even if  $g \neq g'$ . For example, if  $g' = 1$  and  $g \in H$  then both of these are equal to  $H$ .

**Lemma II.3** If  $H < G$  and  $g \in G$  then  $|gH| = |H| = |Hg|$ .

*Proof:* The map  $\mu_g: H \rightarrow gH$  given by  $\mu_g(h) = gh$  is bijective, because it has the two-sided inverse map  $\mu_{g^{-1}}$ .

**Theorem II.4** If  $H < G$  then every element of  $G$  belongs to exactly one left coset of  $H$ . In particular if  $g_1H$  and  $g_2H$  are left cosets, then either  $g_1H = g_2H$  or  $g_1H \cap g_2H = \emptyset$ .

*Proof:* If  $g \in G$  then  $g = g1 \in gH$ , so every  $g \in G$  is contained in at least one coset. Suppose that  $g \in g'H$ : we claim that  $g'H = gH$ , so that  $gH$  is the only coset that contains  $g$ . To check this, first we write  $g = g'h$  (since  $g \in g'H$  this is true for some  $h \in H$ ) and then we consider an arbitrary element  $g'h' \in g'H$ . We have  $g' = gh^{-1}$ , so  $g'h' = gh^{-1}h'$  and that is an element of  $gH$  because  $H$  is a group so  $h^{-1}h' \in H$ . So  $g'H \subseteq gH$ . Similarly, if  $gh' \in gH$  then  $gh' = g'hh' \in g'H$  so  $gH \subseteq g'H$ .

**II.5** Another way to state Theorem II.4 is to say that the cosets *partition*  $G$ : that is,  $G = \coprod_g gH$  where  $g$  runs over a set of representatives, one from each left coset of  $H$ .

A third way to state this is to say that the relation  $g_1 \sim g_2 \iff g_1H = g_2H$  is an equivalence relation.

**Corollary II.6** If  $g_1, g_2 \in G$  and  $H < G$  then  $g_1H = g_2H$  if and only if  $g_1 \in g_2H$ .

**Definition II.7** The number (possibly infinite) of different cosets of  $H$  in  $G$  is called the *index* of  $H$  in  $G$ , written  $|G : H|$ . That is

$$|G : H| = |\{gH \mid g \in G\}|.$$

It is easy to check that if we look at right cosets instead of left cosets the number of cosets is the same, even though the cosets themselves are not ( $gH \neq Hg$  in general, unless  $G$  is abelian).

**Theorem II.8 [Lagrange's Theorem]** If  $G$  is a finite group and  $H$  is a subgroup of  $G$ , then  $|H|$  divides  $|G|$ . In fact,  $|G| = |H| \cdot |G : H|$ , so the index of  $H$  also divides  $|G|$ .

*Proof:* By Theorem II.4,  $H$  has  $r := |G : H|$  different left cosets,  $g_1H, \dots, g_rH$ , and  $G = g_1H \amalg \dots \amalg g_rH$ . By Lemma II.3 we then have

$$|G| = \sum_{i=1}^r |g_iH| = \sum_{i=1}^r |H| = |G : H| \cdot |H|.$$

**Corollary II.9** If  $G$  is a finite group and  $g \in G$  then  $o(g)$  divides  $|G|$ .

*Proof:* By Proposition I.46,  $o(g) = |\langle g \rangle|$  and the latter divides  $|G|$  by Lagrange's theorem.

**II.10** Lagrange's theorem is extremely important in both algebra and number theory. It is one of the few ways of showing that one number divides another without actually knowing what numbers they are.

The converse to Lagrange's theorem is false: it is possible for  $r$  to divide  $|G|$  without  $G$  having a subgroup of order exactly  $r$ . In particular the converse to Corollary II.9 is false. Both converse statements are true, however, if  $r = p$  is prime: this is a consequence of the Sylow theorems, one of which is a partial converse to Lagrange's theorem. The Sylow theorems will not be proved or even stated in this course, but they are not extremely hard.

## Quotient groups

**II.11** We have seen that in general the left cosets and right cosets of a subgroup  $H < G$  are not the same thing, except if  $G$  is an abelian group. There is another important case where this happens, which we have already met: if  $H$  is a normal subgroup of  $G$  (Definition I.16).

**Lemma II.12** Suppose  $G$  is a group and  $H < G$ . Then  $H$  is a normal subgroup if and only if every left coset is also a right coset.

*Proof:* If the left coset  $gH$  is also a right coset then it must be the right coset  $Hg$ , because  $g \in gH$  and  $g \in Hg$  and  $g$  belongs to exactly one coset of each handedness by Theorem II.4. So if that is true for every  $g \in G$  then  $gH = Hg$  for all  $g \in G$ . Now if  $h \in H$  we have  $hg \in Hg = gH$  so  $hg = gh'$  for some  $h' \in H$ , so  $g^{-1}hg \in H$  so  $H \triangleleft G$  according to Definition I.16.

Conversely, if  $H \triangleleft G$  and  $gH$  is a left coset, then I claim that  $gH = Hg$ . But if  $hg \in Hg$  then  $hg = gg^{-1}hg = gh' \in H$  for some  $h' \in H$ , by Definition I.16, so  $Hg \subseteq gH$ . The other inclusion follows by symmetry, or by replacing  $g$  by  $g^{-1}$  (or immediately if  $G$  is finite).

**II.13** The importance of Lemma II.12 is that it allows us to put a group structure on the set of cosets of  $H$  in  $G$ , but only if  $H$  is a normal subgroup. We first make a definition that is valid even if  $H$  is not normal, but gives us only a set, not a group.

**Definition II.14** Let  $G$  be a group and  $H$  a subgroup. The *coset set* of  $G$  with respect to  $H$  is the set  $G/H$  given by

$$G/H := \{gH \mid g \in G\}.$$

**II.15** Of course we can also define the coset set  $H \backslash G$  of right cosets, but we do not need it: notice, however, that it will be the same as the coset set defined above if  $H$  is a normal subgroup. Notice also that  $|G/H| = |G : H|$ . In fact some people prefer to use  $G : H$  for the set defined in Definition II.14 and reserve  $G/H$  for the group that we are about to produce.

**Theorem II.16** Suppose that  $H$  is a normal subgroup of  $G$ . Then we may define a multiplication on  $G/H$  by

$$(g_1H)(g_2H) = (g_1g_2)H.$$

Moreover, with this multiplication  $G/H$  is a group, in which  $e = H$  and  $(gH)^{-1} = g^{-1}H$ .

*Proof:* The main thing that has to be proved here is that the definition of multiplication makes sense. We need to check that if  $g'_1H = g_1H$  and  $g'_2H = g_2H$  then  $g'_1g'_2H = g_1g_2H$ .

By Corollary II.6,  $g'_1H = g_1H$  if and only if  $g'_1 \in g_1H$ , so  $g'_1 = g_1h_1$ , so we assume this and also that  $g'_2 = g_2h_2$ , for  $h_1, h_2 \in H$ . Now we need to show only that  $g'_1g'_2 \in g_1g_2H$ , again by Corollary II.6. But  $g'_1g'_2 = g_1h_1g_2h_2$  and  $h_1g_2 \in Hg_2 = g_2H$  (since  $H$  is normal in  $G$ ). So  $h_1g_2 = g_2h'_1$  for some  $h'_1 \in H$ , so  $g'_1g'_2 = g_1g_2h'_1h_2 \in g_1g_2H$ .

The rest is very similar. Notice that we do not have to check associativity: that is part of the definition.  $H$  is the identity because  $H = 1H$  and  $(1H)(gH) = (1g)H = gH$ , and  $g^{-1}H$  is the inverse of  $gH$  because  $(g^{-1}H)(gH) = (g^{-1}g)H = 1H = H$ . We do, though, have to check that  $g^{-1}H$  is well defined: that is, that if  $g_1H = g_2H$  then  $g_1^{-1}H = g_2^{-1}H$ . But  $g_1H = g_2H$  if and only if  $g_2 \in g_1H$ , so  $g_2 = g_1h$  for some  $h \in H$ : that happens if and only if  $g_2^{-1} = h^{-1}g_1^{-1}$  for some  $h$  (that is, some  $h^{-1}$ ) in  $H$ : but that says  $g_2^{-1} \in Hg_1^{-1} = g_1^{-1}H$  as  $H$  is normal. So  $g_2^{-1}H = g_1^{-1}H$  by Corollary II.6.

**Definition II.17** The group  $G/H$ , with the operations (multiplication and inverse) defined in Theorem II.16, is called the *quotient group* of  $G$  by  $H$ .

**II.18** This is one of the most important concepts in all of mathematics. There is plenty to think about.

One important thing to understand is that  $G/H$  is not a subgroup of  $G$ . (It might happen to be isomorphic to a subgroup of  $G$ , although it also might not, but that is a different thing.)

What we are doing in this construction is pretending that everything in  $H$  is the identity. For example, we might do that because we are not interested in  $H$ , or because  $H$  is acting trivially (see Example III.7(ix)). If you pretend that all elements of  $H$  are trivial, then you also have to think that any two elements

of  $gH$  are the same, and that is exactly what this construction tells you to do. That also explains where the definition of normal subgroup comes from. If you think, however wrongly, that  $h_1$  and  $h_2$  are the identity, then you also have to think that  $h_1h_2^{-1}$  is the identity, but you also have to think that  $gh_1g^{-1}$  is the identity, whether  $g \in H$  or not. We shall make this idea a bit more precise shortly.

**II.19** If  $G$  is a group and  $H$  is a (proper, nontrivial) normal subgroup then one thinks of  $G/H$  and  $H$  as both being smaller than  $G$ : if  $G$  is finite, this is literally true. So we could try to understand finite groups, at least, by looking for normal subgroups inside them and trying to proceed by induction, understanding the smaller “pieces”  $G/H$  and  $H$ . There is still a question about how the pieces fit together. This stops working if  $G$  simply does not have any normal subgroups, apart from 1 and  $G$ . Such a group is called *simple*: they are the building blocks of all finite groups.

One of the major triumphs of 20th-century algebra was the classification, around 1980, of all the finite simple groups. A big first step was the Thompson-Feit theorem from the early 1960s, which says that the only simple groups of odd order are the cyclic groups  $C_p \cong \mathbb{Z}/p$  with  $p$  an odd prime. Of course  $\mathbb{Z}/2$  is also simple. Apart from those, the alternating group  $A_n$  is simple if  $n \geq 5$ , and there are families of finite simple groups called “of Lie type”: these are (more or less) groups of matrices over a finite field, such as  $\text{PSL}(n, \mathbb{F}_p)$ . And then there are twenty-six others, called the sporadic groups: most of them occur as subquotients, i.e. quotients of subgroups, of the biggest of the 26 sporadic groups, the Monster. The order of the Monster is

$$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71.$$

That’s about the mass of the observable universe in grams. (Dark matter not included. Other restrictions may apply.)

**Example II.20** We already know a few examples of quotient groups.

- (i)  $n\mathbb{Z} \triangleleft \mathbb{Z}$  since the additive group  $\mathbb{Z}$  is abelian, and  $\mathbb{Z}/n\mathbb{Z} = \mathbb{Z}/n$ . This is the basic example. When you do addition modulo  $n$ , you are treating  $n$  as 0 (which is the identity) and when you talk about “6 mod 11” what you really mean is “any number that is 6 mod 11” or, what amounts to the same thing in that it carries the same information, “the class of numbers that are 6 mod 11”. That’s the coset  $6 + 11\mathbb{Z}$  (additive notation). Notice that  $\mathbb{Z}/n\mathbb{Z}$  is definitely not isomorphic to any subgroup of  $\mathbb{Z}$ , because every element in  $\mathbb{Z}/n\mathbb{Z}$  has finite order but no element of  $\mathbb{Z}$  apart from 0 has finite order.
- (ii) There are two trivial cases:  $G/G$  is the trivial group  $\{1\}$  and  $G/\langle 1_G \rangle$  is  $G$  (strictly speaking “is naturally isomorphic to  $G$ ” but even algebraists won’t worry about that).
- (iii) The order 3 subgroup of rotations in  $D_6$  is normal, and the quotient is (isomorphic to)  $\mathbb{Z}/2$ , telling you which side of the triangle is facing you.
- (iv) More generally,  $A_n \triangleleft S_n$  with quotient  $\mathbb{Z}/2$ .

- (v) If  $G$  is a group and  $H < G$  is of index 2, then  $H \triangleleft G$  and  $G/H \cong \mathbb{Z}/2$ .
- (vi) The group  $\text{Aff}(\mathbb{R}^2)$  of affine linear transformations of a plane,  $\mathbf{x} \mapsto M\mathbf{x} + \mathbf{a}$  where  $M \in \text{GL}(2, \mathbb{R})$  and  $\mathbf{a} \in \mathbb{R}^2$ , has a normal subgroup consisting of the translations. The quotient is isomorphic to  $\text{GL}(2, \mathbb{R})$ , but the same thing cannot be done in the other direction because  $\text{GL}(2, \mathbb{R})$  is not a normal subgroup of  $\text{Aff}(\mathbb{R}^2)$ .

**Lemma II.21** If  $G$  is a group and  $K$  is a normal subgroup of  $G$ , then the map  $\pi: G \rightarrow G/K$  given by  $\pi(g) = gK$  is a surjective group homomorphism whose kernel is  $K$ . In particular, every normal subgroup is the kernel of a group homomorphism.

*Proof:* There is actually nothing to prove.  $\pi$  is surjective by the definition of  $G/K$  as a set, Definition II.14, and it is a group homomorphism by Theorem II.16: the definition of multiplication there exactly matches what is needed for  $\pi$  to be a group homomorphism according to Proposition I.23. The statement about the kernel is simply the statement that  $gK = K$  if and only if  $g \in K$ , which is a special case of Corollary II.6.

**II.22** Notice that Lemma II.21 and Lemma I.27 between them say that kernels and normal subgroups are exactly the same thing.

In vector spaces, homomorphisms are linear maps and kernels are null spaces. Any linear subspace can be the null space of something, and null spaces are linear subspaces. The difference in the case of groups is that only some subgroups, the normal subgroups, are eligible to be kernels.

**Definition II.23** Let  $G$  be a group and  $K$  a subgroup. If  $K \triangleleft G$  then the map  $\pi: G \rightarrow G/K$  defined in Lemma II.21 is called the *quotient map*. The map  $\iota: K \rightarrow G$  defined by  $\iota(h) = h$  is called the *inclusion map*.

It is obvious that  $\iota$  is an injective group homomorphism.

**Theorem II.24** Let  $\varphi: G \rightarrow H$  be a group homomorphism and let  $K$  be a normal subgroup in  $G$  satisfying  $K \subseteq \text{Ker } \varphi$ . Then there exists a unique group homomorphism  $\bar{\varphi}: G/K \rightarrow H$  such that the diagram

$$\begin{array}{ccc}
 G & \xrightarrow{\varphi} & H \\
 \pi \downarrow & \nearrow \bar{\varphi} & \\
 G/K & & 
 \end{array}$$

commutes, i.e.,  $\bar{\varphi} \circ \pi = \varphi$ .

*Proof:* The map  $\bar{\varphi}: G/K \rightarrow H$  is defined by setting  $\bar{\varphi}(gK) = \varphi(g)$ . To see that this map is well-defined, independent of any choices, notice that

$$\begin{aligned} g_1K = g_2K &\iff g_1^{-1}g_2 \in K \\ &\implies g_1^{-1}g_2 \in \text{Ker}(\varphi) \\ &\iff 1_H = \varphi(g_1^{-1}g_2) = \varphi(g_1)^{-1}\varphi(g_2) \\ &\iff \varphi(g_1) = \varphi(g_2). \end{aligned}$$

In particular,  $g_1K = g_2K \implies \varphi(g_1) = \varphi(g_2)$ , so  $\bar{\varphi}$  does not depend on the choice of representative  $g$  in the coset  $gK$ .

To see that  $\bar{\varphi}$  is a group homomorphism, notice that

$$\bar{\varphi}((g_1K)(g_2K)) = \bar{\varphi}(g_1g_2K) = \varphi(g_1g_2) = \varphi(g_1)\varphi(g_2) = \bar{\varphi}(g_1K)\bar{\varphi}(g_2K).$$

The group homomorphism  $\bar{\varphi}$  satisfies  $\bar{\varphi} \circ \pi = \varphi$ , because for all  $g \in G$  we have

$$(\bar{\varphi} \circ \pi)(g) = \bar{\varphi}(gK) = \varphi(g)$$

as required.

Finally,  $\bar{\varphi}$  is the unique map (in particular, the unique homomorphism) satisfying the conditions. For if  $\theta: G/K \rightarrow H$  satisfies  $\theta \circ \pi = \varphi$  then  $(\theta \circ \pi)(g) = \varphi(g)$  for all  $g \in G$ , and since  $\pi(g) = gK$ , this implies that  $\theta(gK) = \varphi(g)$  for all  $g \in G$ : that is,  $\theta = \bar{\varphi}$ .

**Theorem II.25 [First Isomorphism Theorem]** Let  $\varphi: G \rightarrow H$  be a group homomorphism with kernel  $\text{Ker } \varphi = K$ . Then there is a group isomorphism

$$\bar{\varphi}: G/K \longrightarrow \text{Im } \varphi.$$

*Proof:* Applying the universal property from Theorem II.24 to the normal subgroup  $K$  gives a homomorphism  $\bar{\varphi}: G/K \rightarrow H$  given by  $\bar{\varphi}(gK) = \varphi(g)$ . From this we get a surjective homomorphism (with the same name!)

$$\bar{\varphi}: G/K \longrightarrow \text{Im } \varphi$$

simply by changing the target of the morphism, making it be the image of  $\varphi$  rather than  $H$ .

To see that  $\bar{\varphi}$  is injective, suppose  $\bar{\varphi}(g_1K) = \bar{\varphi}(g_2K)$ . Then  $\varphi(g_1) = \varphi(g_2)$ , so  $\varphi(g_1^{-1}g_2) = \varphi(g_1)^{-1}\varphi(g_2) = 1_H$ , giving  $g_1^{-1}g_2 \in \text{Ker } \varphi = K$  and hence  $g_1K = g_2K$  as required.

Therefore the map  $\bar{\varphi}: G/K \rightarrow \text{Im } \varphi$  is a bijective group homomorphism, so it is an isomorphism by Proposition I.31.

**Corollary II.26** Every homomorphism can be written as the composition of a surjective group homomorphism, then an isomorphism, and finally an injective group homomorphism.

*Proof:* We factorise  $\varphi: G \rightarrow H$  as shown below:



$$\begin{array}{ccc}
G & \xrightarrow{\varphi} & H \\
\pi \downarrow & & \uparrow \iota \\
G/K & \xrightarrow{\bar{\varphi}} & \text{Im } \varphi
\end{array}$$

where  $\pi$  is the quotient map,  $\bar{\varphi}$  is the map from the First Isomorphism Theorem, and  $\iota$  is the inclusion map. These maps have all the required properties.

**Lemma II.27** Suppose that  $H < G$  and  $N \triangleleft G$ . Then  $HN = \{hn \mid h \in H, n \in N\}$  is a subgroup of  $G$ .

*Proof:* Notice that since  $N$  is a normal subgroup,  $HN = NH$ . Moreover  $1_G \in HN$  so  $HN \neq \emptyset$ . Suppose that  $h_1n_1 \in HN$  and  $h_2n_2 \in HN$ . Then

$$(h_1n_1)(h_2n_2)^{-1} = h_1n_1n_2^{-1}h_2^{-1} \in HNH = HHN = HN.$$

**Theorem II.28 [Second Isomorphism Theorem]** If  $H < G$  and  $N \triangleleft G$  then  $H \cap N \triangleleft H$  and

$$H/(H \cap N) \cong HN/N.$$

*Proof:* The map  $\varphi: H \rightarrow HN/N$  given by  $\varphi(h) = hN$  is a homomorphism, and it is surjective because if  $hnN \in HN/N$  then  $hnN = hN = \varphi(h)$ . Moreover, if  $h \in N$  then  $\varphi(h) = hN = 1N$ , so  $H \cap N \subset \text{Ker } \varphi$ . Conversely, if  $h \in \text{Ker } \varphi$  then  $hN = N$  so  $h \in N$  so  $h \in H \cap N$ . Thus  $\text{Ker } \varphi = H \cap N$ , so the result follows from the First Isomorphism Theorem.

**Theorem II.29 [Third Isomorphism Theorem]** If  $H \triangleleft G$  and  $K \triangleleft G$  and  $K < H$  then  $H/K \triangleleft G/K$  and

$$(G/K)/(H/K) \cong G/H.$$

*Proof:* We consider the map  $\varphi: G/K \rightarrow G/H$  given by  $\varphi(gK) = gH$ . This is well defined: if  $g_1K = g_2K$  then  $g_1^{-1}g_2 \in K$  so  $g_1^{-1}g_2 \in H$  so  $g_1H = g_2H$ . It is also a group homomorphism because  $\varphi((g_1K)(g_2K)) = \varphi(g_1g_2K) = g_1g_2H = (g_1H)(g_2H)$ . Clearly, if  $gK \in \text{Ker } \varphi$  then  $gH = H$  so  $g \in H$  so  $gK \in H/K$ . On the other hand, suppose  $hK \in H/K$ ; then  $\varphi(hK) = hH = H$  so  $hK \in \text{Ker } \varphi$ . Again the result follows from the First Isomorphism Theorem.

### III Group actions

We shall make groups work as symmetries of sets, and relate this idea to what has gone before.

## Basics of group actions

**III.1** The idea is that, given a group  $G$ , we want to realise it as the group of symmetries of something. Or, failing that, we want to get a relation between it and a group of symmetries of something: that is, a homomorphism  $G \rightarrow \text{Sym } X$  for some  $X$ . Actions are very far from unique and they can tell us a great deal about  $X$  as well as  $G$ , especially if  $X$  has some extra structure and  $G$  preserves it. For example, if we can write  $G$  as a group of matrices then we can make  $G$  act on a vector space so that the elements of  $G$  become linear maps.

**Definition III.2** Suppose  $G$  is a group and  $X$  is a set. An *action* of  $G$  on  $X$  is a map  $a: G \times X \rightarrow X$ , in which we normally denote  $a(g, x)$  by  $g(x)$  or even  $gx$ , such that if  $g_1, g_2 \in G$  and  $x \in X$  then  $g_1(g_2(x)) = (g_1g_2)(x)$  and  $1_G(x) = x$ . In this case we say that  $G$  *acts on*  $X$ .

**Lemma III.3** If  $G$  acts on  $X$  and  $g \in G$  then the map  $a_g: X \rightarrow X$  given by  $x \mapsto g(x)$  is a bijection.

*Proof:* The inverse map is given by the map  $a_{g^{-1}}: x \mapsto g^{-1}(x)$ .

**Proposition III.4** If  $G$  acts on  $X$  then the map  $\alpha: G \rightarrow \text{Sym } X$  given by  $g \mapsto a_g$  is a group homomorphism. Conversely, any group homomorphism  $\alpha: G \rightarrow \text{Sym } X$  induces an action of  $G$  on  $X$  by setting  $g(x) = \alpha(g)(x)$ .

*Proof:* For the first part, we need to check that  $\alpha(g_1g_2) = \alpha(g_1)\alpha(g_2)$ . But

$$\alpha(g_1g_2)(x) = a_{g_1g_2}(x) = (g_1g_2)(x) = g_1(g_2(x)) = a_{g_1}(a_{g_2}(x)) = \alpha(g_1)(\alpha(g_2)(x))$$

as required. The second part leads to the same computation.

**III.5** These are, technically, left actions. Left actions are pleasanter than right actions because then composition of maps goes the way you expect if you write maps on the left ( $f(x)$  rather than  $xf$ ).

Although  $a_g$  is a bijection,  $\alpha$  is in general not: it need be neither injective nor surjective.

If  $G$  acts on  $X$  and  $H$  is a subgroup of  $G$  then  $H$  also acts on  $X$ , simply by restricting  $a$  to  $H \times X$ .

**Definition III.6** If  $\alpha$  is injective, so  $a_g = \text{id}_X$  only if  $g = 1_G$ , we say that  $G$  acts *faithfully* on  $X$ .

**Example III.7** We have already seen many examples of group actions.

- (i)  $\mathbb{Z}$  acts on  $\mathbb{R}$ , by translations.
- (ii)  $D_6$  acts on the vertices of a triangle.
- (iii)  $D_6$  acts on the faces of a (cardboard) triangle.

- (iv) The cube group (the group of symmetries of a cube) acts on the cube, but also on the octohedron.
- (v)  $\text{GL}(2, \mathbb{R})$  acts on  $\mathbb{R}^2$ : more generally, any subgroup of  $\text{GL}(n, \mathbb{F})$  acts on  $\mathbb{F}^n$ .
- (vi)  $\text{SL}(2, \mathbb{R})$  acts on the upper half-plane  $\mathbb{H} = \{z \in \mathbb{C} \mid \text{Im } z > 0\}$  by Möbius transformations:  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} (z) = \frac{az+b}{cz+d}$ . (This action, and especially its restriction to  $\text{SL}(2, \mathbb{Z})$ , is extremely important in number theory.)
- (vii) Any group  $G$  acts on itself (i.e. we take  $X = G$ ) by (say) left multiplication,  $a(g, h) = gh$ , but notice that  $a_g: G \rightarrow G$  is not a group homomorphism except when  $g = 1$ .
- (viii) Any group  $G$  acts on itself by conjugation,  $a(g, h) = ghg^{-1}$ : this time  $a_g$  is a homomorphism.
- (ix) Any group  $G$  acts on any set  $X$  by the trivial action,  $g(x) = x$  for all  $g \in G$  and all  $x \in X$ .

**Definition III.8** Suppose that  $G$  acts on  $X$  and  $x \in X$ . The *orbit* of  $x$ , denoted  $\text{orb}(x)$ , is the set

$$\text{orb}(x) = \{gx \mid g \in G\} \subseteq X.$$

We will sometimes refer to  $\text{orb}(x)$  as the  $G$ -orbit of  $x$  and write  $\text{orb}_G(x)$ . A set that is  $\text{orb}_G(x)$  for some  $x \in X$  is called a  $G$ -orbit.

**III.9** The orbit of  $x$  is the set of places that  $G$  can take  $x$  to.

Obviously  $|\text{orb}(x)| \leq |G|$  but different orbits can be of different sizes. Consider, for example, the action of  $\mathbb{C}^*$  on  $\mathbb{C}$  given by multiplication:  $a(w, z) = wz$ . The orbit of  $z \in \mathbb{C}$  is  $\{wz \mid w \in \mathbb{C}^*\}$  and that is  $\mathbb{C}^*$  if  $z \neq 0$  but  $\{0\}$  if  $z = 0$ .

Another example is conjugacy classes: these are the orbits of the action in Example III.7(viii). If  $G$  is not abelian, there is some pair of elements  $h$  and  $g$  such that  $ghg^{-1} \neq h$ , and then if  $G$  acts on itself by conjugation the orbit  $\text{orb}(h)$  contains at least those two elements. But  $\text{orb}(1) = \{1\}$  for this action, since  $g1g^{-1} = 1$ .

On the other hand, in the action in Example III.7(vii) the orbits are all the same size. In fact there is only one orbit, because any  $g \in G$  is in  $\text{orb}(1) = \{g1 \mid g \in G\} = G$ .

If we restrict the action in Example III.7(vii) to a subgroup  $H < G$  then the orbit of  $g$  is  $\{hg \mid h \in H\}$  which is the (right) coset  $Hg$ . In this case the orbits are all the same size by Lemma II.3.

**Theorem III.10 [Orbit partition theorem]** If  $G$  acts on  $X$ , then every  $x \in X$  belongs to exactly one  $G$ -orbit.

*Proof:* We shall prove that  $G$ -orbits are equivalence classes for an equivalence relation. We define the relation by  $x \sim y \iff \exists g \in G$  such that  $g(y) = x$ . This is reflexive (take  $g = 1_G$ ), symmetric (because  $y = g^{-1}(x)$ ) and transitive because if  $x \sim y$  and  $y \sim z$  then  $x = g(y)$  and  $y = g'(z)$  for some  $g, g' \in G$  and then  $x = (gg')(z)$  by Definition III.2.

**III.11** We can now see Theorem II.4 and II.5 as a special case: in view of III.9 and Example III.7(vii) we can describe cosets as orbits.

**Definition III.12** We say that the action of  $G$  on  $X$  is *transitive* (or  $G$  acts transitively on  $X$ ) if  $\text{orb}_G(x) = X$  for some  $x \in X$ .

**III.13** In this case there is only one orbit so  $\text{orb}_G(x) = X$  for any  $x \in X$ .

We saw in III.9 that the action of any  $G$  on itself by multiplication, Example III.7(vii), is transitive. Also any action on a 1-point set must be transitive.

It is easy to make many examples of transitive group actions. Take any group action of  $G$  on  $X$ , and pick a  $G$ -orbit  $X_0$ . Then  $G$  acts transitively on  $X_0$ . Sometimes this is interesting: for instance,  $\text{GL}(n, \mathbb{F})$  acts on  $\mathbb{F}^n$  (Example III.7(v)) and it acts transitively on  $\mathbb{F}^n \setminus \{0\}$ , which is the orbit of any non-zero vector.

**Definition III.14** Let  $G$  be a group acting on a set  $X$  and suppose  $x \in X$ . The *stabiliser*  $\text{Stab}_G(x)$  of  $x$  in  $G$  is

$$\text{Stab}_G(x) := \{g \in G \mid g(x) = x\}.$$

**Lemma III.15** The stabiliser  $\text{Stab}_G(x)$  is a subgroup of  $G$ .

*Proof:* Notice that  $1 \in \text{Stab}_G(x)$ , so  $\text{Stab}_G(x) \neq \emptyset$ . So according to Lemma I.14, we need to show that  $g_1 g_2^{-1} \in \text{Stab}_G(x)$  if  $g_1, g_2 \in \text{Stab}_G(x)$ . If  $g_i \in \text{Stab}_G(x)$  then  $g_i(x) = x$  and therefore also  $g_i^{-1}(x) = g_i^{-1}g_i(x) = x$ , so in this case we have  $(g_1 g_2^{-1})(x) = g_1(g_2^{-1}(x)) = g_1(x) = x$  as required.

**III.16** A very useful way to find subgroups of a group  $G$  is to make  $G$  act on something and look at the stabiliser, often choosing the element  $x$  carefully so as to get an interesting stabiliser. Unlike kernels, which will only find normal subgroups, this sees all the subgroups of  $G$ .

In particular, the stabiliser is not usually a normal subgroup, so you cannot say “choose  $x \in X$  and take the quotient of  $G$  by  $\text{Stab}_G(x)$ ”.

**Example III.17** We again know many examples.

- (i)  $\mathbb{Z}$  acting on  $\mathbb{R}$  has trivial stabilisers.
- (ii)  $D_6$  acting on the vertices of a triangle has stabilisers of order 2.
- (iii)  $D_6$  acting on the faces of a (cardboard) triangle has stabilisers of order 3.
- (iv) In the action of  $\text{GL}(2, \mathbb{R})$  on  $\mathbb{R}^2$ , the stabiliser of the origin is  $\text{GL}(2, \mathbb{R})$  and the stabiliser of  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$  is  $\left\{ \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix} \mid d \neq 0, b \in \mathbb{R} \right\}$ .

- (v) In the action of  $\mathrm{SL}(2, \mathbb{Z})$  on the upper half-plane  $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im} z > 0\}$ , the stabiliser of most  $z \in \mathbb{H}$  is  $\pm I$ , but the stabiliser of  $i \in \mathbb{H}$  is a group of order 4 generated by  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  and the stabiliser of  $\omega = e^{2\pi i/3}$  is of order 6, generated by  $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ . You wouldn't believe how much trouble, or the reverse, is caused by these two groups.

**Proposition III.18** Suppose  $H < G$ . Then there exists a set  $X$ , an element  $x \in X$  and an action of  $G$  on  $X$  such that  $H = \mathrm{Stab}_G(x)$ .

*Proof:* Take  $X = \{gH \mid g \in G\}$  to be the coset set of  $H$  in  $G$  (Definition II.14). Then  $G$  acts on  $X$  by left multiplication, i.e.  $a(g, g'H) = gg'H$ . It is simple to check that this is indeed a group action, and the stabiliser of  $H \in X$  (that is, the coset  $1_G H$ ) is  $H$ .

**Theorem III.19 [Orbit-Stabiliser Theorem]** Suppose  $G$  is a group acting on a set  $X$  and  $x \in X$ . Then the size of the orbit of  $x$  is the index of its stabiliser:

$$|\mathrm{orb}_G(x)| = |G : \mathrm{Stab}_G(x)|.$$

*Proof:* For brevity, write  $S = \mathrm{Stab}_G(x)$ . We need to find a bijection between  $\mathrm{orb}_G(x)$  and the coset set  $G/S$  of  $S$  in  $G$  (this is only a set, in general, not a group: see II.15 and Definition II.14). An element of the coset set is  $gS$  for some  $g \in G$  and we define  $\omega: G/S \rightarrow \mathrm{orb}_G(x)$  by  $\omega(gS) = g(x)$ . We need to check that this is well-defined, i.e. that if  $g'S = gS$  then  $g'(x) = g(x)$ , but  $g'S = gS$  if and only if  $g' \in gS$ , i.e. if and only if  $g' = gh$  for some  $h \in S$ , and then  $g'(x) = gh(x) = g(h(x)) = g(x)$  since  $h$  stabilises  $x$ .

We also need to check that  $\omega$  is bijective. We do this by writing down its inverse. If  $y \in \mathrm{orb}_G(x)$  then  $y = g(x)$  for some  $g \in G$  and we consider the map  $\gamma: \mathrm{orb}_G(x) \rightarrow G/S$  given by  $\gamma(y) = gS$ . Again we must check that  $\gamma$  is well-defined. If  $y = g(x) = g'(x)$  then  $g^{-1}g'(x) = x$  so  $g^{-1}g' \in S$  so  $g' = gg^{-1}g' \in gS$ , so  $g'S = gS$  as required. Then  $\gamma(\omega(gS)) = \gamma(g(x)) = gS$  and  $\omega(\gamma(y)) = \omega(gS) = g(x) = y$ , so the maps are inverse so they must both be bijections.

**III.20** Note that we have not assumed that either  $G$  or  $X$  is finite.

The orbit-stabiliser theorem is very useful for counting, which is difficult in general. It is often much easier to compute the index of a group than to compute the size of an orbit, or vice versa.

## Representations

**III.21** Suppose that  $G$  is a group: in principle any group, but we shall mainly want this construction for finite groups. Then  $G$  acts on itself by left multiplication (Example III.7(vii)) and this gives a homomorphism  $\alpha: G \rightarrow \mathrm{Sym} G$  by Proposition III.4.

Recall (Example I.12(xii)) that the set of bijections from a set  $X$  to itself forms a group under composition. Such a group is called a permutation group.

**Theorem III.22 [Cayley's Theorem]** Every group is isomorphic to a subgroup of a permutation group.

*Proof:* Consider the left action of  $G$  on itself, so  $a_g = \alpha(g): h \rightarrow gh$  for  $g, h \in G$ . We claim that  $\alpha$  is injective: then  $G \cong \text{Im } \alpha < \text{Sym } G$  by Theorem II.25.

According to Lemma I.28 it is sufficient to check that  $\text{Ker } \alpha = 1$ . But if  $g \in \text{Ker } \alpha$  then  $a_g = \text{id}$ : that is,  $gh = h$  for all  $h \in G$ . In particular if we take  $h = 1_G$  we get  $g = 1_G$ , as required.

**III.23** Despite its short proof, Cayley's theorem is important. It is the formal statement that captures the idea that groups are all about symmetry. It exhibits every possible group as a group of symmetries of something; namely, itself. Not the full group of symmetries:  $\text{Sym } G$  is typically much bigger than  $G$ .

**Corollary III.24** Every finite group is isomorphic to a subgroup of  $\text{GL}(|G|, \mathbb{F})$ , where  $\mathbb{F}$  is any field. In particular, any finite group is isomorphic to a group of matrices.

*Proof:* In view of Cayley's Theorem it is enough to prove that  $S_n$  is isomorphic to a subgroup of  $\text{GL}(n, \mathbb{F})$ . (Note that this is not the special case of this corollary for  $G = S_n$ : that says that  $S_n$  is isomorphic to a subgroup of  $\text{GL}(n!, \mathbb{F})$ .) We take the usual basis  $e, \dots, e_n$  for  $\mathbb{F}^n$  and we define  $M: S_n \rightarrow \text{GL}(n, \mathbb{F})$  by

$$M(\sigma)e_i = e_{\sigma(i)}$$

and extending by linearity to other elements of  $\mathbb{F}^n$ . This is a group homomorphism because

$$M(\sigma\sigma')(i) = e_{(\sigma\sigma')(i)} = e_{\sigma(\sigma'(i))} = M(\sigma)(e_{\sigma'(i)}) = M(\sigma)(M(\sigma')(i))$$

for every  $i$ , and since linear maps are determined by their effect on a basis, that proves that  $M(\sigma\sigma') = M(\sigma)M(\sigma')$ . It is injective because if  $M(\sigma) = \text{id}$  then  $e_i = M(\sigma)(e_i) = e_{\sigma(i)}$  for all  $i$ , so  $\sigma$  is the trivial permutation.

**III.25** This is completely false for infinite groups. They could, for example, simply be bigger than the set of all matrices over  $\mathbb{F}$ , but in fact it can fail even for countable groups.

**III.26** A homomorphism  $\varphi: G \rightarrow \text{GL}(n, \mathbb{F})$  for some  $n$  and some field  $\mathbb{F}$  is called a representation of  $G$  over the field  $\mathbb{F}$ . A representation is called faithful if its kernel is trivial, i.e. its image is isomorphic to  $G$ . Corollary III.24 says that every finite group has a faithful representation. Studying representations of groups is one of the main ways to get information about them.

## IV Rings

We introduce rings, which have both an addition and a multiplication, but not multiplicative inverses.

### Basic theory of rings

**IV.1** The basic idea of a ring is that you are allowed to add and subtract and multiply, but not necessarily divide: addition is commutative, but multiplication might not be. Then there have to be rules about how multiplication and addition interact. So, looking at I.10, we expect two binary operations, one unary operation (subtraction, or rather minus), and one or two nullary operations (0 and, optionally, 1).

**IV.2** The basic example of some things that you can add and multiply by not necessarily divide is the integers. You can't expect to be able to divide an integer by 2, not if you want the answer to be an integer. So  $\mathbb{Z}$  should be one basic example of a ring. In that case multiplication is commutative. Another good example is the ring of polynomials with (say) real coefficients in one variable,  $\mathbb{R}[t]$ . You can add two of these and you can multiply them, but you can't divide:  $\frac{1}{t}$  isn't a polynomial. In fact, these two turn out to have much more in common than just being rings: they are very similar rings. In other words, a polynomial may be nothing like an integer, but polynomials collectively behave remarkably like integers collectively.

If you want an example of a noncommutative ring, consider the set of (say)  $2 \times 2$  (say) real matrices,  $M_2(\mathbb{R})$ . Here you have addition and multiplication, and you can take inverses as long as the determinant isn't zero; but sometimes it is zero.

**Definition IV.3** A *ring* is a triple  $(R, +, \cdot)$ , where  $R$  is a set with two binary operations called addition (denoted  $+$ ) and multiplication (denoted  $\cdot$ , or just nothing), such that the following axioms hold:

- (i) **(addition)**  $(R, +)$  is an abelian group.
- (ii) **(associative)** The binary operation  $\cdot$  is associative.
- (iii) **(distributive)** Multiplication is *distributive over addition*, from the left or the right: that is

$$\begin{aligned} a \cdot (b + c) &= (a \cdot b) + (a \cdot c) && \text{for all } a, b, c \in R; \\ (b + c) \cdot a &= (b \cdot a) + (c \cdot a) && \text{for all } a, b, c \in R. \end{aligned}$$

- (iv) **(identity)** There is a *multiplicative identity*, an element  $1 \in R$  such that  $1 \cdot a = a \cdot 1 = a$  for all  $a \in R$ .

We write 0 for the (unique) additive identity, and  $-a$  for the (unique) additive inverse of  $a \in R$ .

**IV.4** The last axiom (**identity**) in IV.3 is in a sense optional. Many writers omit it, and allow rings without 1. The disadvantage is that when you want to talk about a ring with a 1 (also called a unital ring), you have to say so; and that happens a lot, so you would rather keep the simpler phrasing for the commonest case. Here we have made the opposite decision, and the disadvantage is that we now have no name for a ring without a 1, i.e. something that satisfies IV.3(i)–(iii) but not necessarily IV.3(iv).

**IV.5** We often omit  $\cdot$  and write  $ab$  instead of  $a \cdot b$ . Sometimes we will need to write  $0_R$  and  $1_R$  for the identities in  $R$ , to distinguish them from identities in other rings or groups.

As with groups, we will very often know what the operations are and just talk about the ring  $R$ , rather than calling it  $(R, +, \cdot)$ .

For simplicity we often avoid brackets when there is no ambiguity. Here the same conventions hold as for real numbers:  $\cdot$  has priority over  $+$ . For example  $ab + ac$  stands for  $(a \cdot b) + (a \cdot c)$  and not  $(a \cdot (b + a)) \cdot c$ . One also writes  $a^2$  for  $a \cdot a$  and (see V.30 below)  $2a$  for  $a + a$  and so on.

**Lemma IV.6** In any ring  $(R, +, \cdot)$ , we have

- (i)  $a \cdot 0 = 0$  and  $0 = 0 \cdot a$  for all  $a \in R$ ; and
- (ii)  $a \cdot (-b) = -(a \cdot b)$  and  $-(a \cdot b) = (-a) \cdot b$  for all  $a, b \in R$ .

**Definition IV.7** Let  $R$  be a ring. An element  $a \in R$  is called a *unit* if it has a multiplicative inverse: i.e. if there exists  $b \in R$  such that  $a \cdot b = b \cdot a = 1$ .

**Lemma IV.8** Let  $R$  be a ring. Then the multiplicative identity is unique, and if  $a \in R$  is a unit then the multiplicative inverse of  $a$  is unique.

*Proof:* The same argument as in I.9 still works.

**IV.9** Notice that if  $R$  is a ring then  $(R, \cdot)$  is *never* a group, except in the trivial case where  $R = \{0\}$  and  $0 = 1$ . This is because 0 cannot possibly have a multiplicative inverse, because of IV.6(i).

The set of units in  $R$  is denoted by  $R^*$  or sometimes  $R^\times$ . It is easy to see that  $R^*$  does form a group under multiplication (but it behaves very badly under addition: the sum of two units does not have to be a unit). In fact, there is nothing to check because associativity and identity are already in the ring axioms.

**Example IV.10** Some examples of rings and their units are:

- (i)  $R = \mathbb{Z}$ ; then  $\mathbb{Z}^* = \{\pm 1\}$ .
- (ii)  $R = \mathbb{C}$ ; then  $\mathbb{C}^* = \{z \in \mathbb{C} \mid z \neq 0\}$ .
- (iii)  $R = M_2(\mathbb{R})$ ; then  $R^* = \{A \in M_2(\mathbb{R}) \mid \det A \neq 0\}$ .



Notice how different these three are. In (i) there are very few units, but more than just 1. In (ii) everything is a unit, except for 0 of course. In (iii) most elements are units, but there are still exceptions.

## Types of rings

**Definition IV.11** A ring  $R$  is a *commutative ring* if  $a \cdot b = b \cdot a$  for all  $a, b \in R$ .

One does not say “abelian ring”! In contexts where only commutative rings are important, one sometimes drops the word “commutative”, and explicitly says “noncommutative ring” where necessary, but that will not be done here.

**Definition IV.12** A ring  $R$  is an *integral domain* if it is a commutative ring in which  $0 \neq 1$ , such that if  $a, b \in R$  and  $ab = 0$ , then  $a = 0$  or  $b = 0$ .

Sometimes an integral domain is just called a domain.

**Definition IV.13** A ring  $R$  is a *division ring* if  $0 \neq 1$ , and every non-zero element is a unit.

That is,  $a^{-1}$  exists as long as  $a \neq 0$ . Note that a division ring is not required to be commutative.

**Definition IV.14**  $R$  is a *field* if it is a commutative division ring.

**IV.15** Informally, in a field you can add and multiply and divide by anything that isn’t actually zero, and multiplication is commutative.

Fields are often called  $K$  (German *Körper*). Every field  $K$  is an integral domain: if  $a, b \in K$  and  $ab = 0$ , then if  $a \neq 0$  we have  $b = 1 \cdot b = a^{-1}ab = a^{-1} \cdot 0 = 0$ .

**Example IV.16** We start with a few familiar examples.

- (i) Every field is a commutative ring. In particular  $\mathbb{Q}$ ,  $\mathbb{R}$ , and  $\mathbb{C}$  are commutative rings.
- (ii) Division rings need not be commutative, so division rings need not be fields. An example of this is the quaternions.
- (iii) The ring  $\mathbb{Z}$  is an integral domain (that’s why they are called “integral”), but it is not a division ring, so it is not a field.
- (iv) The set  $\mathbb{R}[t]$  of polynomials in one variable with real coefficients is also an integral domain, but not a field.
- (v) The commutative ring  $\mathbb{Z}/4$  consists of the integers with arithmetic mod 4. It is a commutative ring, but it is not an integral domain because  $2 \times 2 = 4 = 0$  but  $2 \neq 0$ .
- (vi) On the other hand,  $\mathbb{Z}/5$  is a field; in fact  $\mathbb{Z}/p\mathbb{Z}$  is a field, called  $\mathbb{F}_p$ , whenever  $p$  is prime.

There is also a field with 4 elements, called  $\mathbb{F}_4$ , but it is not the same as  $\mathbb{Z}/4\mathbb{Z}$ . They are both commutative rings and they both have four elements, but they are otherwise different.

**Definition IV.17** If  $K$  is a field, a  $K$ -algebra is a ring that is also a  $K$ -vector space, and such that  $(\lambda a)(\mu b) = (\lambda\mu)(ab)$  for any  $\lambda, \mu \in K$  and  $a, b \in A$ .

**Example IV.18** For any ring  $R$ , let  $M_n(R)$  denote the set of all  $n \times n$  matrices with coefficients in the ring  $R$ . Then  $M_n(R)$  is a ring with respect to usual addition and multiplication of square matrices. However, even if  $R$  is a commutative ring,  $M_n$  is not commutative, unless  $n = 1$  (when it is just  $R$  in a very light disguise).

We saw this ring in IV.10(iii) in the case  $R = \mathbb{R}$ . Note that if  $R = K$  is a field then  $M_n(K)$  is a  $K$ -algebra.

**Example IV.19** Let  $R$  be a ring and let  $t$  be a variable. Let  $d \geq 0$  be a non-negative integer. A *polynomial  $f$  over  $R$  of degree  $d$*  is a formal expression

$$f = \sum_{k=0}^d a_k t^k = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots + a_d t^d,$$

with  $a_k \in R$  for  $0 \leq k \leq d$  and  $a_d \neq 0$ . The  $a_k$  are called the *coefficients*.

We let  $R[t]_d$  denote the set of all polynomials of degree  $d$ , and we set

$$R[t] = \{0\} \cup \bigcup_{d=0}^{\infty} R[t]_d.$$

Thus  $1 + t$ ,  $39 - 62t + 3t^{19}$  and  $94$  are all elements of  $\mathbb{Z}[t]$ , and so is  $0$ , but  $\frac{1}{t}$  is not and neither is  $e^t$ .

The *degree* of a polynomial is the highest  $k$  such that  $a_k \neq 0$ , i.e. the highest power of  $t$  that actually occurs: a polynomial of degree 0 is called a non-zero constant (it is just a non-zero element of  $R$ ). The degree of the polynomial 0 is not defined – sometimes one can save some writing by declaring that it is 0, or that it is  $-1$  or even  $-\infty$ , but only temporarily. 0 is also called a constant, so the *constants* are just the elements of  $R$ .

A polynomial of degree  $d$  is called a *monic polynomial* if  $a_d = 1$ .

It is easy to check that  $R[t]$  is a ring. If  $R = K$  is a field, then  $K[t]$  is a  $K$ -algebra.

**Definition IV.20** A nonempty subset  $S$  of a ring  $R$  is called a *subring* if and only if, for any  $a, b \in S$  we have both  $a - b \in S$  and  $ab \in S$ .

**Lemma IV.21** Let  $S$  be a subset of a ring  $(R, +, \cdot)$ . Then  $S$  is a subring of  $R$  if and only if  $(S, +, \cdot)$  satisfies IV.3(i)–(iii) of a ring.

In other words, a subring is a subset that happens to be a ring, but not necessarily with a 1. This is an inconvenience of our convention that “ring” means “unital ring”, but it is what we want.

**Example IV.22** We have seen several examples of subrings already.

- (i) For any ring  $R$ , both  $\{0\}$  and  $R$  are subrings of  $R$ .
- (ii) The ring  $\mathbb{Z}$  is a subring of  $\mathbb{Q}$  which is a subring of  $\mathbb{R}$  which is a subring of  $\mathbb{C}$ , under the usual operations of addition and multiplication.
- (iii) The even integers  $2\mathbb{Z}$  are a subring of  $\mathbb{Z}$ . By Lemma IV.21 the even integers therefore form a ring but not necessarily with a 1: and indeed 1 is odd. (But 0 is even, a fact that seems to be poorly understood.) So a subring does not have to be a ring in the full sense of satisfying IV.3(iv) as well.
- (iv) If  $R$  is any ring and  $t$  is a variable, then  $R$  is a subring of  $R[t]$ . In this case, the subring really is a ring.
- (v) The Gaussian integers  $\mathbb{Z}[i] = \{a + bi \in \mathbb{C} \mid a, b \in \mathbb{Z}\}$  form a subring of the field  $\mathbb{C}$ : also  $1 = 1 + 0i \in \mathbb{Z}[i]$ , so  $\mathbb{Z}[i]$  is a ring.

The notation  $\mathbb{Z}[i]$  is not incompatible with the notation for polynomial rings: unlike  $t$ ,  $i$  is not a variable, but in any case,

$$\sum_{k=0}^d a_k i^k = \left( \sum_{l=0}^{\lceil d/4 \rceil} (a_{4l} - a_{4l+2}) \right) + i \left( \sum_{l=0}^{\lceil d/4 \rceil} (a_{4l+1} - a_{4l+3}) \right) \in \mathbb{Z}[i].$$

**Lemma IV.23** If a subring  $S$  of an integral domain  $R$  contains the element  $1 \in R$ , then  $S$  is an integral domain.

## Maps between rings

**Definition IV.24** Let  $R$  and  $S$  be rings. A map  $\varphi: R \rightarrow S$  is said to be a *ring homomorphism* if and only if for all  $a, b \in R$ , we have

$$\varphi(a + b) = \varphi(a) + \varphi(b) \quad \text{and} \quad \varphi(a \cdot b) = \varphi(a) \cdot \varphi(b).$$

**Example IV.25** Here are a few examples and unexamples.

- (i) The map  $\varphi: \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$  defined by

$$\varphi(n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

is a ring homomorphism. Indeed, if we compare the rules for adding and multiplying even and odd integers with the addition and multiplication tables for  $\mathbb{Z}/2\mathbb{Z}$ , we see that computing in  $\mathbb{Z}$  and then applying  $\varphi$  is the same as applying  $\varphi$  and then computing in  $\mathbb{Z}/2\mathbb{Z}$ .

- (ii) More generally, the map  $\varphi: \mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$  that takes an integer  $a$  to its residue class  $[a] \bmod n$  is a ring homomorphism, for any  $n \in \mathbb{N}$ .

- (iii) The map  $\varphi: \mathbb{Z} \rightarrow 2\mathbb{Z}$  defined by  $\varphi(n) = 2n$  is *not* a ring homomorphism, because  $\varphi(nm) = 2nm$  is typically not equal to  $4nm = (2n)(2m) = \varphi(n)\varphi(m)$ .
- (iv) Let  $R$  be a commutative ring and choose  $r \in R$ . If we take any polynomial  $f \in R[t]$  with coefficients in  $R$ , then  $f(r) \in R$ , so we obtain a map

$$\text{ev}_r: R[t] \longrightarrow R$$

by taking  $\text{ev}_r(f) = f(r)$ . In other words, we *evaluate* each polynomial at  $r \in R$ , i.e., substitute  $r \in R$  into each polynomial. We shall see shortly in Proposition IV.29 that  $\text{ev}_r$  is a ring homomorphism, for any  $r \in R$ .

- (v) The map  $\varphi: \mathbb{R} \rightarrow M_{2 \times 2}(\mathbb{R})$  given by  $\varphi(x) = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$  is a ring homomorphism.

**Lemma IV.26** The composition of two ring homomorphisms is a ring homomorphism.

**Lemma IV.27** If  $\varphi: R \rightarrow S$  is a ring homomorphism then

- (i) for  $a, b \in R$ , we have  $\varphi(b - a) = \varphi(b) - \varphi(a)$ ;
- (ii)  $\varphi(0_R) = 0_S$ ;
- (iii) for  $a \in R$ , we have  $\varphi(-a) = -\varphi(a)$ .

*Proof:* The fact that in Definition IV.24 we required  $\varphi(a + b) = \varphi(a) + \varphi(b)$  tell us that a ring homomorphism is in particular a group homomorphism (for addition), by Proposition I.23, and that also gives us (i) and (ii) because they are parts (i) and (ii) of Definition I.20.

For part (iii), substitute  $b = 0$  into (i) and use (ii) to obtain

$$\varphi(-a) = \varphi(0_R - a) = \varphi(0_R) - \varphi(a) = 0_S - \varphi(a) = -\varphi(a)$$

as required.

**IV.28** Notice that in Lemma IV.27 we have not claimed that  $\varphi(1_R) = 1_S$ , and indeed that does not have to happen. For an example, see IV.25(vi). In that case we have  $\varphi(1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ , which is not the identity matrix.

**Proposition IV.29** Let  $R$  be a commutative ring and  $r \in R$ . Then the evaluation map  $\text{ev}_r: R[t] \rightarrow R$  is a ring homomorphism.

*Proof:* Given any two polynomials  $f = \sum_{k=0}^{d_1} a_k t^k$  and  $g = \sum_{k=0}^{d_2} b_k t^k$ , we have

$$\begin{aligned}
 \text{ev}_r(f + g) &= \text{ev}_r \left( \sum_{k=0}^{\max\{d_1, d_2\}} (a_k + b_k) t^k \right) \\
 &= \sum_{k=0}^{\max\{d_1, d_2\}} (a_k + b_k) r^k \\
 &= \sum_{k=0}^{d_1} a_k r^k + \sum_{k=0}^{d_2} b_k r^k \\
 &= \text{ev}_r(f) + \text{ev}_r(g),
 \end{aligned}$$

using commutativity of addition and distributivity in  $R$  to get from the second line to the third. Similarly

$$\begin{aligned}
 \text{ev}_r(fg) &= \text{ev}_r \left( \sum_{k=0}^{d_1+d_2} \left( \sum_{i+j=k} a_i b_j \right) t^k \right) \\
 &= \sum_{k=0}^{d_1+d_2} \left( \sum_{i+j=k} a_i b_j \right) r^k \\
 &= \sum_{i=0}^{d_1} a_i r^i \cdot \sum_{j=0}^{d_2} b_j r^j \\
 &= \text{ev}_r \left( \sum_{i=0}^{d_1} a_i t^i \right) \cdot \text{ev}_r \left( \sum_{j=0}^{d_2} b_j t^j \right) \\
 &= \text{ev}_r(f) \cdot \text{ev}_r(g),
 \end{aligned}$$

where the first line comes from the definition of multiplication in  $R[t]$  and getting from the second line to the third uses everything: the distributive laws, commutativity of addition and associativity of both addition and multiplication in the ring  $R$ .

## V Modules, ideals and quotient rings

In this section we introduce modules, ideals and quotients, all of which are analogous to structures that are already familiar.

### Modules

**V.1** A module is to a ring what a vector space is to a field: that is, it is an abelian group on which the ring acts linearly. For simplicity, we shall limit ourselves to the case of commutative rings.

**Definition V.2** Let  $R$  be a commutative ring. An  $R$ -module is an abelian group  $M$  (written additively) together with a map  $R \times M \rightarrow M$ , called *scalar multiplication* and written  $(r, m) \mapsto rm$ , such that for all  $r, s \in R$  and  $m, n \in M$  we have

- (a)  $r(m + n) = rm + rn$ ;
- (b)  $r(sm) = (rs)m$ ;
- (c)  $(r + s)m = rm + sm$ ;
- (d)  $1m = m$ .

Some texts omit the last condition. Leaving it out allows the possibility that there is a non-zero subgroup  $N < M$  for which multiplication by  $r \in R$  is simply 0: that is,  $rn = 0$  for every  $n \in N$  and every  $r \in R$ . In practice this makes very little difference.

**V.3** Informally (but completely accurately) Definition V.2 means that a module is a structure in which we are able to take linear combinations with coefficients taken from  $R$ : if  $r, s \in R$  and  $m, n \in M$  then  $rm + sn \in M$ . Addition in  $M$  is the case  $r = s = 1_R$  and scalar multiplication is the case  $s = 0$ .

**Example V.4** We know a few examples.

- (i) If  $R = K$  is a field, then a  $K$ -module is the same thing as a  $K$ -vector space.
- (ii) Any commutative ring  $R$  is an  $R$ -module, by taking scalar multiplication to be ring multiplication. This is the same as thinking of  $K$  as a 1-dimensional  $K$ -vector space.
- (iii) More generally,  $R^n$  is an  $R$ -module by multiplication in each factor: that is,  $r(r_1, \dots, r_n) = (rr_1, \dots, rr_n)$ : if  $R = K$  is a field, this is the same as thinking of  $K^n$  as an  $n$ -dimensional vector space. Such a module is called a *free  $R$ -module*: see Definition VII.6.
- (iv) If  $S$  is a subring of  $R$  (and  $1_R \in S$ ) then  $R$  is an  $S$ -module, again by taking scalar multiplication to be ring multiplication in  $R$ . In particular,  $R[t]$  is an  $R$ -module.
- (v)  $\mathbb{Z}/N$  is a  $\mathbb{Z}$ -module, where the multiplication is given by multiplication in  $\mathbb{Z}$ . That is, if  $r \in \mathbb{Z}$  and  $[m]$  is the class of  $m \bmod N$ , then we define  $r[m] = [rm]$ : since  $[r(m + kN)] = [m]$  this is well defined, and the module axioms follow at once from the ring axioms in  $\mathbb{Z}$ .
- (vi) Abelian groups and  $\mathbb{Z}$ -modules are the same thing: a  $\mathbb{Z}$ -module is an abelian group already, and if  $A$  is an abelian group we define multiplication by an element of  $\mathbb{Z}$  by repeated addition, as in V.30 below.

**V.5** Notice that Example V.4(v) shows that even for  $R = \mathbb{Z}$  there is not such a simple classification theorem for  $R$ -modules as there is for vector spaces: there is no such thing as the dimension of an  $R$ -module. We shall see what the classification of  $\mathbb{Z}$ -modules looks like in VII.30.

**Definition V.6** If  $N$  is a subgroup of  $M$  that is closed under scalar multiplication (that is,  $rn \in N$  if  $r \in R$  and  $n \in N$ ) then  $N$  is called a *submodule* of  $M$ .

**Definition V.7** An *ideal* of  $R$  is an  $R$ -submodule of  $R$ .

**V.8** A more direct way to give Definition V.7 is to say that an ideal is a subgroup of  $(R, +)$  such that  $ra \in I$  and  $ar \in I$  for any  $a \in I$  and  $r \in R$ . Another way to say the same this is to say that  $I$  is a subring of  $R$  with the extra property that  $ar \in I$  if just one of  $a$  and  $r$  belongs to  $I$  (the definition of subring only guarantees this if they both do).

**V.9** To check that a nonempty set  $I \subset R$  is an ideal, it is enough to check that

- (i)  $a - b \in I$  if  $a, b \in I$ , and
- (ii)  $ra \in I$  if  $a \in I$  and  $r \in R$ .

or more simply still, that linear combinations stay in  $I$ : that is,  $ra + sb \in I$  if  $r, s \in R$  and  $a, b \in I$ . This is a short cut similar to Lemma I.14.

**V.10** Ideals play for rings the role that normal subgroups play for groups. Indeed we shall write  $I \triangleleft R$  to mean that  $I$  is an ideal of  $R$  (one also says “ideal in  $R$ ”).

**V.11** If  $R$  is non-commutative there are left and right modules and left and right ideals, and one wants to work mostly with two-sided ideals, i.e. left ideals that are also right ideals.

**Example V.12** There is one motivating example of an ideal, and we do know some others.

- (i) Suppose  $N \in \mathbb{Z}$  and  $N \geq 0$ . Then  $N\mathbb{Z}$ , the set of multiples of  $N$ , is an ideal. This is the example everybody knows, and the historically first one.
- (ii) More generally,  $R$  be a commutative ring and let  $a \in R$ . The set  $aR = \{ar \mid r \in R\}$ , also sometimes denoted  $\langle a \rangle$  if the ring  $R$  is clear from the context, is an ideal in  $R$ . As we are assuming that  $R$  is a commutative ring, we may also write  $Ra$  if that is convenient.
- (iii) Let  $R = K[t]$  for some field  $K$  (say  $K = \mathbb{C}$ ). Then the set of polynomials with  $a_0 = 0$  forms an ideal. It is in fact the ideal  $\langle t \rangle = tR$ . More generally, for any  $d$ , the set of polynomials with  $a_0 = \dots = a_{d-1} = 0$  is the ideal  $\langle t^d \rangle$ .
- (iv)  $\{0\}$  is an ideal of  $R$ , usually called simply  $0$ : it is equal to  $0R$ .

- (v)  $R$  is an ideal of  $R$ : it is equal to  $1R$ .
- (vi) Consider the ring  $\mathbb{C}[x, y]$  of polynomials in two variables. Again the set of polynomials whose constant term is zero forms an ideal, but this is not of the form  $pR$  for any polynomial  $p \in R$ .
- (vii) Again with  $R = \mathbb{C}[x, y]$ , we may regard any  $p \in R$  as a function  $p: \mathbb{C}^2 \rightarrow \mathbb{C}$ . If we take  $X$  to be the union of the  $x$ - and  $y$ -axes in  $\mathbb{C}^2$  and consider

$$I(X) = \{p \in R \mid p(a, b) = 0 \text{ if } (a, b) \in X\}$$

then  $I(X)$  is an ideal of  $R$ .

- (viii) In (vii), if we instead take  $X = \{(0, 0)\}$  then  $I(X)$  is the set of polynomial functions on  $\mathbb{C}^2$  that vanish at the origin, which is the same ideal as in (vi).

**Definition V.13** If  $a \in R$  then  $\langle a \rangle$  (also written  $aR$  or  $Ra$ ) is called the *ideal generated by  $a$* , and every ideal of this form is called a *principal ideal*.

**Definition V.14** Let  $I$  be an ideal in a commutative ring  $R$ . The *quotient ring*  $R/I$  is the set

$$R/I = \{a + I \mid a \in R\}$$

of cosets of  $I$  in  $R$ . Addition and multiplication in the ring  $R/I$  are defined by

$$\begin{aligned} (a + I) + (b + I) &= (a + b) + I \\ (a + I) \cdot (b + I) &= (a \cdot b) + I. \end{aligned}$$

That is, we add and multiply cosets by choosing representatives of them and doing the addition and multiplication in  $R$ , and then taking the coset of the result.

**Example V.15** We already know several examples of quotient rings.

- (i) If  $n \in \mathbb{Z}$  then  $n\mathbb{Z}$  is an ideal in  $\mathbb{Z}$  (Example V.12(i)) and  $\mathbb{Z}/n\mathbb{Z} = \mathbb{Z}/n$ .
- (ii) If  $R = \mathbb{C}[t]$  then the quotient by the ideal  $\langle t^2 \rangle$  from Example V.12(iii) is described as follows. Any polynomial  $f$  can be written in the form  $f = t^2h + at + b$  for unique  $a, b \in \mathbb{C}$  (and  $h \in \mathbb{C}[t]$ ), so  $f + \langle t^2 \rangle = (at + b) + \langle t^2 \rangle$  for some unique  $a, b \in \mathbb{C}$ . Therefore  $\mathbb{C}[t]/\langle t^2 \rangle = \{[at + b] \mid a, b \in \mathbb{C}\}$ , where  $[f]$  denotes  $f + \langle t^2 \rangle$ , and addition and multiplication are given by

$$[at + b] + [ct + d] = [(a + c)t + (b + d)]$$

and

$$[at + b] \cdot [ct + d] = [act^2 + (ad + bc)t + bd] = [(ad + bc)t + bd].$$

respectively. In other words we work with polynomials and then discard all the terms involving  $t^2$ , because that is zero in  $\mathbb{C}[t]/\langle t^2 \rangle$ .

**V.16** V.15(ii) provides a respectable way to say the thing that you said when learning calculus: “ $\varepsilon^2$  is small so we’ll ignore it”. You were actually working in  $\mathbb{R}[\varepsilon]/\langle \varepsilon^2 \rangle$ .



## Kernel and image

**Definition V.17** Let  $\varphi: R \rightarrow S$  be a ring homomorphism. The *kernel* of  $\varphi$  is the subset  $\text{Ker } \varphi$  of  $R$  given by

$$\text{Ker } \varphi = \{a \in R \mid \varphi(a) = 0\}.$$

This is very similar to the definition of null space in vector spaces, or kernel in groups. Remember, though, that the kernel consists of all those elements that are mapped to 0, not 1. This is because the only operation that makes a ring into a group is addition: we are missing multiplicative inverses.

**Definition V.18** Again let  $\varphi: R \rightarrow S$  be a ring homomorphism. The *image* of  $\varphi$  is the subset  $\text{Im } \varphi$  of  $S$  given by

$$\text{Im } \varphi = \{\varphi(a) \in S \mid a \in R\}.$$

**Lemma V.19** Let  $\varphi: R \rightarrow S$  be a ring homomorphism. Then  $\text{Ker } \varphi$  is an ideal of  $R$ . Moreover,  $\varphi$  is injective if and only if  $\text{Ker } \varphi = \{0\}$ .

*Proof:* A ring homomorphism is in particular an additive group homomorphism, so  $\text{Ker } \varphi$  is an additive subgroup by Lemma I.27 and the statement about injectivity follows from Lemma I.28. Also, if  $r \in R$  and  $a \in \text{Ker } \varphi$  then  $\varphi(ra) = \varphi(r)\varphi(a) = \varphi(r) \cdot 0$  and similarly for  $ar$ , so  $\text{Ker } \varphi$  is an ideal.

**Lemma V.20** Let  $\varphi: R \rightarrow S$  be a ring homomorphism. Then  $\text{Im } \varphi$  is a subring of  $S$ . Moreover,  $\varphi$  is surjective if and only if  $\text{Im } \varphi = S$ .

This is trivial, but notice that  $\varphi(1_R) \neq 1_S$  in general.

**Definition V.21** Let  $I$  be an ideal in a ring  $R$ . The *quotient map*  $\pi: R \rightarrow R/I$  is defined by setting  $\pi(a) = a + I$  (exactly as in Definition II.23).

**Lemma V.22** Let  $R$  be a ring, let  $I$  be an ideal in  $R$  and let  $S$  be a subring of  $R$  with  $1 \in S$ . Then

- (i)  $\pi: R \rightarrow R/I$  is a surjective ring homomorphism, so  $\text{Im } \pi = R/I$ , and  $\text{Ker } \pi = I$ ;
- (ii)  $\iota: S \rightarrow R$  is an injective ring homomorphism, so  $\text{Ker } \iota = \{0\}$ , and  $\text{Im } \iota = S$ .

*Proof:*  $\pi$  is a ring homomorphism, because

$$\pi(a + b) = (a + b) + I = (a + I) + (b + I) = \pi(a) + \pi(b),$$

and

$$\pi(ab) = ab + I = (a + I)(b + I) = \pi(a) \cdot \pi(b).$$

It is clearly surjective, and  $\pi(a) = 0 \iff a \in I$ . Therefore  $\text{Im } \pi = R/I$  and  $\text{Ker } \pi = I$ .

For  $\iota$  there is nothing to prove.

**Theorem V.23** Let  $\varphi: R \rightarrow S$  be a ring homomorphism and let  $I$  be an ideal in  $R$  satisfying  $I \subseteq \text{Ker } \varphi$ . Then there exists a unique ring homomorphism  $\bar{\varphi}: R/I \rightarrow S$  such that the diagram

$$\begin{array}{ccc}
 R & \xrightarrow{\varphi} & S \\
 \pi \downarrow & \nearrow \bar{\varphi} & \\
 R/I & & 
 \end{array}$$

commutes, i.e.,  $\bar{\varphi} \circ \pi = \varphi$ .

*Proof:* This follows from II.24 applied to the additive groups, except that we need to check that  $\bar{\varphi}$  is a ring homomorphism, i.e. that  $\bar{\varphi}$  preserves multiplication. But

$$\bar{\varphi}((a + I) \cdot (b + I)) = \bar{\varphi}(ab + I) = \varphi(ab) = \varphi(a) \cdot \varphi(b) = \bar{\varphi}(a + I) \cdot \bar{\varphi}(b + I).$$

**Definition V.24** Let  $R, S$  be rings. A homomorphism  $\varphi: R \rightarrow S$  is called an *isomorphism* if there is a ring homomorphism  $\psi: S \rightarrow R$  such that  $\psi(\varphi(r)) = r$  for all  $r \in R$  and  $\varphi(\psi(s)) = s$  for all  $s \in S$ . Given an isomorphism  $\varphi: R \rightarrow S$ , we say that  $R$  is *isomorphic* to  $S$  and write  $R \cong S$ .

**V.25** As with groups, it is in fact the case that a bijective ring homomorphism is an isomorphism. We are fully entitled to use this fact, but it is not the definition.

**Theorem V.26** Let  $\varphi: R \rightarrow S$  be a ring homomorphism. Then there is a ring isomorphism

$$\bar{\varphi}: (R/\text{Ker } \varphi) \longrightarrow \text{Im } \varphi.$$

*Proof:* Exactly as for Theorem II.25.

**Corollary V.27** Every ring homomorphism can be written as the composition of a surjective ring homomorphism, then an isomorphism, and finally an injective ring homomorphism.

*Proof:* Exactly as for Theorem II.26.

**Corollary V.28** Let  $\varphi: K \rightarrow R$  be a ring homomorphism where  $K$  is a field. Then  $\varphi$  is either the zero map or an isomorphism from  $K$  to a subring of  $R$ .

*Proof:* The kernel of  $\varphi$  is an ideal in  $K$ , and a field can only have two ideals:  $\{0\}$  (when  $\varphi$  is an isomorphism onto its image) and  $K$  (when  $\varphi = 0$ ). For if  $I$  is a nonzero ideal in  $K$  then we have  $0 \neq b \in I$ , so  $1 = b^{-1}b \in I$ , so for any  $a \in K$  we have  $1a \in I$ , so  $I = K$ .

## Characteristic

**Definition V.29** The *characteristic* of a ring  $R$ , denoted  $\text{char } R$ , is the order  $o(1_R)$  of  $1_R$  in the abelian group  $(R, +)$ , if that is finite. If  $o(1_R) = \infty$  we say that  $\text{char } R = 0$ .

**V.30** Given a ring  $R$ , an element  $a \in R$  and  $n \in \mathbb{Z}$  we define an element  $na \in R$  by

$$na = \underbrace{a + \cdots + a}_n \text{ if } n \geq 0, \text{ and } (-n)a = -(na).$$

In particular, zero copies of an element  $a \in R$  is the zero element  $0_R \in R$ , that is  $0a = 0_R$ , where  $0$  is the zero element in  $\mathbb{Z}$ .

This is just notation. It is tempting to using the phrase “multiply  $a$  by  $n$ ” for it, and everybody does, but what it really means is “add together  $n$  copies of  $a$ ”. We are not doing any ring multiplication, only addition: we could replace  $R$  with an abelian group  $A$  here.

Slightly more formally, we could say that we are defining a map  $\mathbb{Z} \times R \rightarrow R$  and calling this multiplication, even though we shouldn’t do because we have already used the word “multiplication” as the name of a map  $\cdot : R \times R \rightarrow R$ .

Notice that  $0_R \cdot a = 0_R$  is a fact that we proved in Lemma IV.6 but  $0a = 0_R$  is just a natural notation when  $0$  is the zero integer.

**Example V.31** It is usually obvious what the characteristic is.

- (i) The zero ring  $R = \{0\}$  has  $\text{char } R = 1$ , because  $1_R = 0_R = 0$ . No other ring has  $\text{char } R = 1$ .
- (ii) For any  $n \in \mathbb{N}$  we have  $\text{char } \mathbb{Z}/n\mathbb{Z} = n$ .
- (iii) The field  $\mathbb{C}$  has characteristic zero, and hence so do  $\mathbb{Z}$ ,  $\mathbb{Q}$  and  $\mathbb{R}$ .

**Lemma V.32** Let  $R$  be a ring of positive characteristic  $n > 0$ . Then  $na = 0$  for all  $a \in R$ .

*Proof:* For  $a \in R$ , we have

$$na = \underbrace{a + \cdots + a}_n = \underbrace{(1_R \cdot a + \cdots + 1_R \cdot a)}_n = \underbrace{(1_R + \cdots + 1_R)}_n \cdot a = 0_R \cdot a = 0_R$$

as required.

**Definition V.33** Let  $R$  be a ring. Then

$$\mathbb{Z}1_R = \{n1_R \mid n \in \mathbb{Z}\} = \{\dots, -2 \cdot 1_R, -1_R, 0_R, 1_R, 2 \cdot 1_R, \dots\}.$$

is (obviously) a subring of  $R$  (with  $1$ ), called the *prime subring* of  $R$ .

**Lemma V.34** Let  $R$  be a ring. Then either:

- (i)  $\text{char } R = 0$ , in which case  $\mathbb{Z}1_R$  is isomorphic to  $\mathbb{Z}$ ; or
- (ii)  $\text{char } R = n > 0$ , in which case  $\mathbb{Z}1_R$  is isomorphic to  $\mathbb{Z}/n\mathbb{Z}$ .

*Proof:* The map  $\varepsilon: \mathbb{Z} \rightarrow R$  given by  $\varepsilon(n) = n1_R$  is a ring homomorphism because

$$\varepsilon(n + m) = (n + m)1_R = n1_R + m1_R = \varepsilon(n) + \varepsilon(m),$$

and the distributive law gives

$$\varepsilon(nm) = nm1_R = n1_R \cdot m1_R = \varepsilon(n) \cdot \varepsilon(m).$$

Moreover, the image of  $\varepsilon$  is clearly  $\mathbb{Z}1_R$ .

Suppose first that  $\text{char } R = 0$ . Then  $\varepsilon(n) = n1_R$ , which equals  $0_R$  if and only if  $n = 0$ , so  $\text{Ker } \varepsilon = \{0\}$ . Applying V.26 to  $\varepsilon$  gives  $\mathbb{Z} \cong \mathbb{Z}1_R$  which proves (i).

Otherwise,  $\text{char}(R) = n > 0$ . Then  $\varepsilon(m) = m1_R$ , which equals  $0_R$  if and only if  $n|m$ , so  $\text{Ker } \varepsilon = n\mathbb{Z}$ . Applying V.26 to  $\varepsilon$  gives  $\mathbb{Z}_n \cong \mathbb{Z}1_R$ , which proves (ii).

**Proposition V.35** The characteristic of an integral domain is either 0 or a prime.

*Proof:* Let  $R$  be an integral domain. Notice first that  $\text{char } R \neq 1$ , because  $R \neq \{0\}$ . So if  $n = \text{char } R$  is neither 0 nor a prime, it must be composite: that is, we can write  $n = rs$  for some  $1 < r \leq s < n$ . Then

$$0_R = n1_R = (rs)1_R = (r1_R) \cdot (s1_R),$$

but since  $R$  is an integral domain it follows that either  $r1_R = 0$  or  $s1_R = 0$ . But then we have found  $k$  with  $0 < k < n$  such that  $k1_R = 0_R$ , which is impossible because  $n$  was supposed to be the least such positive integer.

## The Chinese remainder theorem

**Definition V.36** Let  $I$  and  $J$  be ideals of  $R$ .

- (i) The *sum*  $I + J$  of  $I$  and  $J$  is the subset

$$I + J := \{a + b \in R \mid a \in I, b \in J\}.$$

- (ii) The *product*  $IJ$  of  $I$  and  $J$  is the subset

$$IJ := \left\{ \sum_{i=1}^k a_i b_i \in R \mid k \in \mathbb{N}, a_i \in I, b_i \in J \text{ for all } 1 \leq i \leq k \right\}.$$

- (iii) The *intersection*  $I \cap J$  of  $I$  and  $J$  is the subset

$$I \cap J := \{a \in R \mid a \in I \text{ and } a \in J\}.$$

**Lemma V.37** If  $I$  and  $J$  are ideals of  $R$  then  $I + J$ ,  $IJ$  and  $I \cap J$  are all ideals of  $R$ . Moreover,  $IJ \subseteq I \cap J \subseteq I + J$ .

**V.38**  $IJ$  is the smallest ideal containing the set of products  $ab$  for  $a \in I$  and  $b \in J$ . The set of such products fails to be an ideal itself because it is not closed under addition.

**Example V.39** Take  $R = \mathbb{Z}$  and consider  $m, n \in \mathbb{Z}$ . If we take  $I = m\mathbb{Z} = \langle m \rangle$  and  $J = n\mathbb{Z} = \langle n \rangle$ , then  $IJ = \langle mn \rangle$ ,  $I + J = \langle \text{hcf}(m, n) \rangle$  and  $I \cap J = \langle \text{lcm}(m, n) \rangle$ .

Notice that  $mn \geq \text{lcm}(m, n) \geq \text{hcf}(m, n)$ : compare this with V.37.

**Definition V.40** Let  $R$  and  $S$  be rings. The *direct product* of  $R$  and  $S$  is the ring

$$R \times S = \{(r, s) \mid r \in R, s \in S\},$$

where the operations are  $(a, b) + (c, d) = (a + c, b + d)$  and  $(a, b) \cdot (c, d) = (ac, bd)$ .

**Theorem V.41 (Chinese Remainder Theorem)** Let  $I, J$  be ideals in a ring  $R$  satisfying  $I + J = R$ . Then there is a ring isomorphism

$$\frac{R}{I \cap J} \cong \frac{R}{I} \times \frac{R}{J}.$$

*Proof:* Consider the map  $\varphi: R \rightarrow R/I \times R/J$  defined by setting  $\varphi(a) = (a + I, a + J)$ . This is a ring homomorphism because

$$\begin{aligned} \varphi(a + b) &= (a + b + I, a + b + J) \\ &= ((a + I) + (b + I), (a + J) + (b + J)) && \text{by V.14} \\ &= (a + I, a + J) + (b + I, b + J) && \text{by V.40} \\ &= \varphi(a) + \varphi(b) \end{aligned}$$

and

$$\begin{aligned} \varphi(a \cdot b) &= (a \cdot b + I, a \cdot b + J) \\ &= ((a + I) \cdot (b + I), (a + J) \cdot (b + J)) && \text{by V.14} \\ &= (a + I, a + J) \cdot (b + I, b + J) && \text{by V.40} \\ &= \varphi(a) \cdot \varphi(b). \end{aligned}$$

We now compute the kernel of  $\varphi$ . For this, notice that

$$a \in \text{Ker } \varphi \iff (a + I, a + J) = (0 + I, 0 + J) \iff a \in I \cap J,$$

so  $\text{Ker } \varphi = I \cap J$ . The first isomorphism theorem, V.26, applied to  $\varphi$  gives an isomorphism  $\bar{\varphi}: R/(I \cap J) \rightarrow \text{Im } \varphi$ .

So it remains to show that  $\varphi$  is surjective, i.e. that  $\text{Im } \varphi = R/I \times R/J$ , so we choose an arbitrary  $(a + I, b + J) \in R/I \times R/J$  and we need to show that this

is in the image of  $\varphi$ . Since  $R = I + J$ , there exist  $x \in I$  and  $y \in J$  such that  $1 = x + y$ : we set  $r = ay + bx \in R$ . Then

$$\begin{aligned}\varphi(r) &= (ay + bx + I, ay + bx + J) \\ &= (ay + I, bx + J) && \text{as } bx \in I \text{ and } ay \in J \\ &= (a(1-x) + I, b(1-y) + J) && \text{as } 1 = x + y \\ &= (a - ax + I, b - by + J) \\ &= (a + I, b + J) && \text{as } x \in I \text{ and } y \in J,\end{aligned}$$

as required.

**Corollary V.42** Let  $m, n \in \mathbb{N}$  be coprime natural numbers: that is, there exist  $\lambda, \mu \in \mathbb{Z}$  such that  $\lambda m + \mu n = 1$ . Then  $\mathbb{Z}/mn\mathbb{Z} \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ .

*Proof:* In this case we have  $\mathbb{Z} = m\mathbb{Z} + n\mathbb{Z}$  (because the right-hand side is an ideal that contains 1). Now V.41 gives the isomorphism: this is the Chinese Remainder Theorem for the integers.

## Prime ideals and maximal ideals

**Definition V.43** Suppose that  $R$  is a commutative ring. A proper ideal  $I$  in  $R$  is called a *prime ideal* if, whenever  $ab \in I$ , then  $a \in I$  or  $b \in I$ .

**V.44** This is very similar to the condition in Definition IV.12. It is also the condition satisfied by prime numbers in  $\mathbb{Z}$ : an integer  $p > 1$  is prime if and only if  $ab \in p\mathbb{Z}$  implies  $a \in p\mathbb{Z}$  or  $b \in p\mathbb{Z}$ . Thus  $p\mathbb{Z}$  is a prime ideal of  $\mathbb{Z}$ . Note that in Definition V.43 we do not allow  $I = R$  (that is, we insist that  $I$  should be a proper ideal) but we do allow  $I = 0$ .

**Theorem V.45** Let  $R$  be a commutative ring and  $I$  an ideal of  $R$ . Then  $I$  is a prime ideal if and only if  $R/I$  is an integral domain.

*Proof:* If  $I$  is a prime ideal and  $(a+I)(b+I) = 0_{R/I}$  then  $ab+I = I$  so  $ab \in I$  so  $a \in I$  or  $b \in I$ , meaning that  $a+I$  or  $b+I$  is  $0_{R/I}$ . Therefore  $R/I$  is an integral domain.

Conversely, if  $R/I$  is an integral domain and  $ab \in I$ , then  $(a+I)(b+I) = 0_{R/I}$  so  $a+I = 0_{R/I}$  or  $b+I = 0_{R/I}$ , i.e.  $a \in I$  or  $b \in I$ .

**Definition V.46** Let  $R$  be a commutative ring. A proper ideal  $I$  is called a *maximal ideal* if every ideal  $J \supseteq I$  is either  $I$  or  $R$ .

**Theorem V.47** Let  $R$  be a commutative ring and  $I$  an ideal of  $R$ . Then  $I$  is a maximal ideal if and only if  $R/I$  is a field.

*Proof:* Suppose that  $I$  is maximal: we shall show that  $a+I$  has an inverse unless  $a \in I$ . Consider the ideal  $J = \langle a, I \rangle$ : that is, the smallest ideal of  $R$  that contains both  $a$  and all of  $I$ . Note that  $J = \{ra + b \mid r \in R, b \in I\}$ . Since  $I$  is

maximal,  $J$  must be equal to either  $R$  or  $I$ . If  $J = I$  then  $a \in I$ . If  $J = R$  then  $1_R \in J$  so  $1_R = ra + b$  so  $1 + I = ra + I = (r + I)(a + I)$  so  $r + I$  is the inverse of  $a + I$ .

Conversely, suppose that  $R/I$  is a field and that  $J \supseteq I$  is an ideal containing  $I$ . If  $J \neq I$  choose  $a \in J \setminus I$  and consider  $a + I \in R/I$ . Since  $a + I \neq 0_{R/I}$  it has an inverse  $r + I$ , say; but then  $1_R \in (r + I)(a + I) = ra + I$  and  $ra + I \subseteq J$  so  $1_R \in J$  so  $J = R$ .

**Corollary V.48** If  $I$  is a maximal ideal then  $I$  is a prime ideal.

## Field of fractions

**V.49** The basic integral domain is  $\mathbb{Z}$  and we can think of  $\mathbb{Z}$  as determining  $\mathbb{Q}$ . We want to do something similar starting with an arbitrary integral domain.

Consider the set  $T = R \times R \setminus \{0\}$ : a typical member of this is  $(a, b)$  with  $b \neq 0$ , but it is preparing to be  $\frac{a}{b}$  really. With this in mind we define two binary operations  $T \times T \rightarrow T$  given by

$$(a, b) + (c, d) := (ad + bc, bd) \quad \text{and} \quad (a, b) \cdot (c, d) := (ac, bd).$$

These operations are well defined – that is, the formulae do define a map from  $T \times T$  to  $T$  – precisely because  $R$  is an integral domain. Indeed, suppose otherwise, i.e., suppose that  $bd = 0$ . The fact that  $R$  is an integral domain forces either  $b = 0$  or  $d = 0$ , but then either  $(a, b) \notin T$  or  $(c, d) \notin T$  which is absurd.

Notice that  $T$  with these operations is *not* a ring: for instance, if  $b \in R$  is not a unit then  $(1, b)$  does not have an additive inverse. This is because we are treating fractions as numerator/denominator pairs, without cancelling: we still think that  $\frac{1}{2}$  and  $\frac{2}{4}$  are different.

**Lemma V.50** Define a relation  $\sim$  on  $T$  by setting  $(a, b) \sim (c, d) \iff ad = bc$ . Then for all  $a, a', b, b', c, c', d, d' \in R$  with  $b, b', d, d' \neq 0$ , we have

$$(a, b) \sim (a', b') \text{ and } (c, d) \sim (c', d') \Rightarrow \begin{cases} (a, b) + (c, d) \sim (a', b') + (c', d') \\ (a, b) \cdot (c, d) \sim (a', b') \cdot (c', d') \end{cases}$$

*Proof:* Notice that

$$\begin{aligned} (ad + bc)b'd' &= ab'dd' + bb'cd' \\ &= a'bdd' + bb'c'd \\ &= (a'd' + b'c')bd. \end{aligned}$$

(The second line follows from the first by using the conditions  $(a, b) \sim (a', b')$ , i.e.  $ab' = a'b$ , and the same for  $c$  and  $d$ .) But this says that  $(ad + bc, bd) \sim (a'd' + b'c', b'd')$  and those are  $(a, b) + (c, d)$  and  $(a', b') + (c', d')$ , so we have proved the part about addition.

Similarly for multiplication we notice that  $ab'cd' = a'bcd' = a'bc'd$ , and that says that  $(ac, bd) \sim (a'c', b'd')$ , i.e., that  $(a, b) \cdot (c, d) = (a', b') \cdot (c', d')$  as required.

**Definition V.51** We define  $\mathcal{Q}(R) = T/\sim$ , and we give the name  $a/b$  (or  $\frac{a}{b}$ ) to the equivalence class  $[(a, b)]$ .

$a/b$  is not quite the same thing as  $ab^{-1}$ , even if  $b^{-1}$  happens to exist, but it almost is.

**Theorem V.52** Let  $R$  be an integral domain. The operations  $+$  and  $\cdot$  on  $T$  induce operations (also called  $+$  and  $\cdot$ ) on  $\mathcal{Q}(R) = T/\sim$ . These rules make  $\mathcal{Q}(R)$  into a field, called the *field of fractions* of  $R$ , and the map  $R \rightarrow \mathcal{Q}(R)$  defined by  $a \mapsto \frac{a}{1}$  is an injective ring homomorphism.

*Proof:* It is easy to check that the operations  $+$  and  $\cdot$  are well defined. However, we are not taking the quotient of a ring by an ideal ( $T$  is not even a ring) so we have to check the ring axioms in  $\mathcal{Q}(T)$  by hand.

Our convention that the classes are called  $\frac{a}{b}$  does mean that addition and multiplication, as we have just defined them, are expressed in the usual way as addition and multiplication of fractions:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \quad \text{and} \quad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

To check that  $(\mathcal{Q}(R), +)$  is an abelian group, take  $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in \mathcal{Q}(R)$ : then

$$\left(\frac{a}{b} + \frac{c}{d}\right) + \frac{e}{f} = \frac{ad + bc}{bd} + \frac{e}{f} = \frac{adf + bcf + bde}{bdf} = \frac{a}{b} + \frac{cf + de}{df} = \frac{a}{b} + \left(\frac{c}{d} + \frac{e}{f}\right)$$

so addition is associative. Addition is commutative in  $\mathcal{Q}(R)$  because multiplication in the integral domain  $R$  is commutative (and addition is commutative) and hence

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} = \frac{cb + da}{db} = \frac{c}{d} + \frac{a}{b}$$

The zero element is  $\frac{0}{1}$  because

$$\frac{a}{b} + \frac{0}{1} = \frac{a \cdot 1 + b \cdot 0}{b \cdot 1} = \frac{a}{b} = \frac{0 \cdot b + 1 \cdot a}{1 \cdot b} = \frac{0}{1} + \frac{a}{b},$$

and the additive inverse of  $\frac{a}{b}$  is  $\frac{-a}{b}$  because  $0 \cdot 1 = 0 = b^2 \cdot 0$  and hence in  $\mathcal{Q}(R)$  we have

$$\frac{a}{b} + \frac{-a}{b} = \frac{ab + (-a)b}{b^2} = \frac{0}{b^2} = \frac{0}{1} = \frac{-ab + ab}{b^2} = \frac{-a}{b} + \frac{a}{b}.$$

Next multiplication is associative because multiplication in  $R$  is associative, so

$$\frac{a}{b} \cdot \left(\frac{c}{d} \cdot \frac{e}{f}\right) = \frac{a}{b} \cdot \frac{ce}{df} = \frac{a(ce)}{b(df)} = \frac{(ac)e}{(bd)f} = \frac{ac}{bd} \cdot \frac{e}{f} = \left(\frac{a}{b} \cdot \frac{c}{d}\right) \cdot \frac{e}{f}.$$

For the distributive laws,  $b^2df(acf + ade) = bdf(abcf + abde)$ , so in  $\mathcal{Q}(R)$  we have

$$\begin{aligned} \frac{a}{b} \cdot \left(\frac{c}{d} + \frac{e}{f}\right) &= \frac{a}{b} \cdot \frac{cf + de}{df} = \frac{a(cf + de)}{bdf} = \frac{acf + ade}{bdf} \\ &= \frac{abcf + abde}{b^2df} = \frac{ac}{bd} + \frac{ae}{bf} = \frac{a}{b} \cdot \frac{c}{d} + \frac{a}{b} \cdot \frac{e}{f}. \end{aligned}$$



The other distributive law is similar, and the multiplicative identity is  $\frac{1}{1}$ . This proves that  $\mathcal{Q}(R)$  with the given operations is a ring.

To check that  $\mathcal{Q}(R)$  is a field, we need to check that it is commutative and not  $\{0\}$  and nontrivial and that every nonzero element is a unit. Commutativity is easy:  $\frac{a}{b} \frac{c}{d} = \frac{ac}{bd} = \frac{ca}{db} = \frac{c}{d} \frac{a}{b}$ . It is not  $\{0\}$  because  $\frac{0}{1} \neq \frac{1}{1}$  (otherwise  $0 = 1$  in  $R$  which is excluded).

It remains to show that every nonzero element  $\frac{a}{b}$  has a multiplicative inverse. But

$$\frac{a}{b} \cdot \frac{b}{a} = \frac{ab}{ba} = \frac{1}{1}$$

so  $\mathcal{Q}(F)$  is a field.

Finally,  $a \mapsto \frac{a}{1}$  is a homomorphism because  $\frac{a}{1} + \frac{a'}{1} = \frac{a+1a'}{1 \cdot 1} = \frac{a+a'}{1}$  and  $\frac{a}{1} \frac{a'}{1} = \frac{aa'}{1}$ , and it is injective because  $a \in R$  is in its kernel if and only if  $\frac{a}{1} = \frac{0}{1}$ , which immediately gives  $a = 0$ .

**Example V.53** Apart from  $\mathcal{Q}(\mathbb{Z}) = \mathbb{Q}$ , the most familiar example of this is  $\mathcal{Q}(K[t])$  for  $K$  a field, which is the field  $K(t)$  of rational functions.

## VI Factorisation in integral domains

Unique factorisation of integers into primes is a very useful feature of  $\mathbb{Z}$ . We formulate an analogous statement for integral domains, and establish that it holds in some cases (but not all).

### Primes and irreducibles

**VI.1** Integral domains are very common and are significantly better behaved than general rings. For this whole section,  $R$  denotes an integral domain: in particular,  $0_R \neq 1_R$ .

**Example VI.2** We've already seen many examples of integral domains:

- (i) any field  $\mathbb{F}$  is an integral domain (see IV.15);
- (ii) the ring  $\mathbb{Z}$  and the ring of Gaussian integers  $\mathbb{Z}[i]$  are both integral domains by Lemma IV.23 because they are subrings of  $\mathbb{C}$  and contain 1.
- (iii) the ring  $R[t]$  associated to an integral domain  $R$  is an integral domain.

**Lemma VI.3** Let  $R$  be a commutative ring such that  $0 \neq 1$ . Then  $R$  is an integral domain if and only if it satisfies the *cancellation property*:

$$\text{if } a, b, c \in R \text{ and } a \neq 0, \text{ then } ab = ac \Rightarrow b = c.$$

*Proof:* First, let  $R$  be an integral domain, and suppose  $ab = ac$  and  $a \neq 0$ . Then  $0 = ab + (-ac) = ab + a(-c) = a(b + (-c))$ . Since  $R$  is an integral domain and  $a \neq 0$ , we have  $b + (-c) = 0$ , that is  $b = c$ .

In the other direction, let  $R$  be a commutative ring such that  $0 \neq 1$ , and assume the cancellation property. Suppose  $ab = 0$  and  $a \neq 0$ . Then  $ab = 0 = a \cdot 0_R$ , and since  $a \neq 0$  the cancellation property gives  $b = 0$ , which shows that  $R$  is an integral domain.

**Definition VI.4** Let  $a, b \in R$ . We say that  $a$  *divides*  $b$  (or that  $b$  is *divisible by*  $a$ ), and write  $a|b$ , if there exists  $c \in R$  such that  $b = ac$ .

The  $|$  in  $a|b$  is a verb, and  $a|b$  is a sentence which may be true or false, but is not, ever, the name of an element of  $R$ .

**Lemma VI.5** Suppose  $a, b \in R$ . The following are equivalent:

- (i)  $a|b$
- (ii)  $b \in aR$
- (iii)  $bR \subseteq aR$ .

*Proof:* (i) $\Rightarrow$ (ii): If  $a|b$  then there exists  $c \in R$  such that  $b = ca = ac \in aR$ .  
(ii) $\Rightarrow$ (iii)  $aR$  is an ideal so if  $b \in aR$  then  $br \in aR$  for all  $r \in R$ , so  $bR \subseteq aR$ .  
(iii) $\Rightarrow$ (i) If  $bR \subseteq aR$  then  $b = b1_R \in aR$  so  $b = ac$  for some  $c \in R$ , so  $a|b$ .

**VI.6** Lemma VI.5 says that “dividing” means “generating a bigger ideal”: for example, 6 divides 12 and every multiple of 12 is also a multiple of 6

**Lemma VI.7** Let  $R$  be an integral domain and let  $a, b \in R$ . Then  $aR = bR$  if and only if  $a = ub$  for some unit  $u \in R^*$ . In particular,  $uR = R$  if and only if  $u \in R^*$ .

*Proof:* If  $aR = bR$  then  $aR \subseteq bR$  and  $bR \subseteq aR$ , so by Lemma VI.5  $b|a$  and  $a|b$ . Thus there exist  $u, v \in R$  such that  $a = ub$  and  $b = va$ , so  $a = uva$ . If  $a = 0$ , then  $b = 0$  and there is nothing to prove. Otherwise, the cancellation law gives  $uv = 1$ , so  $u$  is a unit in  $R$ .

Conversely, if  $a = ub$  for some unit  $u \in R^*$  then  $a \in bR$ , so  $aR \subseteq bR$ . Since  $u$  is a unit, we may multiply  $a = ub$  by  $u^{-1}$  to obtain  $b = u^{-1}a = au^{-1}$ : this gives  $b \in aR$  and hence  $bR \subseteq aR$ . Hence  $aR = bR$  as required. The final statement is just the case  $a = 1$ .

**Definition VI.8** Let  $R$  be an integral domain. Let  $p \in R$  be nonzero and not a unit. Then we say

- (i)  $p$  is a *prime* if  $p|ab \Rightarrow p|a$  or  $p|b$ .
- (ii)  $p$  is *irreducible* if  $p = ab \Rightarrow a \in R^*$  or  $b \in R^*$ .

We say that  $p$  is *reducible* if it is not irreducible, i.e., if there exist  $a, b \in R$  such that  $p = ab$  where neither  $a$  nor  $b$  is a unit: this is what is usually called “composite” in the case  $R = \mathbb{Z}$ .

**Example VI.9** (i) The prime elements in  $\mathbb{Z}$  are

$$\{\dots, -11, -7, -5, -3, -2, 2, 3, 5, 7, 11, \dots\},$$

i.e.,  $\pm 1$  (a unit!) times the (positive) prime numbers. The irreducible elements are the same ones.

- (ii) If  $R = \mathbb{F}$  is a field then every nonzero element is a unit, so  $\mathbb{F}$  contains neither primes nor irreducibles.
- (iii)  $t^2 + 1 \in \mathbb{R}[t]$  is irreducible (and, in fact, prime), but  $t^2 + 1 \in \mathbb{C}[t]$  is reducible because  $t^2 + 1 = (t + i)(t - i)$ .
- (iv) If  $R$  is an integral domain and  $p \in R$  is nonzero then  $p$  is a prime if and only if  $pR$  is a prime ideal. (But  $0R = \{0\}$  is a prime ideal, even though we have defined 0 not to be a prime: this just turns out to be convenient most of the time.)

**Proposition VI.10** Let  $R$  be an integral domain. Then every prime element is irreducible.

*Proof:* Let  $p \in R$  be prime, and suppose  $p = ab$ . Then either  $p|a$  or  $p|b$ , so without loss of generality  $p|a$ , say  $a = pc$ . Then  $p = p \cdot 1_R = ab = pcb$ , and the cancellation property gives  $cb = 1$ , so  $b$  is a unit. This shows that  $p$  is irreducible.

The converse is not true in general.

## Euclidean domains and PIDs

**Definition VI.11** Let  $R$  be an integral domain. A *Euclidean valuation* on  $R$  is a map  $\nu: R \setminus \{0\} \rightarrow \mathbb{Z}_{\geq 0}$  such that:

- (i) if  $f, g \in R \setminus \{0\}$  then  $\nu(f) \leq \nu(fg)$ ; and
- (ii) for all  $f, g \in R$  with  $g \neq 0$ , there exists  $q, r \in R$  such that  $f = qg + r$  and either  $r = 0$  or  $r \neq 0$  and  $\nu(r) < \nu(g)$ .

A *valuation* is a function satisfying (i) but not necessarily (ii).

**Definition VI.12** We say that  $R$  is a *Euclidean domain* if it has a Euclidean valuation.

**Example VI.13** (i) Any field is trivially a Euclidean domain because we may take  $\nu(a) = 1$  for all  $a \in \mathbb{F}$ .

- (ii) The absolute value  $\nu(n) = |n|$  provides a Euclidean valuation on  $\mathbb{Z}$ , so  $\mathbb{Z}$  is a Euclidean domain.

- (iii) For  $\mathbb{F}$  a field, the degree of a polynomial  $\nu(f(t)) = \deg f(t)$  provides a Euclidean valuation on  $\mathbb{F}[t]$ , so  $\mathbb{F}[t]$  is a Euclidean domain. This is the reason for the choice of the letters  $f$ ,  $g$ ,  $q$  and  $r$  in Definition VI.11.
- (iv)  $\mathbb{Z}[i]$  is a Euclidean domain.

**Definition VI.14** Recall from Definition V.13 that an ideal  $I$  of a commutative ring  $R$  is a *principal ideal* if  $I = aR$  (also denoted  $\langle a \rangle$  if  $R$  is understood) for some  $a \in R$ . An integral domain  $R$  is called a *principal ideal domain* or *PID* if and only if every ideal in  $R$  is principal.

**Lemma VI.15** Let  $R$  be a nonzero commutative ring. Then  $R$  is a field if and only if the only ideals of  $R$  are  $\{0_R\} = \langle 0_R \rangle$  and  $R = \langle 1_R \rangle$ .

*Proof:* If  $R$  is a field and  $I$  is a nonzero ideal then choose any  $0 \neq u \in I$ . Since  $R$  is a field,  $u$  is a unit so  $uR = R$  by VI.7. But  $uR \subseteq I \subseteq R$ , so  $I = R$ .

Conversely, if  $R$  is not a field then let  $a \neq 0$  be any nonzero non-unit. Then  $aR \neq R$  by VI.7 and  $aR \neq \{0\}$  either.

**Theorem VI.16** Let  $R$  be a Euclidean domain. Then  $R$  is a PID.

*Proof:* Denote the Euclidean valuation on  $R$  by  $\nu$  and suppose  $I$  is a nonzero ideal in  $R$ .

Consider the image  $\nu(I \setminus \{0\})$ , i.e.  $\{\nu(a) \in \mathbb{Z}_{\geq 0} \mid a \in I, a \neq 0\}$ . This is a nonempty subset of  $\mathbb{Z}_{\geq 0}$ , so it has a least element  $\sigma$ .

Choose  $g \in I$  such that  $\nu(g) = \sigma$ : in other words, we choose  $0 \neq g \in I$  with  $\nu(g)$  as small as possible, so  $\nu(f) \geq \nu(g)$  for all  $0 \neq f \in I$ . If we take any  $f \in I$ , then since  $R$  is a Euclidean domain there exist  $q, r \in R$  such that  $f = qg + r$ , and  $r = 0$  or  $\nu(r) < \nu(g) = \sigma$ . But if  $r \neq 0$  then  $r = f - qg \in I$ , so  $\nu(r) \geq \sigma$ . This is a contradiction, so we must have  $r = 0$ , but then  $f = qg \in gR$ . Since  $f$  was arbitrary, that means  $I \subseteq gR$ ; but  $g \in I$  so we also have  $gR \subseteq I$ . Hence  $I = gR$  and so  $I$  is a principal ideal.

**Example VI.17** Theorem VI.16 implies that the following rings are PIDs:

- (i) any field (this also follows from VI.15);
- (ii) the ring of integers  $\mathbb{Z}$ ;
- (iii) the polynomial ring  $\mathbb{F}[t]$  with coefficients in a field  $\mathbb{F}$ ;
- (iv) the ring of Gaussian integers  $\mathbb{Z}[i]$ .

**Example VI.18** The integral domain  $R = \mathbb{Z}[t]$  is not a PID, so it is also not a Euclidean domain. It is slightly harder to produce a PID that is not a Euclidean domain. One example is  $\mathbb{Z}[\theta]$  for  $\theta = \frac{1}{2} + i\frac{\sqrt{19}}{2}$ . We shall not prove this, but for reasons of space, not difficulty.

**VI.19** The fact that  $\mathbb{F}[t]$  is a PID has an important practical consequence: if we can associate to some object an ideal in  $\mathbb{F}[t]$  then that object has a *minimal polynomial*, namely the unique monic generator of the corresponding ideal.

**Example VI.20** There are two especially important examples of minimal polynomials:

- (i) Suppose that  $A$  is an  $n \times n$  matrix with entries in  $\mathbb{F}$ . Then one can evaluate  $f \in \mathbb{F}[t]$  at  $A$ , obtaining a map  $\text{ev}_A: \mathbb{F}[t] \rightarrow M_{n \times n}(\mathbb{F})$  given by

$$\text{ev}_A: f = \sum_{i=0}^d a_i t^i \mapsto \sum_{i=0}^d a_i A^i.$$

It is easy to check that this is a ring homomorphism (for fixed  $A$ ), so we say that the unique monic generator of  $\text{Ker } \text{ev}_A$  is the minimal polynomial of  $A$ . One can do the same thing starting with  $\alpha \in \text{End}(V)$  where  $V$  is a fixed finite-dimensional  $\mathbb{F}$ -vector space.

- (ii) Suppose that  $\xi \in \mathbb{C}$ , and consider  $\text{ev}_\xi: \mathbb{Q}[t] \rightarrow \mathbb{C}$ : note that we have restricted to  $\mathbb{Q}[t]$ , so this is not quite the same as the map from Example IV.25(iv), but it is still a ring homomorphism. If  $\text{Ker } \text{ev}_\xi = 0$  then  $\xi$  is said to be *transcendental*; otherwise it is *algebraic*. In fact the set of all algebraic numbers forms a field called  $\overline{\mathbb{Q}}$ , the algebraic closure of the rationals. If  $\xi \in \overline{\mathbb{Q}}$  then we say that the unique monic generator of  $\text{Ker } \text{ev}_\xi$  is the minimal polynomial of  $\xi$ . For example, the minimal polynomial of  $i$  is  $t^2 + 1$ .

## Properties of PIDs

**Definition VI.21** Let  $R$  be a PID. Two elements  $a, b \in R$  are said to be *coprime* if every common factor is a unit; by this, we mean that if  $d|a$  and  $d|b$ , then  $d$  is a unit.

**Lemma VI.22** Let  $R$  be a PID and let  $a, b \in R$  be coprime. There exist  $r, s \in R$  such that  $1 = ra + sb$ .

*Proof:* Consider the ideal  $aR + bR$ . Since  $R$  is a PID, there exists  $d \in R$  such that  $aR + bR = dR$ : then  $aR \subset dR$  so  $d|a$  by VI.5 and similarly  $d|b$ . By hypothesis, then  $d$  is a unit, so  $dR = R$  by VI.7: in particular,  $1 \in dR = aR + bR$ , which is to say that there exist  $r, s \in R$  such that  $1 = ra + sb$ .

**Proposition VI.23** Let  $R$  be a PID. Then every irreducible element in  $R$  is prime.

*Proof:* Suppose that  $p$  is irreducible and that  $p|ab$ , and that  $p$  does not divide  $a$ . We need to show that  $p|b$ .

We claim first that  $a$  and  $p$  are coprime. To see this, let  $d$  be a common factor of  $a$  and  $p$ : say  $p = cd$  and  $a = ed$ . Since  $p$  is irreducible, we know that either  $c$  or  $d$  is a unit. But if  $c$  is a unit, then  $a = ed = ec^{-1}cd = ec^{-1}p$  so  $p|a$  which is contrary to the assumptions. Therefore  $d$  is a unit, so  $a$  and  $p$  are coprime.

Now by VI.22 there exist  $r, s \in R$  such that  $1 = ra + sp$ . But now  $b = 1 \cdot b = (ra + sp) \cdot b = rab + psb$ , and we assumed that  $ab$  is divisible by  $p$ : so  $b$  is divisible by  $p$ , as required.

**Theorem VI.24** Let  $R$  be a PID. If  $p$  is irreducible then  $R/pR$  is a field.

*Proof:* The ring  $R/pR$  is commutative because  $R$  is commutative, and it is nonzero because  $pR \neq R$  by VI.7. It remains to show that every nonzero element of  $R/pR$  is a unit.

Choose  $a \in R$  and consider the ideal  $pR + aR = dR$  for some  $d$  (since  $R$  is a PID). This tells us that  $pR \subseteq dR$ , that is  $d|p$ , so  $p = de$  for some  $e$ , and also that  $d|a$ . Since  $p$  is irreducible, either  $d$  or  $e$  must be a unit.

If  $e$  is a unit, then  $a \in dR = deR = pR$  so  $p|a$  and in that case  $a + pR \in R/pR$  is zero.

If  $e$  is not a unit, then  $d$  is a unit; but  $d \in pR + aR$  so  $d = rp + sa$  for some  $r$  and  $s$ . So then  $1 = d^{-1}rp + d^{-1}sa$ , which means that  $d^{-1}s + pR$  is an inverse of  $a + pR$  in  $R/pR$ .

## Unique factorisation domains

**Definition VI.25** An integral domain  $R$  is called a *unique factorisation domain* or *UFD* if

- (i) every nonzero nonunit element in  $R$  can be written as the product of finitely many irreducibles in  $R$ ; and
- (ii) given two such decompositions, say  $r_1 \cdots r_s = r'_1 \cdots r'_t$  we have that  $s = t$  and, after renumbering if necessary, we have  $r_i R = r'_i R$  for  $1 \leq i \leq s$ .

**VI.26** By VI.7, the condition VI.25(ii) means that the factorisation into irreducibles is unique up to reordering and multiplying the factors by units.

**Example VI.27** It is not usually easy to tell whether a given ring is a UFD. Some well-known examples are

- (i)  $\mathbb{Z}$  is a UFD;
- (ii)  $\mathbb{R}[t]$  is a UFD;
- (iii)  $\mathbb{Z}[t]$  is a UFD;
- (iv)  $\mathbb{Z}[i]$  is a UFD;
- (v)  $\mathbb{Z}[\sqrt{-5}]$  is not a UFD – in fact  $6 = 2 \cdot 3 = (1 + \sqrt{-5}) \cdot (1 - \sqrt{-5})$  is an example of two different factorisations of the same element into irreducibles.

**Proposition VI.28** Let  $R$  be a UFD. Then  $p \in R$  is irreducible if and only if it is prime.

*Proof:* Every prime is irreducible by Proposition VI.10, since  $R$  is an integral domain.

Conversely, let  $p \in R$  be irreducible, and suppose that  $p|ab$ : say  $ab = cp$  for some  $c \in R$ . We want to show that  $p|a$  or  $p|b$ . We may as well assume that neither  $a$  nor  $b$  is a unit, since if  $a$  is a unit then  $b = a^{-1}cp$  and so  $p|b$ . Also  $p|0$  so we may assume that  $a$  and  $b$  are nonzero.

Using Definition VI.25(i), we may take irreducible factorisations of  $a$  as  $a = p_1 \cdots p_k$ , of  $b$  as  $b = q_1 \cdots q_l$  and of  $c$  as  $c = r_1 \cdots r_m$ . (Note that these irreducibles are not necessarily distinct!) Now we have two factorisations of  $ab$ :

$$p_1 \cdots p_k \cdot q_1 \cdots q_l = ab = cp = r_1 \cdots r_m \cdot p.$$

According to Definition VI.25(ii) and Lemma VI.7, there must be a unit  $u \in R$  such that either  $p_i = up$  for some  $1 \leq i \leq k$  or  $q_j = up$  for some  $1 \leq j \leq l$ . In the first case  $p|a$ , and in the second case  $p|b$ .

**Theorem VI.29** Let  $R$  be a PID. Then  $R$  is a UFD.

*Proof:* We need to check Definition VI.25(i) and Definition VI.25(ii). Both parts are nontrivial. We begin with VI.25(i), which says that factorisations exist.

Let  $a \in R$  be a nonzero, nonunit element. Let us say, temporarily, that an element of  $R$  is *factorisable* if it can be written as the product of finitely many irreducibles, and suppose that  $a$  is unfactorisable. Then it is certainly reducible, so we can write  $a = a_0 = a_1 b_1$ , with  $a_1$  and  $b_1$  non-units: moreover, at least one of  $a_1$  and  $b_1$  must be unfactorisable as well. Without loss of generality we may assume that  $a_1$  is unfactorisable, and we continue in this way: each  $a_{j+1}$  is chosen to be an unfactorisable factor of  $a_j$ , and  $a_j = a_{j+1} b_{j+1}$  with  $b_{j+1}$  not a unit.

So we have  $\dots a_n | a_{n-1} | a_{n-2} | \dots | a_1 | a = a_0$ , or, using Lemma VI.5

$$a_0 R \subseteq a_1 R \subseteq \dots \subseteq a_{n-2} R \subseteq a_{n-1} R \subseteq a_n R \subseteq \dots$$

Next, we take  $I = \bigcup_{j=0}^{\infty} a_j R$ . We claim that  $I$  is an ideal in  $R$ : it obviously contains  $a_0$  so it is not empty. Suppose that  $c, d \in I$  and  $r \in R$ : it is enough to show that  $rc \in I$  and  $c - d \in I$ , by V.9. But since  $c \in I$  we know that  $c \in a_n R$  for some  $n$ , and similarly  $d \in a_m R$  for some  $m$ . Without loss of generality we assume  $m \geq n$ . Then  $a_n R \subseteq a_m R$ , so  $c \in a_m R$  as well as  $d \in a_m R$ . But  $a_m R$  is an ideal, so  $rc \in a_m R \subseteq I$  and  $c - d \in a_m R \subseteq I$ , so  $I$  is an ideal.

Because  $I$  is an ideal and  $R$  is a PID, we have  $I = eR$  for some  $e \in R$ . But then  $e \in I$ , so  $e \in a_i R$  for some  $i$ , and then

$$a_{i+1} R \subseteq I = eR \subseteq a_i R \subseteq a_{i+1} R$$

so  $a_{i+1} R = a_i R$ . According to Lemma VI.7 that implies that  $a_i = u a_{i+1}$  for some unit  $u$ , but we also know that  $a_i = b_{i+1} a_{i+1}$ . By the cancellation property Lemma VI.3, that implies  $b_{i+1} = u$ ; but  $b_{i+1}$  was chosen to be not a unit.

This is a contradiction, so no unfactorisable elements exist, and we have checked Definition VI.25(i).

Now we check Definition VI.25(ii). Suppose that there do exist elements of  $R$  violating this condition, so that we can find equations

$$vp_1 \cdots p_s = q_1 \cdots q_t$$

where the  $p_i$  and the  $q_j$  are irreducibles,  $v$  is a unit and the left-hand side is not just a rearrangement of the right-hand side, and we assume without loss of generality that  $0 < s \leq t$ . Among all such equations, we choose a shortest one: one for which  $t$  is as small as possible.

We have  $p_s | q_1 \cdots q_t$ . By Proposition VI.28,  $p_s$  is prime, so it divides one of the  $q_j$ : without loss of generality we may assume that  $p_s | q_t$ . However,  $q_t$  is irreducible, so  $q_t = up_s$  for some unit. So now we have

$$u^{-1}vp_1 \cdots p_s = q_1 \cdots q_{t-1} \cdot p_s.$$

Cancelling the  $p_s$  gives a shorter violation of Definition VI.25(ii): one with only  $t - 1$  irreducibles  $q_j$ . This is a contradiction, so no such violations exist.

**VI.30** It is somewhat easier to prove that a Euclidean domain is a UFD, because we can use the valuation for inductions, but the argument is essentially the same and it is worth taking the extra care to prove this stronger result.

**Corollary VI.31** If  $a \in \mathbb{Z}$  and  $a > 1$  then  $a$  can be written as  $a = \prod p_i^{n_i}$ , where the  $p_i$  are distinct prime numbers and the  $n_i$  are positive integers. The primes  $p_i$  and their exponents  $n_i$  are uniquely determined (up to order).

This follows from Theorem VI.29 and the facts that  $\mathbb{Z}$  is a Euclidean domain, hence a PID, and that the units in  $\mathbb{Z}$  are  $\pm 1$  so every nonzero ideal has a unique positive generator.

**Example VI.32** Not every UFD is a PID: in fact  $\mathbb{Z}[t]$  is a UFD but not a PID.

**VI.33** So far we have seen that shown that

- (i) Fields are Euclidean domains — Example VI.13(i)
- (ii) Euclidean domains are PIDs — Theorem VI.16
- (iii) PIDs are UFDs — Theorem VI.29
- (iv) UFDs are integral domains — Definition VI.25

and examples to show that the reverse implications do not hold.

## UFDs and polynomial rings

**VI.34** We want to investigate polynomial rings where the coefficients come from a UFD. For the rest of this section, we assume that  $R$  is a UFD.

**Definition VI.35** An ideal in a commutative ring  $A$  is said to be *finitely generated* if it is of the form  $a_1A + \cdots + a_kA$  (also written  $\langle a_1, \dots, a_k \rangle$ ) for some finite set  $\{a_1, \dots, a_k\} \subset A$ .



**Lemma VI.36** Let  $R$  be a UFD and suppose that  $I = \langle a_1, \dots, a_k \rangle$  is a finitely generated ideal. Then there is a unique smallest principal ideal  $C$  containing  $I$ : that is,  $C$  is principal,  $C \supseteq I$ , and if  $C' \supseteq I$  is another principal ideal containing  $I$ , then  $C' \supseteq C$ .

*Proof:* We proceed by induction on  $k$ . If  $k = 1$  there is nothing to prove. Otherwise, suppose that  $C_1 = \langle c_1 \rangle$  is the least principal ideal containing  $I_1 = \langle a_1, \dots, a_{k-1} \rangle$ , which exists by the induction hypothesis. (Note that  $I_1 \subseteq I$ .) Since  $R$  is a UFD, there are finitely many irreducibles  $p_1, \dots, p_m$  such that  $p_j | c_1$ . For each  $p_j$ , we let  $n_j \in \mathbb{Z}_{\geq 0}$  be the largest integer such that  $p_j^{n_j} | c_1$  and  $p_j^{n_j} | a_k$ , and we take  $c = p_1^{n_1} \cdots p_m^{n_m}$ . I claim that  $C = \langle c \rangle$  has the required properties.

Firstly,  $C$  is a principal ideal. Second, again because  $R$  is a UFD,  $c | c_1$  so  $C \supseteq C_1$ , so  $C \supseteq I_1$ . Third,  $c | a_k$ , so  $C \supseteq I$ . Finally, suppose  $C' = \langle c' \rangle \supseteq I$ . Then  $C' \supseteq I_1$  because  $I \supseteq I_1$ , so  $C' \supseteq C_1$  by definition of  $C_1$ ; so  $c' | c_1$ . Also,  $C' \supseteq \langle a_k \rangle$ , so  $c' | a_k$ . Now if  $p^n$  is a power of an irreducible and  $p^n | c'$  then  $p^n | c_1$  and  $p^n | a_k$ , so  $p^n | p_j^{n_j}$  for some  $j$ : in other words, all the irreducible factors of  $c'$  divide  $c$  to at least the same power. So, by unique factorisation,  $c' | c$ : that is,  $C' \supseteq C$ .

**VI.37** The element  $c$  in Lemma VI.36 is the product of all the powers of irreducibles that divide all of the  $a_i$ , so it should be thought of as the hcf of the  $a_i$ : we can't say it that way because we do not have a notion of "highest" until Lemma VI.36 gives us one.

**Definition VI.38** A nonconstant polynomial  $g = \sum_{i=0}^n a_i t^i \in R[t]$  is *primitive* if the only common divisors of all the coefficients of  $g$  are units in  $R$ .

**VI.39** In light of unique factorisation in  $R$ , it is equivalent to say that  $g$  is primitive if and only if no irreducible element of  $R$  divides all coefficients of  $g$ .

If  $f$  is an arbitrary polynomial, then according to Lemma VI.36 the smallest principal ideal  $C_f$  containing all its coefficients is generated by an element  $c \in R$ , unique up to a unit factor, which is called the *content* of  $f$ , and then  $f = cg$  with  $g$  primitive.

Another way to express Definition VI.38 is to say that  $g$  is primitive if the content of  $g$  is a unit (so the ideal  $C_g$  is  $R$ ).

**Example VI.40**  $t^3 + 2t - 1 \in \mathbb{Z}[t]$  is primitive and so is  $t^2 + 6t - 3$ , whereas  $3t^3 + 6t - 3 \in \mathbb{Z}[t]$  is not.

**Proposition VI.41** Let  $R$  be a UFD. The product of finitely many primitive polynomials in  $R[t]$  is primitive.

*Proof:* It suffices to prove the result for two primitive polynomials  $f = \sum_{i=0}^n a_i t^i$  and  $g = \sum_{i=0}^m b_i t^i$ . Suppose that  $fg = \sum_{l=0}^{m+n} d_l t^l$  is not primitive: then the content  $c \in R$  of  $fg$  is not a unit, so it has an irreducible factor  $p | c$ . Thus  $p | d_l$  for any  $l$ . Since  $f$  and  $g$  are primitive,  $p$  does not divide all of the coefficients of either  $f$  or  $g$ . Let  $a_{i_0}$  and  $b_{j_0}$  be the first coefficients of  $f$  and  $g$  respectively that are not divisible by  $p$ , so  $p \nmid a_i$  if  $i < i_0$  and  $p \nmid b_j$  if  $j < j_0$ .

Now we take  $l_0 = i_0 + j_0$  and consider the coefficient of  $t^{l_0}$  in the product  $fg$ , namely

$$d_{l_0} = (a_0b_{l_0} + \cdots + a_{i_0-1}b_{j_0+1}) + a_{i_0}b_{j_0} + (a_{i_0+1}b_{j_0-1} + \cdots + a_{l_0}b_0).$$

Rearranging this, we get

$$\begin{aligned} a_{i_0}b_{j_0} &= -d_{l_0} + (a_0b_{l_0} + \cdots + a_{i_0-1}b_{j_0+1}) + (a_{i_0+1}b_{j_0-1} + \cdots + a_{l_0}b_0) \\ &= -d_{i_0+j_0} + \sum_{i < i_0} a_i b_{l_0-i} + \sum_{j < j_0} a_{l_0-j} b_j, \end{aligned}$$

and  $p$  divides all the terms on the right-hand side. So  $p|a_{i_0}b_{j_0}$ ; but  $p$  is irreducible and hence prime by Proposition VI.28, so  $p$  must divide either  $a_{i_0}$  or  $b_{j_0}$ , which is a contradiction.

**Corollary VI.42** Let  $R$  be a UFD with field of fractions  $F = \mathcal{Q}(R)$ , and let  $f \in R[t]$ . Then  $f$  is irreducible if and only if either

- (i)  $f$  is an irreducible element of  $R$ , or
- (ii)  $f$  is primitive in  $R[t]$  and irreducible in  $F[t]$ .

*Proof:* ( $\Rightarrow$ ) From VI.39 we have  $f = c \cdot g$  for  $c \in R$  and  $g \in R[t]$  primitive. Since  $f$  is irreducible, either  $c$  or  $g$  must be a unit in  $R[t]$ .

- (i) If  $g$  is a unit in  $R[t]$ , then  $g \in R$  and hence  $f \in R$ . Then since  $f$  is irreducible in  $R[t]$  it is also irreducible in  $R$ .
- (ii) If  $c$  is a unit then  $f = cg$  is primitive because  $g$  is primitive. Moreover,  $f \notin R$ , since otherwise it would be primitive and thus a unit in  $R$ , so not irreducible. So  $f$  is primitive and irreducible in  $R[t]$ , and of positive degree. But such an  $f$  is irreducible in  $F[t]$ . To see this, suppose that  $f = g \cdot h$  with  $g, h \in F[t]$  nonunits (so of positive degree). Then, after multiplying by a suitable  $r \in R$ , we have  $a \cdot f = g_0 \cdot h_0$  with  $g_0, h_0 \in R[t]$  of positive degree: for example, take  $r$  to be the product of all the denominators of coefficients of  $g$  and  $h$ . Among all such expressions, take one in which  $r$  has the smallest possible number of irreducible factors: this makes sense because  $R$  is a UFD. Then  $r$  is not a unit, because then  $f$  would be reducible. So  $r \cdot f$  is not primitive, so by Proposition VI.41, one of  $g_0$  and  $h_0$  is not primitive either. So some irreducible  $p \in R$  divides  $g_0$ , say: write  $g_0 = pg_1$ . So  $p$  divides  $r \cdot f$ , but  $f$  is primitive so there is a coefficient  $a_i$  of  $f$  such that  $p$  does not divide  $a_i$ . However,  $p|ra_i$ , and  $p \in R$  is prime because  $R$  is a UFD, so  $p|r$ , say  $r = pr'$ . Now we have  $r' \cdot f = g_1 \cdot h_0$ , contradicting the choice of  $r$ .

( $\Leftarrow$ ) If  $f \in R$  is irreducible then it is irreducible in  $R[t]$ , so it remains to prove the result when  $f \in R[t]$  is primitive in  $R[t]$  and irreducible in  $F[t]$ . Since  $f$  is irreducible in  $F[t]$ , it is nonzero and a nonunit in  $F[t]$ , so it has positive degree and is therefore nonzero and a nonunit in  $R[t]$  also.

Suppose now that there exist  $g, h \in R[t]$  such that  $f = g \cdot h$ . We know  $f$  is irreducible in  $F[t]$ , so without loss of generality  $g$  is a unit in  $F[t]$ : that is,  $g = a \in F^*$  because the units of  $F[t]$  are the units of  $F$ . Since  $F \cap R[t] = R$  that tells us that  $a \in R$ . From  $f = ah$  we see that  $a$  divides the content of  $f$ , but  $f$  is primitive in  $R[t]$ , so its content is a unit. Hence  $a = g$  is a unit in  $R$ , and thus a unit in  $R[t]$ , and we have shown that  $f$  is irreducible in  $R[t]$ .

**Theorem VI.43** If  $R$  is a UFD, then the polynomial ring  $R[t]$  is also a UFD.

*Proof:* For Definition VI.25(i), let  $f \in R[t]$  be a nonzero nonunit. If  $f \in R$ , then it is a nonzero nonunit in  $R$ , and since  $R$  is a UFD,  $f$  can be written as the product of finitely many irreducibles in  $R$  which are necessarily irreducibles in  $R[t]$ .

Otherwise,  $f$  has positive degree. We may regard  $f$  as an element of  $F[t]$  where  $F = \mathcal{Q}(R)$ . Since  $F$  is a field,  $F[t]$  is a UFD by Example VI.17 and Theorem VI.29. Accordingly, we write  $f = q_1 q_2 \cdots q_k$  for irreducible elements  $q_1, \dots, q_k \in F[t]$ . We clear denominators: take  $d_i$  to be the product of all the denominators of all the coefficients of  $q_i$  and put  $f_i = d_i q_i \in R[t]$ . Then  $df = f_1 f_2 \cdots f_k$  for some  $d = d_1 \cdots d_k \in R$ .

Then we use VI.39 to write  $f_i = c_i g_i$  with  $c_i \in R$  (the content of  $f_i$ ) and  $g_i \in R[t]$  primitive, and similarly  $f = cg$ , so that

$$dcg = df = f_1 \cdots f_k = (c_1 \cdots c_k) \cdot g_1 \cdots g_k.$$

Notice that in  $F[t]$  we have  $g_i = \frac{d_i}{c_i} q_i$  which is irreducible (in  $F[t]$ ) as  $q_i$  is irreducible and  $\frac{d_i}{c_i} \neq 0$  is a unit. By Corollary VI.42, therefore,  $g_i$  is irreducible in  $R[t]$ , and the product  $g_1 \cdots g_k$  is primitive by Proposition VI.41, so the uniqueness part of Lemma VI.36 shows that there exists a unit  $u \in R$  such that  $dcu = c_1 \cdots c_k$ .

So  $df = dcug_1 \cdots g_k$ , so  $f = ucg_1 \cdots g_k$  and since  $c \in R$  which is a UFD we can factorise  $cu = r_1 \cdots r_l$  with  $r_i \in R$  irreducible or a unit, and thus

$$f = r_1 \cdots r_l \cdot g_1 g_2 \cdots g_k,$$

with each factor irreducible or a unit in  $R[t]$  as required for Definition VI.25(i).

To show Definition VI.25(ii), suppose  $f \in R[t]$  admits two such decompositions

$$r_1 \cdots r_l \cdot g_1 \cdots g_k = r'_1 \cdots r'_m \cdot g'_1 \cdots g'_n.$$

These two polynomials have the same content (up to a unit factor), so  $r_1 \cdots r_l = u \cdot r'_1 \cdots r'_m$  for some unit  $u \in R$ . Since  $R$  is a UFD, we have  $l = m$  and (after permuting indices)  $r_i R = r'_i R$  for  $1 \leq i \leq l$ . Similarly, the primitive part of  $f$  is unique up to multiplication by a unit, so there exists a unit  $u' \in R$  such that  $g_1 \cdots g_k = u' \cdot g'_1 \cdots g'_n$ .

By Corollary VI.42, each  $g_i, g'_j \in F[t]$  is irreducible: but  $F[t]$  is a UFD because  $F$  is a field, so  $k = n$  and (after permuting indices)  $g_i F[t] = g'_i F[t]$  for  $1 \leq i \leq k$ . Now Lemma VI.7 gives a unit  $u_i \in F$  such that  $g_i = u_i g'_i \in R[t]$ . But  $g_i$  and  $g'_i$  are primitive, so comparing contents shows that  $u_i$  is a unit in  $R$ .

**Corollary VI.44** Let  $K$  be a field. Then  $K[x_1, \dots, x_n]$  is a UFD.

*Proof:*  $K$  is a PID, so it is a UFD. The result follows immediately by induction using Theorem VI.43.

**Example VI.45**  $K[x_1, \dots, x_n]$  is not a PID for  $n \geq 2$ .

## Irreducible polynomials

**VI.46** Depending on the field  $K$ , we may be able to tell whether certain polynomials in  $K[x]$  are irreducible or not. If  $K$  is a finite field, then deciding whether  $f \in K[x]$  is irreducible is a finite search (we simply check, for each polynomial  $g$  of degree less than  $\deg f$ , whether  $g|f$  or not). If  $K$  or  $\deg f$  is very large this may be impractical, but it is possible in many useful cases.

**VI.47** If  $f \in K[x]$  is an irreducible polynomial of degree  $d$ , then the ideal  $\langle f \rangle$  generated by  $f$  is a maximal ideal so  $K[x]/\langle f \rangle$  is a field, according to Theorem V.47. Suppose that  $K = \mathbb{F}_p$ . It is fairly easy to see that  $\mathbb{F}_p[x]/\langle f \rangle$  has  $p^d$  elements. Two other facts are less obvious and will merely be stated here. First, the isomorphism class of  $\mathbb{F}_p[x]/\langle f \rangle$  depends only on  $d$  (and of course  $p$ , which is the characteristic): that is, all irreducible polynomials of degree  $d$  give the same quotient field by this construction. Second, for each  $p$  and  $d$ , irreducible polynomials of degree  $d$  do exist. Moreover, any finite field can be constructed in this way. It follows that for each  $q = p^d$  there is up to isomorphism a unique finite field  $\mathbb{F}_q$  with  $q$  elements, of characteristic  $p$ , and that there are no other finite fields. Notice that if  $d > 1$  the rings  $\mathbb{Z}/q$  and  $\mathbb{F}_q$  are very different: the former is not even an integral domain.

**VI.48** We should expect actually factorising polynomials over  $\mathbb{Q}$  (or over  $\mathbb{Z}$ , which is the same thing by Corollary VI.42) to be quite difficult, but we might hope to be able to test individual polynomials for irreducibility.

**Proposition VI.49** Suppose that  $f \in \mathbb{Z}[x]$  and  $p \in \mathbb{Z}$  is a prime. There map  $\text{red}_p: \mathbb{Z}[x] \rightarrow \mathbb{F}_p[x]$  given by  $\text{red}_p(\sum_{i=0}^d a_i x^i) = \sum_{i=0}^d \bar{a}_i x^i$ , where  $\bar{a}_i \in \mathbb{F}_p = \mathbb{Z}/p$  is the reduction mod  $p$  of  $a_i \in \mathbb{Z}$ , is a ring homomorphism and if  $\text{red}_p(f)$  is irreducible in  $\mathbb{F}_p[x]$  and of the same degree as  $f$ , then  $f$  is irreducible in  $\mathbb{Z}[x]$ .

*Proof:*  $\text{red}_p$  is simply the homomorphism whose kernel is  $\langle p \rangle \triangleleft \mathbb{Z}[x]$ . Suppose that  $f = gh$  is reducible in  $\mathbb{Z}[x]$ . Then  $\text{red}_p(f) = \text{red}_p(g) \text{red}_p(h)$  so since  $\text{red}_p(f)$  is irreducible one of the factors, say  $\text{red}_p(g)$ , must be constant. However,  $g$  itself is not constant so its leading coefficient (and indeed all its coefficients apart from the constant) must be divisible by  $p$ . But then the leading coefficient of  $f$  is divisible by  $p$  as it is the product of the leading coefficients of  $g$  and  $h$ , so  $\deg \text{red}_p(f) < \deg f$ .

**Theorem VI.50 Eisenstein's criterion** Suppose that  $f = \sum_{i=0}^d a_i x^i \in \mathbb{Z}[x]$  is of degree  $d$  and for some prime  $p \in \mathbb{Z}$  we have  $p|a_i$  for  $0 \leq i < d$  but  $p$  does not divide  $a_d$  and  $p^2$  does not divide  $a_0$ . Then  $f$  is irreducible in  $\mathbb{Z}[x]$  (and therefore irreducible in  $\mathbb{Q}[x]$ ).

*Proof:* Suppose that  $f$  is reducible in  $\mathbb{Z}[x]$ , so  $f = gh$ . Then, as in Proposition VI.49,  $\text{red}_p f = (\text{red}_p g)(\text{red}_p h)$ , but  $\text{red}_p f = \overline{a_d}x^d$  by the hypotheses. Since  $\mathbb{F}_p[x]$  is a UFD it follows that  $\text{red}_p g = bx^{\deg g}$  and  $\text{red}_p h = cx^{\deg h}$  for some  $b, c \in \mathbb{F}_p$ . In particular the constant terms of  $\text{red}_p g$  and  $\text{red}_p h$  are both zero, so the constant terms of  $g$  and  $h$  are both divisible by  $p$ . But then the constant term  $a_0$  of  $f$  is divisible by  $p^2$ .

**VI.51** Eisenstein's criterion can often be applied directly but a useful trick is to apply it not to  $f(x)$  but to  $f(x+n)$  for some  $n \in \mathbb{Z}$ . The map that sends  $f(x)$  to  $f(x+n)$  is an isomorphism from  $\mathbb{Z}[x]$  to  $\mathbb{Z}[x]$  so  $f(x)$  is irreducible if and only if  $f(x+n)$  is irreducible.

Thus, for example,  $x^5 - 76x^4 - 1002x^3 - 4630x^2 - 9437x + 194$  is irreducible because replacing  $x$  with  $x-3$  gives  $x^5 - 91x^4 + 7x^3 + 14x^2 - 98x + 7147$  which satisfies the conditions of Eisenstein's criterion with  $p = 7$ .

## VII Structure of modules

In this section we shall look more closely at modules over a commutative ring. We shall prove a structure theorem for finitely-generated modules over a principal ideal domain.

### Noetherian rings

**Definition VII.1** A commutative ring  $R$  is said to be *noetherian* if every ideal of  $R$  is finitely generated.

**VII.2** This is a very common condition: often it is sufficient to consider only noetherian rings. In particular, a PID is noetherian: also, Hilbert's Basis Theorem says that if  $R$  is noetherian then so is  $R[t]$ . Consequently, if  $K$  is a field (or indeed any noetherian ring) then  $K[t_1, \dots, t_n]$  is noetherian.

**Proposition VII.3** Let  $R$  be a commutative ring. The following conditions are equivalent.

- (i)  $R$  is noetherian.
- (ii) Any ascending chain of ideals of  $R$  is bounded above. That is, if  $I_j$  is a proper ideal of  $R$ , for each  $j \in \mathbb{N}$ , and  $I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots$ , then there exists  $n \in \mathbb{N}$  such that  $I_j = I_n$  if  $j \geq n$ .
- (iii) If  $\mathcal{X}$  is a non-empty collection of ideals of  $R$  then  $\mathcal{X}$  contains a maximal element: that is, there is an ideal  $J \in \mathcal{X}$  such that if  $I \in \mathcal{X}$  and  $I \supseteq J$  then  $I = J$ .

*Proof:* (i)  $\implies$  (ii): suppose  $I_j$  is an ascending chain of ideals and consider  $I = \bigcup_{j \in \mathbb{N}} I_j$ . This is an ideal of  $R$ , so it is generated by finitely many elements

$a_1, \dots, a_r \in I$ . For each  $a_i$ , since  $a_i \in I$  there is some  $n_i$  such that  $a_i \in I_{n_i}$ . It is sufficient to take  $n = \max(n_1, \dots, n_r)$ : for then  $a_i \in I_n$  for all  $i = 1, \dots, r$ , so  $I \subseteq I_n$  (in fact,  $I = I_n$ ), so  $I_j \subseteq I_n$  for all  $j \in \mathbb{N}$ . But if  $j \geq n$  we also have  $I_j \supseteq I_n$ , so  $I_j = I_n$  as claimed.

(ii)  $\iff$  (iii): it is clear that (iii)  $\implies$  (ii) (just take  $\mathcal{X} = \{I_j\}$ ) and the converse is exactly the statement of Zorn's Lemma, which is an equivalent of the Axiom of Choice.

(ii)  $\implies$  (i): suppose that  $R$  is not noetherian, and let  $I$  be an ideal of  $R$  that is not finitely generated. Then for any finite subset  $A \subset I$  we have that  $\langle A \rangle$  is strictly contained in  $I$ . We choose  $a_i \in I$ , for  $i \in \mathbb{N}$ , as follows:  $a_1 \in I$ , and  $a_r \in I \setminus \langle a_1, \dots, a_{r-1} \rangle$  (which cannot be empty). Then we take  $I_j = \langle a_1, \dots, a_j \rangle$ , for  $j \in \mathbb{N}$ : this evidently does not satisfy the conclusion of (ii).

## Free modules

**Definition VII.4** Let  $R$  be a commutative ring and let  $M$  and  $N$  be  $R$ -modules. A map  $\varphi: M \rightarrow N$  is said to be  $R$ -linear if and only if it is an abelian group homomorphism, and for all  $r \in R$  and all  $m \in M$  we have

$$\varphi(rm) = r\varphi(m).$$

An  $R$ -linear map is also known as an  $R$ -module homomorphism.

**VII.5**  $R$ -modules behave collectively exactly like abelian groups. Kernels of  $R$ -linear maps are submodules and conversely: if  $U$  is a submodule of  $M$  then the abelian group quotient  $M/U$  is also an  $R$ -module, with multiplication given by  $r(m + U) = rm + U$ ; the First Isomorphism Theorem holds (and so do the Second and Third).

**Definition VII.6** Let  $X$  be a set and  $R$  a commutative ring. A *free  $R$ -module* with basis  $X$  (also called a free  $R$ -module on  $X$ ) is an  $R$ -module  $F_X$  together with a map (of sets)  $i: X \rightarrow F_X$  such that, if  $M$  is an  $R$ -module and  $f: X \rightarrow M$  is a map, then there is a unique  $R$ -linear map  $\varphi: F_X \rightarrow M$  such that  $f = \varphi i$ .

**VII.7** This defines a free  $R$ -module by a universal property: it says that any map from  $X$  to an  $R$ -module goes via  $F_X$ . It is best shown by means of a diagram.

$$\begin{array}{ccc} X & \xrightarrow{i} & F_X \\ & \searrow f & \downarrow \exists! \varphi \\ & & M \end{array}$$

That teaches us how to recognise free modules: it does not guarantee that they exist.

**Definition VII.8** Suppose that  $A$  is a set and  $\{M_\alpha \mid \alpha \in A\}$  is a set of  $R$ -modules. The *external direct sum* of the  $M_\alpha$  is the  $R$ -module

$$\bigoplus_{\alpha \in A} M_\alpha := \{(m_\alpha)_{\alpha \in A} \in \prod_{\alpha \in A} M_\alpha \mid m_\alpha = 0 \text{ for all but finitely many } \alpha\}.$$

**Definition VII.9** Suppose that  $M$  is an  $R$ -module and  $P$  and  $Q$  are submodules such that  $P \cap Q = 0$ . The the sum  $P + Q = \{p + q \mid p \in P, q \in Q\}$  is called an *internal direct sum*, and may also be written  $P \oplus Q$ .

**VII.10** We usually write  $\sum m_\alpha$  rather than  $(m_\alpha)$  in this situation. The external direct sum is the same as the internal direct sum of the submodules  $M_\alpha$  (that is,  $\{(0, 0, \dots, 0, m_\alpha, 0, \dots)\}$  of  $\prod M_\alpha$ , so it is not a new object: it is just a standardised way of taking the direct sum of modules that may not all happen to be submodules of a module to start with. Note that under the conditions of Definition VII.9 the internal and external direct sums agree: the map  $p + q \mapsto (p, q)$  is an isomorphism from the internal to the external direct sum.

**Definition VII.11** Suppose that  $A$  is a set and  $\{M_\alpha \mid \alpha \in A\}$  is a set of  $R$ -modules. The *direct product* of the  $M_\alpha$  is the  $R$ -module

$$\prod_{\alpha \in A} M_\alpha := \{(m_\alpha)_{\alpha \in A}\}.$$

**VII.12** Clearly, if  $A$  is a finite set then  $\prod_{\alpha \in A} M_\alpha = \bigoplus_{\alpha \in A} M_\alpha$ . For this reason we shall sometimes use product notation instead of direct sum notation if convenient. If  $A$  is infinite we need to be careful; but observe that  $\prod_{\alpha \in A} M_\alpha \ni (0, 0, 0, \dots)$  so it is not empty, even without the Axiom of Choice.

**Lemma VII.13** If  $X$  is any set then the  $R$ -module  $\bigoplus_{x \in X} R \cdot x$  is a free module with basis  $X$ , where  $R \cdot x$  is the set of formal symbols  $rx$  for  $r \in R$ , and  $i(x) = 1_R x$ .

*Proof:* Suppose  $f: X \rightarrow M$  is a map of sets. Then the requirement that  $f = \varphi i$  implies that  $\varphi: \bigoplus_{x \in X} R \cdot x \rightarrow M$  has  $\varphi(1_R x) = f(x)$ . Then the requirement that  $\varphi$  is  $R$ -linear implies that  $\varphi(\sum_{x \in X} r_x x) = \sum_{x \in X} r_x f(x)$ , but that is an  $R$ -linear map with  $f = \varphi i$  and we are done.

**VII.14** Lemma VII.13 tells us that free modules do exist. If  $X$  is finite, say  $X = \{1, \dots, m\}$ , then  $\bigoplus_{x \in X} R \cdot x = R^m$  as in Example V.4(iii).

Note that  $R \cdot x$  is not always quite the same thing as  $Rx$ . For example, if  $R$  is not an integral domain,  $x \in R$  and  $rx = 0$  the symbol  $r \cdot x$  is still just  $r \cdot x$ , unevaluated.

**VII.15** Free modules behave very like vector spaces, even for arbitrary commutative rings  $R$ .

- (i) If  $F_X$  is free with basis  $X$  with respect to the map  $i: X \rightarrow F_X$  then  $i$  is injective.

- (ii) If  $F$  and  $F'$  are free modules with basis  $X$ , then  $F \cong F'$ . More precisely, there is a unique isomorphism  $\eta: F \rightarrow F'$  such that  $i' = \eta i$ .
- (iii) If  $X$  and  $Y$  are the same size then the corresponding free modules  $F_X$  and  $F_Y$  are isomorphic. More precisely, if  $b: X \rightarrow Y$  is a bijection then there is a unique isomorphism  $\beta: F_X \rightarrow F_Y$  such that  $\beta i_X = i_Y b$  (it is the linear extension of  $b$ ).
- (iv) In view of (i)–(iii) we may usually drop the mention of  $i$  or even of  $X$  and talk about a free  $R$ -module  $F$  on  $n$  elements, where  $n = \#X$  (even if  $X$  is not finite).
- (v) The basis  $X$  (identified with  $i(X) \subset F$ ) of a free module is a linearly independent spanning set: that is,  $\sum r_x x = 0$  implies  $r_x = 0$  for all  $x$ , and every element of  $F$  can be written in this form. This follows simply because those things are true for  $\bigoplus Rx$  and are preserved by isomorphism.

**Corollary VII.16** Any  $R$ -module is a quotient of a free  $R$ -module.

*Proof:* For instance, we could take  $X = M$  and  $f = \text{id}$ : then the definition of free module gives a surjective map from the free module  $F_M$  with basis  $M$  to  $M$ .

**Proposition VII.17** Suppose  $R$  is an integral domain and  $n \in \mathbb{N}$ : denote by  $R^n$  the free module on  $n$  elements. Then any set of  $n + 1$  distinct elements of  $R^n$  is linearly dependent over  $R$ .

*Proof:* We prove this by induction on  $n$ . For  $n = 0$  it is trivial:  $R^0$  is the trivial module and the only subset with  $n + 1 = 1$  elements is  $\{0\}$ , which is linearly dependent.

Suppose that the statement is true for all  $n < d$ , and let  $x_0, \dots, x_d \in R^d$  be distinct, say  $x_i = (a_{i1}, \dots, a_{id})$ . We may assume that  $a_{0d} \neq 0$ , since if  $a_{1d} = a_{2d} = \dots = a_{dd} = 0$  then  $\{x_1, \dots, x_d\}$  is contained in  $R^{d-1}$  and therefore linearly dependent by the induction hypothesis.

Now we use  $a_{0d}$  as a pivot: consider the elements  $a_{id}x_0 - a_{0d}x_i$  for  $1 \leq i \leq d$ . These are all contained in  $R^{d-1}$  so they are linearly dependent, so there are elements  $r_1, \dots, r_d \in R$ , not all zero, such that  $\sum_{i=1}^d r_i (a_{id}x_0 - a_{0d}x_i) = 0$ . Without loss of generality, assume  $r_1 \neq 0$ . Then in this relation the coefficient of  $x_1$  is  $r_1 a_{0d}$  which is non-zero since  $R$  is an integral domain: hence  $\{x_0, \dots, x_d\}$  is linearly dependent over  $R$ .

**Corollary VII.18** If  $R$  is an integral domain and  $F$  is a free  $R$ -module on a finite set  $X$  then the cardinality  $n$  of  $X$  is determined by  $F$ .

*Proof:* By Proposition VII.17, we can characterise  $n$  as being the size of the largest linearly independent subset of  $F$ .

**Definition VII.19** The number  $n$  in Corollary VII.18 is called the *rank* of the  $R$ -module  $F$ .



**VII.20** If  $R$  is a field then  $F$  is a vector space and the rank is the same as the dimension. For this reason the word “dimension” is sometimes used, even if  $R$  is only an integral domain, instead of “rank”.

**Definition VII.21** If  $M$  is an  $R$ -module and  $m \in M$  we say that  $m$  is a *torsion element* if there exists a non-zero  $r \in R$  such that  $rm = 0$ . If every element of  $M$  is a torsion element we say that  $M$  is a *torsion module*; if there are no non-zero torsion elements of  $M$  (that is,  $rm = 0$  implies  $r = 0$  or  $m = 0$ ) we say that  $M$  is *torsion-free*.

**Definition VII.22** If  $M$  is an  $R$ -module and  $P < M$ , we say that  $P$  is a *direct summand* of  $M$  if there exists a submodule  $Q < M$  such that  $M = P \oplus Q$ .

**Example VII.23** If  $R$  is a field then every  $R$ -module is free and torsion-free. For other rings, the position is more complicated.

- (i) If  $a \in R$  is a zero divisor then the ideal  $aR$  is not torsion-free, and not free either.
- (ii) If  $R$  is an integral domain, then every free module is torsion-free. The converse is false, even for finitely-generated  $R$ -modules.
- (iii) A linearly independent set in an  $R$ -module  $M$  cannot always be extended to a basis: for example, the only bases for the  $\mathbb{Z}$ -module  $\mathbb{Z}$  are  $\{1\}$  and  $\{-1\}$ , so  $\{2\}$ , which is a linearly independent set, is not part of any basis.
- (iv) A set of generators for a free module need not contain a basis:  $\{2, 3\}$  generates  $\mathbb{Z}$  because  $n = 3n - 2n$ , but does not contain 1 or  $-1$ .
- (v) Not every submodule is a direct summand: for example,  $2\mathbb{Z}$  is not a direct summand of  $\mathbb{Z}$ .
- (vi) If  $P$  is a submodule of an  $R$ -module  $M$  and  $M/P$  is free, then  $P$  is a direct summand of  $M$ : a more precise version of this statement is Proposition VII.24.

**Proposition VII.24** Suppose that  $M$  is a finitely generated  $R$ -module and  $\pi: M \rightarrow L$  is an  $R$ -linear map with kernel  $\text{Ker } \pi = P$ . Suppose that  $\pi(M) \cong M/P$  is free. Then there is a left inverse to  $\pi$ , an  $R$ -linear map  $\psi: \pi(M) \rightarrow M$  such that  $\pi \circ \psi: \pi(M) \rightarrow \pi(M)$  is the identity, with image  $Q = \psi(\pi(M))$ , and  $M = P \oplus Q$ .

*Proof:* Since  $M/P$  is free we may take a basis  $x_1 + P, \dots, x_n + P$  of  $M/P$ . Then the map  $\psi$  is defined by taking  $x_i + P \mapsto x_i \in M$ : by freeness, this extends to an  $R$ -linear map  $M/P \rightarrow P$ . Then  $\psi$  is defined by composing this with the isomorphism  $\bar{\pi}^{-1}: \pi(M) \rightarrow M/P$  from the First Isomorphism Theorem. From this it follows immediately that  $\pi \circ \psi$  is the identity (and in particular,  $\psi$  is injective), and  $Q = Rx_1 + \dots + Rx_n$ .

Evidently  $M$  is generated by  $P$  and  $Q$ , so  $M = P + Q$ : moreover,  $P \cap Q = 0$  because  $P \cap Q = \text{Ker } \pi \cap \text{Im } \psi$  which is trivial because  $\pi \circ \psi$  is injective.

**VII.25** This is a special case of an important notion.

- (i) The map  $\psi$  is called a *splitting* of  $\pi$ . It is not unique in general.
- (ii) The situation in Proposition VII.24 is illustrated by the following diagram:

$$\begin{array}{ccc}
 X & \xrightarrow{\pi} & L \\
 \downarrow & \swarrow \psi & \uparrow \\
 M/P & \xrightarrow{\quad} & \text{Im } \pi
 \end{array}$$

which is the diagram from Corollary II.26 with an extra arrow.

- (iii) We can also express this by (using  $F$  for the free module  $\text{Im } \pi \cong M/P$ ) writing a short exact sequence

$$0 \longrightarrow P \longrightarrow M \longrightarrow F \longrightarrow 0.$$

This conventionally means that the image of each map is (exactly) the kernel of the next one: so  $P \rightarrow M$  is injective (in fact it is simply an inclusion) and  $P$  is the kernel of the map  $M \rightarrow F$ , which is  $\pi$  and is surjective.

- (iv) We can also include  $\psi$  in this picture:

$$0 \longrightarrow P \longrightarrow M \begin{array}{c} \xleftarrow{\psi} \\ \xrightarrow{\pi} \end{array} F \longrightarrow 0$$

which conventionally means that also  $\pi\psi$  is the identity on  $F$ .

## Modules over a PID

For the rest of this section, we assume that  $R$  is a PID. We aim to classify finitely-generated modules over a PID.

**Theorem VII.26** If  $M$  is a free  $R$ -module of finite rank  $m$  and  $N$  is a submodule, then  $N$  is free and  $\text{rank } N \leq m$ .

*Proof:* This is trivially true if  $m = 0$  (then  $M$  is the trivial module). Suppose that  $m \geq 1$  and that the statement is true for smaller rank: that is, if  $M'$  is free of rank  $m' < m$  and  $N' < M'$ , then  $N'$  is free of rank  $\leq m'$ .

Since  $M$  is free of rank  $m$  we may choose a basis  $v_1, \dots, v_m$  of  $M$ , and we consider the map  $\pi: M \rightarrow R$  given by  $\pi(\sum a_i v_i) = a_m$ , which is a surjective  $R$ -linear map. Notice that  $\text{Ker } \pi = \bigoplus_{i=1}^{m-1} R \cdot v_i$ , which is free of rank  $m - 1$ .

If we put  $I = \pi(N)$  then  $I$  is an ideal (a submodule) of  $R$ . Since  $R$  is a PID we have  $I = aR$  for some  $a \in R$ , so  $I$  is a free module of rank 0 (if  $a = 0$ ) or 1 (if  $a \neq 0$ ).

Now we apply the first isomorphism theorem to the restriction  $\pi|_N: N \rightarrow R$ . This tells us that, as  $R$ -modules, we have

$$I \cong N / \text{Ker}(\pi|_N) = N / (N \cap \text{Ker } \pi).$$

So Proposition VII.24 tells us that  $N \cap \text{Ker } \pi$  is a direct summand of  $N$ , say  $N = (N \cap \text{Ker } \pi) \oplus L$ : also, this shows that  $L \cong I$  so  $L$  is free of rank  $\leq 1$ .

But  $N \cap \text{Ker } \pi < \text{Ker } \pi$  and  $\text{Ker } \pi$  is free of rank  $m - 1$ , so by the induction hypothesis  $N \cap \text{Ker } \pi$  is free of rank  $\leq m - 1$ . Therefore  $N$  is free of rank  $\leq m$ .

**Corollary VII.27** Suppose that  $M$  is an  $R$ -module generated by  $m$  elements and  $N$  is a submodule. Then  $N$  is also finitely generated, and has a generating set with at most  $m$  elements.

*Proof:* The hypothesis on  $M$  means that there is a surjection  $f: R^m \rightarrow M$ . Put  $\tilde{N} = f^{-1}(N)$ . Then  $\tilde{N}$  is a submodule of  $R^m$ , so it is free on  $\leq m$  generators by Theorem VII.26. The images of these generators generate  $N$ .

**Example VII.28** The picture is not exactly as for vector spaces.

- (i) In Theorem VII.26 we cannot conclude that a proper submodule has strictly lower rank. For example, the  $\mathbb{Z}$ -submodule  $2\mathbb{Z}$  of  $\mathbb{Z}$  also has rank 1.
- (ii) If  $M$  is free of rank  $m$  and  $X \subset M$  is a generating set with more than  $m$  elements, it is nevertheless possible that  $X$  does not contain a basis of  $M$ . For example,  $\{2, 3\}$  is a generating set for the  $\mathbb{Z}$ -module  $\mathbb{Z}$ , but no proper subset generates  $\mathbb{Z}$ . In fact, we saw in Example VII.23(iv) that in this case no subset is even part of a basis.

**Theorem VII.29** Suppose that  $M$  is a free  $R$ -module of rank  $m \in \mathbb{N}$  (for  $R$  a PID) and that  $N$  is an  $R$ -submodule of  $M$ . Then there exist an  $R$ -basis  $e_1, \dots, e_m$  of  $M$  and elements  $a_1, \dots, a_m \in R$ , such that  $a_i | a_{i+1}$  for  $1 \leq i < m$  and

$$N = Ra_1e_1 \oplus \cdots \oplus Ra_me_m.$$

*Proof:* We prove this by induction on  $m$ . The case  $m = 1$  is just the statement that  $R$  is a PID (and the case  $m = 0$  is trivial). Fix  $m$  and suppose that the statement holds for all  $R$ -modules of rank strictly less than  $m$ .

Let  $\pi: M \rightarrow R$  be an  $R$ -linear surjection (these exist because  $M \cong R^m$ ) and put  $I_\pi = \pi(N)$ . This is an ideal in  $R$ . As such, it is principal. Let  $\mathcal{X}$  be the collection of all  $I_\pi$ , for all possible  $R$ -linear surjections  $\pi: M \rightarrow R$ .

According to Proposition VII.3(iii), there is a maximal element  $I_\rho \in \mathcal{X}$ , say  $I_\rho = Ra$ . Put  $P = \text{Ker } \rho < M$ . According to Proposition VII.24 we have  $M \cong P \oplus Q$  with  $Q$  isomorphic to  $R$  via  $\rho$ : so if  $x \in N$  we have  $x = u + v$  with  $u \in P$  and  $v \in Q$ , and  $\rho(v) \in Ra$ .

I claim that  $a|p$ ; that is, that  $p = ap'$  for some  $p' \in P$ . Let  $\sigma: P \rightarrow R$  be a surjection and put  $b = \sigma(p)$ .

Consider the ideal  $\langle a, b \rangle$  of  $R$ : it is principal, say  $\langle a, b \rangle = Rc$ . So we can find  $r, s \in R$  such that  $ra + sb = c$ . Note that  $r$  and  $s$  are coprime, because we have  $a = a'c$  and  $b = b'c$  for some  $a', b' \in R$ , so  $c(ra' + sb') = c$ , so  $a'r + b's = 1$ .

We can extend  $\sigma$  to a surjection  $\sigma: M \rightarrow R$  by setting  $\sigma(p + q) = \sigma(p)$ , for any  $p \in P$  and  $q \in Q$ . We consider the map  $\tau = r\rho + s\sigma: M = P \oplus Q \rightarrow R$ , defined by  $\tau(w) = r\rho(w) + s\sigma(w)$ . This is a surjection because we can find  $v \in Q$  with  $\rho(v) = a'$  and  $u \in P$  with  $\sigma(u) = b'$ , and then  $\tau(u + v) = ra' + sb' = 1$ .

So, by the maximality of  $I_\pi$ , we have  $I_\tau \subseteq I_\pi$ , and therefore  $a|c$ . But this was true for every surjection  $\sigma: P \rightarrow R$ , and  $P$  is free so  $P \cong R^n$  for some  $n$ : so we write  $p = (p_1, \dots, p_n)$  and apply this to each projection. We find that  $a|p_i$  for each  $i$ , so  $a|p$ .

Now we put  $a_1 = a$ , and apply Proposition VII.24 to the restriction  $\pi|_N: N \rightarrow I_\pi = Ra_1$ . There is an element  $w_1 \in M$  such that  $\pi(w_1) = 1$ , because  $\pi$  is a surjection, and there is also an element  $v_1 \in N$  such that  $\pi(v_1) = a$  because  $I_\pi = Ra_1$ . So  $\pi(v_1 - a_1w_1) = 0$  so we may write  $v_1 = a_1w_1 + p$  with  $p = v_1 - a_1w_1 \in P$ .

The splitting  $\psi: I_\pi \rightarrow N$  is defined by  $\psi(a_1) = v_1$ , so  $Q = Rv_1$ . By the claim above,  $a_1|p$ , so we have  $v_1 = a_1e_1$  for some  $e_1 \in M$ : note that  $\pi(e_1) = 1$  and  $Q = Ra_1e_1$ .

By Proposition VII.24 we have  $N = Q \oplus (N \cap P) = Ra_1e_1 \oplus (N \cap P)$ . By induction there is a basis  $(e_2, \dots, e_m)$  of  $P$  and elements  $a_2, \dots, a_m \in R$  such that  $N \cap P = Ra_2e_2 + \dots + Ra_me_m$ , with  $a_i|a_{i+1}$  for  $1 < i < m$ , and  $a_1|a_2$  by the claim above. It remains to check that  $e_1, \dots, e_m$  is a basis for  $M$ , but this also follows from Proposition VII.24, applied to  $\pi: M \rightarrow R$  and defining the splitting by sending 1 to  $e_1$ .

**Theorem VII.30** Let  $R$  be a PID and  $M$  a finitely generated  $R$ -module. Then  $M$  may be written as the direct sum

$$M \cong R/Ra_1 \oplus \dots \oplus R/Ra_m$$

with  $a_i|a_{i+1}$  for  $1 \leq i < m$ , for some  $m$ .

*Proof:* Suppose that  $M$  is generated by  $x_1, \dots, x_n$  and let  $F$  be the free module on  $\{x_1, \dots, x_n\}$ , so that there is a surjection  $\varphi: F \rightarrow M$  with kernel  $N = \text{Ker } \varphi$ . According to Theorem VII.29 there is a basis  $f_1, \dots, f_n$  for  $F$  and  $a_i \in R$  such that  $a_i|a_{i+1}$  and  $N = \bigoplus Ra_i f_i$ . Therefore

$$M \cong (Rf_1 \oplus \dots \oplus Rf_n) / (Ra_1f_1 \oplus \dots \oplus Ra_nf_n) \cong \bigoplus R/Ra_i.$$

**VII.31** The decomposition in Corollary VII.31 has a uniqueness property, but one must be careful about what this means. Possibly  $a_1$  is a unit: in that case  $R/Ra_1$  is trivial, so that factor may be discarded.

The condition that  $a_i|a_{i+1}$  is essential for any uniqueness: for example,  $\mathbb{Z}/6 \cong \mathbb{Z}/3 \oplus \mathbb{Z}/2$ .

However, the free part is easily determined: if  $a_r \neq 0$  but  $a_r = 0$  then  $M \cong M_t \oplus M_f$ , where  $M_t$  is torsion and  $M_f$  is free of rank  $n - r$ .

For the torsion part, the Chinese Remainder theorem implies that if we write  $a = up_1^{n_1} \dots p_k^{n_k}$  with  $u$  a unit and  $p_i$  distinct primes then

$$R/Ra \cong R/Rp_1^{n_1} \times \dots \times R/Rp_k^{n_k}.$$

Here we have switched from direct sum notation to direct product notation, which is all right by VII.12, for consistency with Theorem V.41.

**VII.32** It is entirely possible that the rings  $R/Ra$  and  $R/Rb$  are isomorphic but the  $R$ -modules are not. For example, if we take  $R = \mathbb{C}[t]$ , then  $R/\langle t \rangle$  and  $R/\langle t-1 \rangle$  are both isomorphic to  $\mathbb{C}$  as rings, but in the first case  $t$  acts by  $t \cdot z = 0$  and in the second case it acts by  $t \cdot z = z$ .

**Corollary VII.33** If  $M$  is a finitely generated torsion module over a PID  $R$  then there exist primes  $p_1, \dots, p_r$  and positive integers  $n_1, \dots, n_r$  such that

$$M \cong R/Rp_1^{n_1} \times \dots \times R/Rp_r^{n_r}.$$

*Proof:* This follows immediately from the remarks in VII.31 and Theorem VII.30.

**VII.34** Notice that in Corollary VII.33 the  $p_j$  are not supposed to be distinct: the same prime could arise from different  $a_i$ . Remember that

$$\mathbb{Z}/4 \times \mathbb{Z}/4, \quad \mathbb{Z}/2 \times \mathbb{Z}/8, \quad (\mathbb{Z}/2)^4, \quad \mathbb{Z}/16$$

are all different  $\mathbb{Z}$ -modules.

**VII.35** Nevertheless, this allows us to see the uniqueness. The number of factors of  $R/Rp^k$  is the dimension of  $(\text{Ker } p^k)/(\text{Ker } p^{k-1})$  as an  $R/Rp$ -module (vector space, in fact), where  $p^k$  denotes the multiplication-by- $p^k$  map.

Once the  $p_j^{n_j}$  are known it is possible to recover the  $a_i$ .

**Corollary VII.36** Let  $G$  be a finitely generated abelian group. Then  $G$  is a direct sum of cyclic subgroups: there exist non-negative integers  $a_i \neq 1$  such that  $a_i | a_{i+1}$  and

$$G \cong \mathbb{Z}/a_1 \times \dots \times \mathbb{Z}/a_r.$$

*Proof:* Because abelian groups and  $\mathbb{Z}$ -modules are the same, this is an immediate consequence of Theorem VII.30.

**Example VII.37** Let us find all abelian groups of order  $300 = 2^2 \times 3 \times 5^2$ . We seek integers  $a_1, \dots, a_r$  such that  $a_i | a_{i+1}$  for  $i < r$  and  $\prod a_i = 300$ . If  $r \neq 1$  (that is, the group is not  $\mathbb{Z}/300$  and some prime  $p$  divides  $a_1$  more than half as many times as it divides 300, then it does not divide  $300/a_1$  so it certainly does not divide  $a_2$ . So the possible candidates for  $a_1$  are 2, 5 and 10.

So  $G = \mathbb{Z}/300$ , or  $G = \mathbb{Z}/2 \times G'$  with  $|G'| = 150 = 2 \times 3 \times 5^2$ , or  $G = \mathbb{Z}/5 \times G'$  with  $|G'| = 60 = 2^2 \times 3 \times 5$ , or  $G = \mathbb{Z}/10 \times G'$  with  $|G'| = 30 = 2 \times 3 \times 5$ .

So we solve the problem again for 150, 60 and 30.

For 30 there is nothing to be done: it is the product of distinct primes so no prime can divide both  $a_2$  and  $a_3$ , so in that case  $G = \mathbb{Z}/10 \times \mathbb{Z}/30$ .

For 60 one possibility is  $a_2 = 60$  and the only other is  $a_2 = 2$  and  $a_3 = 30$ , but that is not allowed because  $a_1$  does not divide  $a_2$  (all we did was check that it divides  $300/a_1 = a_2a_3$ ). So here the only possibility is  $\mathbb{Z}/5 \times \mathbb{Z}/60$ .

For 150 the possibilities are  $a_2 = 150$  or  $a_2 = 5$ , but again the second one is forbidden because  $a_1$  does not divide  $a_2$ . So here we must have  $\mathbb{Z}/2 \times \mathbb{Z}/150$ .

We conclude that there are four such groups:  $\mathbb{Z}/300$ ,  $G = \mathbb{Z}/10 \times \mathbb{Z}/30$ ,  $\mathbb{Z}/5 \times \mathbb{Z}/60$  and  $\mathbb{Z}/2 \times \mathbb{Z}/150$ .

## VIII Tensor Products

In this section we shall look at another way of combining two or more  $R$ -modules that relates to multilinear maps. In this section,  $R$  is any commutative ring.

### Bilinear maps

**Definition VIII.1** Let  $M$ ,  $N$  and  $W$  be  $R$ -modules. A map  $\mathbf{f}: M \times N \rightarrow W$  is said to be bilinear if

$$\mathbf{f}(ra + sb, y) = r\mathbf{f}(a, y) + s\mathbf{f}(b, y) \text{ and } \mathbf{f}(x, rc + sd) = r\mathbf{f}(x, c) + s\mathbf{f}(x, d)$$

for all  $a, b \in M$ ,  $c, d \in N$  and  $r, s \in R$ .

**Example VIII.2** These are very familiar from vector spaces and they generalise to modules in a straightforward way.

- (i) If  $R = \mathbb{R}$  and  $M = \mathbb{R}^n$  then the dot product  $(x, y) \mapsto x \cdot y = \sum x_i y_i$  is a bilinear map  $\mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ .
- (ii) If  $A$  is any  $n \times n$  real matrix then the map  $(x, y) \mapsto x^T A y$  is a bilinear map  $\mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ .
- (iii) If  $V$  is a  $K$ -vector space and  $V^\vee$  is its dual then the map  $V^\vee \times V \rightarrow K$  given by  $(\alpha, v) \mapsto \alpha(v)$  is a bilinear map.
- (iv) If  $M$  and  $N$  are  $R$ -modules then  $\text{Hom}(M, N)$ , the set of  $R$ -linear maps from  $M$  to  $N$ , is an  $R$ -module and the map  $\text{Hom}(M, N) \times M \rightarrow N$  given by  $(\varphi, v) \mapsto \varphi(v)$  is a bilinear map.

**VIII.3** How much information do we need to tell what a bilinear map does? In the case of linear maps, we can tell what they do if we know what they do to a basis. In the case of maps of sets, we only know what a map does when we know what it does to every element of the set. Bilinear maps should be somewhere between the two.

**Lemma VIII.4** Suppose that  $\mathbf{f}: M \times N \rightarrow V$  is bilinear and  $\varphi: V \rightarrow W$  is  $R$ -linear. Then  $\varphi\mathbf{f}: M \times N \rightarrow W$  is bilinear.

**VIII.5** What we aim for is to answer the question in VIII.3 by finding all bilinear maps from  $M \times N$ : we do this by constructing one particular special one, and then we get all the others by using Lemma VIII.4. The target of this special bilinear map is the tensor product of  $M$  and  $N$ , written  $M \otimes_R N$  or simply  $M \otimes N$ .

**VIII.6** A partial answer to the question in VIII.3 is that we know some relations. For example, if  $\mathbf{f}$  is bilinear, then  $\mathbf{f}(rx, y) = \mathbf{f}(x, ry)$  because they are both equal to  $r\mathbf{f}(x, y)$ . One way to look at this is to say that  $\mathbf{f}$  has the same effect on both of them; but another way is to say that they both have the same effect on  $\mathbf{f}$ , regardless of what  $\mathbf{f}$  is. If we could make sense of the expression  $(rx, y) - (x, ry)$  we would be saying that its effect on  $\mathbf{f}$  is zero.

**Lemma VIII.7** Let  $V$  be an  $R$ -module and denote by  $\mathcal{B}^V$  the set of all  $R$ -bilinear maps  $\mathbf{f}: M \times N \rightarrow V$ . This is an  $R$ -module, and there is a map of sets

$$\text{ev}^V: M \times N \rightarrow \text{Hom}(\mathcal{B}, V)$$

given by  $\text{ev}(x, y)(\mathbf{f}) = \mathbf{f}(x, y)$ . It extends to a linear map  $\text{ev}^V: F_{M \times N} \rightarrow \text{Hom}(\mathcal{B}, V)$  where  $F_{M \times N}$  is the free  $R$ -module on  $M \times N$ .

*Proof:* It is easy to check that  $\mathcal{B}^V$  is an  $R$ -module, and the last part is just the definition of a free module.

**Definition VIII.8** Let  $K^V = \text{Ker ev}^V: F_{M \times N} \rightarrow \text{Hom}(\mathcal{B}, V)$ , and let  $K$  be the submodule  $\bigcap_V K^V$ . The *tensor product*  $M \otimes N$  is the quotient  $F_{M \times N}/K$ . The element  $(x, y) + K \in M \otimes N$  is denoted by  $x \otimes y$ , and the bilinear map  $\mathbf{b}: M \times N \rightarrow M \otimes N$  is defined by  $\mathbf{b}(x, y) = x \otimes y$ .

**VIII.9** There is a set-theoretical difficulty here: we cannot really take the intersection over all  $R$ -modules  $V$ , because the collection of all  $R$ -modules does not form a set. For vector spaces (if  $R$  is a field) it is enough to take  $V = R$ , but there are, for instance, no  $\mathbb{Z}$ -linear maps  $\mathbb{Z}/2 \rightarrow \mathbb{Z}$  at all so we cannot do that even for  $R = \mathbb{Z}$ . In fact we can escape this difficulty because  $K$  can be defined directly by writing down generators, but that obscures what it going on.

**Lemma VIII.10** The map  $\mathbf{b}$  is bilinear.

*Proof:* It is enough to check this for the first variable. We must show that if  $r, s \in R$  then  $\mathbf{b}(rx_1 + sx_2, y) = r\mathbf{b}(x_1, y) + s\mathbf{b}(x_2, y)$ . So we want to show that  $\mathbf{b}(rx_1 + sx_2, y) - r\mathbf{b}(x_1, y) - s\mathbf{b}(x_2, y) \in K$ ; that is, that if  $\mathbf{f}: M \times N \rightarrow R$  is bilinear then  $\mathbf{f}(rx_1 + sx_2, y) - r\mathbf{f}(x_1, y) - s\mathbf{f}(x_2, y) = 0$ , which is simply part of the definition of bilinearity.

**VIII.11** Definition VIII.8 is easier than it looks. A typical element of  $F_{M \times N}$  is a finite  $\sum a_i(x_i, y_i)$ , by Lemma VII.13. It is in  $K$  if  $\text{ev}(\sum a_i(x_i, y_i)) = 0$ : that is, if the map  $\mathbf{f} \mapsto \sum a_i\mathbf{f}(x_i, y_i)$  is the zero map. So the elements of  $K$  are the things that fail to tell us anything about  $\mathbf{f}$ ; they give zero, regardless of what

$\mathbf{f}$  is. So the information we need to reconstruct  $\mathbf{f}$  is in what remains after we have killed  $K$ , i.e. in  $F_{M \times N}/K$ .

**Theorem VIII.12** The tensor product has the following universal property: if  $\mathbf{f}: M \times N \rightarrow V$  is any  $R$ -bilinear map then there is a unique  $R$ -linear map  $\varphi: M \otimes N \rightarrow V$  such that  $\mathbf{f} = \varphi\mathbf{b}$ .

*Proof:* First, notice that the definition of  $K$  did not depend on the fact that that the target of the maps in  $\mathcal{B}$  was  $R$ . We could have replaced  $R$  with  $V$  in the definition of  $\mathcal{B}$ , and  $\text{Hom}(\mathcal{B}, R)$  with  $\text{Hom}(\mathcal{B}, V)$ . Next, let  $\tilde{\varphi}: F_{M \times N} \rightarrow V$  be the  $R$ -linear extension of  $\mathbf{f}$  to  $F_{M \times N}$ , which exists because  $F_{M \times N}$  is free. Then  $\text{Ker } \tilde{\varphi} \supseteq K$  by definition, so by the First Isomorphism Theorem there is a unique  $\varphi: F_{M \times N} = M \otimes N \rightarrow V$  such that  $\tilde{\varphi} = \phi\beta$ , where  $\beta: F_{M \times N} \rightarrow F_{M \times N}/K$  is the quotient map. But  $\mathbf{b}$  is exactly the restriction of  $\beta$  to  $M \times N$ .

**VIII.13** In practice, the universal property in Theorem VIII.12 is what matters. The direct construction with free modules is there to prove that tensor products exist.

## Applications

**Example VIII.14** Tensor products are heavily used in algebra.

- (i) Let us take  $r = \mathbb{R}$  and compute  $\mathbb{R}^2 \otimes \mathbb{R}^2$ . Choose bases  $e_1, e_2$  and  $f_1, f_2$  for the two copies of  $\mathbb{R}^2$ . We want to know what bilinear maps (to  $\mathbb{R}$ ) there are from  $\mathbb{R}^2 \times \mathbb{R}^2$  – they correspond to maps  $\mathbb{R}^2 \otimes \mathbb{R}^2 \rightarrow \mathbb{R}$  so strictly speaking we are computing the dual of the tensor product. We can determine such a map  $\mathbf{f}$  if we know  $\mathbf{f}(e_i, f_j)$  for  $i, j \in \{1, 2\}$ , and there is a bilinear map that has  $\mathbf{f}(e_i, f_j) = 1$  and the others zero: namely, the one given by the  $2 \times 2$  matrix  $E_{ij}$ . So  $\mathbb{R}^2 \otimes \mathbb{R}^2 \cong \mathbb{R}^4$ .
- (ii) Let us take a real vector space  $V$  and compute  $V \otimes \mathbb{C}$ , thinking of  $\mathbb{C}$  as an  $\mathbb{R}$ -vector space with basis  $\{1, i\}$ . If  $V$  has basis  $v_1, \dots, v_n$  then  $V \otimes_{\mathbb{R}} \mathbb{C}$  has  $\mathbb{R}$ -basis  $v_1 \otimes 1, v_1 \otimes i, \dots, v_n \otimes 1, v_n \otimes i$ : but that has a  $\mathbb{C}$ -vector space structure as the space spanned by  $v_1, \dots, v_n$  over  $\mathbb{C}$ . (Strictly speaking one should then write the multiplication by complex constants on the right.) This “change of coefficients” operation is a very common use of tensor product.
- (iii)  $\mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Z}/3$  is zero, because to determine  $\mathbf{f}$  it is enough to know  $\mathbf{f}(1, 1)$ , but then  $2\mathbf{f}(1, 1) = \mathbf{f}(2, 1) = \mathbf{f}(0, 1) = 0$  and  $3\mathbf{f}(1, 1) = \mathbf{f}(1, 3) = \mathbf{f}(1, 0) = 0$ .
- (iv)  $\mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Z} = \mathbb{Z}/2$  because again  $2\mathbf{f}(1, 1) = 0$  but if  $V = \mathbb{Z}/2$  one may still have  $\mathbf{f}(1, 1) = 1$ .
- (v) More generally,  $M \otimes_R R \cong M$ .
- (vi) If  $M$  is a  $\mathbb{Z}$ -module with free and torsion parts  $M_f$  and  $M_t$  then  $M \otimes \mathbb{Q} = M_f \otimes \mathbb{Q}$ , which is a  $\mathbb{Q}$ -vector space whose dimension is the rank of the free part of  $M$ .



- (vii) Tensor product is associative:  $(M \otimes N) \otimes L \cong M \otimes (N \otimes L)$ . This is annoyingly difficult to prove, despite being obvious.
- (viii) If  $V$  and  $W$  are  $K$ -vector spaces then  $W \otimes V^\vee \cong \text{Hom}(V, W)$ , by  $w \otimes \varphi \mapsto (v \mapsto \varphi(v)w)$ .

**VIII.15** The uniqueness in VII.35 can also be seen by tensoring  $M$  with different  $\mathbb{Z}$ -modules and comparing the ranks.

**VIII.16** By thinking about tensor products, we can give precise algebraic descriptions of some familiar objects. From here on we take  $R = k$  to be a field, so that the modules are vector spaces. To avoid some awkwardness, we assume that  $\text{char } k \neq 2$ .

**VIII.17** The tensors, with lower and upper indices, that often occur in applied mathematics, are in fact elements of  $V^{\otimes a} \otimes (V^\vee)^{\otimes b}$  for  $a$  upper and  $b$  lower indices (or vice versa – conventions vary). Here  $V$  is an  $n$ -dimensional vector space, normally over  $\mathbb{R}$ , equipped with a basis  $e_1, \dots, e_n$  with dual basis  $f_1, \dots, f_n$  for  $V^\vee$ , and thus identified with  $\mathbb{R}^n$ . A tensor  $T = T_{j_1, \dots, j_a}^{i_1, \dots, i_b}$  is a way of writing the element

$$T = \sum_{r=1}^a \sum_{s=1}^b T_{j_1, \dots, j_a}^{i_1, \dots, i_b} e_{j_1} \otimes \cdots \otimes e_{j_a} \otimes f_{i_1} \otimes \cdots \otimes f_{i_b}.$$

Very often  $V$  is really the tangent space to something.

**Example VIII.18** The computation of  $\mathbb{R}^2 \otimes \mathbb{R}^2$  in Example VIII.14(i) generalises easily. If  $V$  has a basis  $e_1, \dots, e_n$  then  $V^{\otimes r}$  is spanned by the elements  $e_{i_1} \otimes \cdots \otimes e_{i_r}$  (repetitions are allowed). Consequently,  $\dim V^{\otimes r} = n^r$ .

**VIII.19** The symmetric group  $S_r$  acts on  $V^{\otimes r}$  by permuting the factors: if  $\sigma \in S_r$  then

$$\sigma: e_{i_1} \otimes \cdots \otimes e_{i_r} \rightarrow e_{i_{\sigma(1)}} \otimes \cdots \otimes e_{i_{\sigma(r)}}.$$

**Proposition VIII.20** The action of  $S_r$  on  $V^{\otimes r}$  preserves exactly two potentially nontrivial subspaces

- (1)  $\text{Sym}^r V = \{x \in V^{\otimes r} \mid \sigma(x) = x \text{ for all } \sigma \in S_r\}$ ,
- (2)  $\wedge^r V = \{x \in V^{\otimes r} \mid \sigma(x) = \varepsilon(\sigma)x \text{ for all } \sigma \in S_r\}$ ,

where  $\varepsilon: S_r \rightarrow \pm 1 \leq k^*$  is the signature.

*Proof:* The two subspaces are preserved by definition. (We do not yet know whether they might be trivial.) Suppose that  $S_r$  preserves a subspace  $W < V^{\otimes r}$ . Then we have a map  $\alpha: S_r \rightarrow \text{GL}(W)$  given by  $\alpha(\sigma)(x) = \sigma(x)$ . (This is the same map  $\alpha$  that we saw in Proposition III.4.)

But then  $\text{Ker } \alpha \triangleleft S_r$  so  $(\text{Ker } \alpha) \cap A_r \triangleleft A_r$ . But we asserted in II.19 that  $A_r$  is simple (has no proper normal subgroups) if  $r \geq 5$  (a longer argument is needed for  $r \leq 4$ ). So  $(\text{Ker } \alpha) \cap A_r = 1$ , in which case it is easy to see that  $\text{Ker } \alpha = 1$

and so  $S_r$  acts faithfully and the only invariant subspace is  $V^{\otimes r}$ ; or  $\text{Ker } \alpha \geq A_r$ , in which case either  $\text{Ker } \alpha = S_r$ , which is case (i) ( $\alpha$  is trivial) or  $\text{Ker } \alpha = A_r$ , which is case (ii) ( $\alpha = \varepsilon$ ).

**VIII.21** If  $v, w \in V$  then clearly  $v \otimes w + w \otimes v \in \text{Sym}^2 V$ . More generally, we can find elements of  $\text{Sym}^r V$  by taking  $r$  elements of  $V$  and adding up all the tensors we can make with them in every order.

**Proposition VIII.22**  $\text{Sym}^r V$  is freely generated (i.e. they are a basis) by the elements

$$e_{i_1} \cdots e_{i_r} = \sum_{\sigma \in S_r} e_{i_{\sigma(1)}} \otimes \cdots \otimes e_{i_{\sigma(r)}}$$

for  $i_1, \dots, i_r \in \{1, \dots, n\}$ . In particular,  $\dim \text{Sym}^r V = \binom{n+r-1}{r-1} = \binom{n+r-1}{n}$ .

*Proof:* The first part is clear and the second uses a famous counting argument. We have  $n+r-1$  boxes in a row, and in each box we put either a ball marked with a number  $i \in \{1, \dots, n\}$  or a cube marked with a number  $j \in \{1, \dots, r-1\}$ . We put the cubes in first, and they have to go in order: cube 1 left of cube 2 and so on. The rule is that if the next cube to the right of an unfilled box is marked  $j$  then we should put a ball marked  $j$  in that box; if there is no cube to the right, put a ball marked  $r$ . Then read out the  $i$ s.

**VIII.23** If  $v, w \in V$  then clearly  $v \otimes w - w \otimes v \in \bigwedge^2 V$ . More generally, we can find elements of  $\text{Sym}^r V$  by taking  $r$  elements of  $V$  and adding up all the tensors we can make with them in every order, with a sign given by the signature. However, if two of the elements are the same then this will give zero, because the odd permutation that interchanges them will leave the expression unchanged but multiply the sum by  $-1$ .

**Proposition VIII.24**  $\bigwedge^r V$  is freely generated by the elements

$$e_{i_1} \wedge \dots \wedge e_{i_r} = \sum_{\sigma \in S_r} \varepsilon(\sigma) e_{i_{\sigma(1)}} \otimes \cdots \otimes e_{i_{\sigma(r)}}$$

for  $i_1, \dots, i_r \in \{1, \dots, n\}$  distinct. In particular,  $\dim \text{Sym}^r V = \binom{n}{r}$

*Proof:* This is immediate from the remarks above.

**VIII.25** It is also possible to construct  $\bigwedge^r V$  and  $\text{Sym}^r V$  directly, in the same way that we constructed tensor powers, by considering antisymmetric or symmetric bilinear or multilinear maps. Note that we can take the tensor product of two different modules, but we cannot do that with symmetric or wedge (exterior) products: we need the action of  $S_r$ .

**VIII.26** Notice that if  $r > n$  then  $\bigwedge^r V = 0$ : that is why we wrote “potentially nontrivial” in the statement of VIII.20.

Notice also that  $\dim \bigwedge^n V = 1$ . This has a striking effect.

**Proposition VIII.27** A linear map  $\varphi: V \rightarrow V$  induces linear maps

$$\text{Sym}^r \varphi: S^r V \rightarrow S^r V$$

and

$$\wedge^r \varphi: \wedge^r V \rightarrow \wedge^r V.$$

*Proof:* These are given by

$$\text{Sym}^r \varphi(v_{i_1} \cdots v_{i_r}) = \varphi(v_{i_1}) \cdots \varphi(v_{i_r})$$

and

$$\wedge^r \varphi(v_{i_1} \wedge \cdots \wedge v_{i_r}) = \varphi(v_{i_1}) \wedge \cdots \wedge \varphi(v_{i_r})$$

extended by linearity. (It is enough to do this for the basis elements  $e_i$  but it is not necessary to choose a basis in order to make the definition).

**VIII.28** In particular  $\varphi: V \rightarrow V$  induces  $\wedge^n \varphi: \wedge^n V \rightarrow \wedge^n V$ . But  $\wedge^n V$  is 1-dimensional so  $\text{Hom}(\wedge^n V, \wedge^n V) \cong \text{Hom}(k, k) \cong k$ .

The map  $\wedge^n \varphi \in k = \text{Hom}(k, k)$  is  $\det \varphi$ , which we have now defined directly, without using matrices.

To recover the matrix version, notice that it follows that  $\det$  is, up to scaling, the only  $n$ -linear alternating map from  $V$  to  $k$ . If we regard an  $n \times n$  matrix as consisting of  $n$  elements of  $V$  (the rows) then the usual computation of determinant is  $n$ -linear and alternating (this is how you do row reduction) so it agrees with  $\wedge^n$  up to a scalar: we normalise by agreeing that the identity matrix and the identity map have determinant 1.