ALGEBRA 2B (MA20217)

PROBLEM SHEET 8 WITH SOLUTIONS

1. Verify that the relation defined on T in the construction of $\mathcal{Q}(R)$, in Lemma V.20, is in fact an equivalence relation.

Solution: *The relation is $(a,b) \sim (c,d) \iff ad - bc = 0$, i.e. if and only if $ad = bc$. That is visibly reflexive ($ab = ab$) and symmetric ($ab = cd \iff cd = ab$) so only transitivity needs any thought at all. But even that is trivial: $ab = cd$ and $cd = ef$ implies $ab = ef$,*

2. In this question R is a commutative ring and I and J are ideals in R. Say whether each of the following statements is true or not: give a proof or a counterexample.

(a) If I and J are both prime ideals then $I \cap J$ is a prime ideal.

(b) If I and J are both prime ideals then IJ is a prime ideal.

(c) If I and J are both prime ideals then $I + J$ is a prime ideal.

(d) If I and J are both maximal ideals then $I \cap J$ is a maximal ideal.

(e) If I and J are both maximal ideals then IJ is a maximal ideal.

(f) If I and J are both maximal ideals then $I + J$ is a maximal ideal.

Solution:

(a) This false in $\mathbb{Z}$, because $\langle p\rangle \cap \langle q\rangle = \langle \mathrm{lcm}(p,q)\rangle$ (Example V.39), which is never prime if p and q are different primes: for example $\langle 2\rangle \cap \langle 3\rangle = \langle 6\rangle$. The point is that if $ab \in I \cap J$ then a or b belongs to I and a or b belongs to J, but we don't have a way to exclude the possibility that $a \in I$ and $b \in J$ but neither is in $I \cap J$.

(b) Again this fails in $\mathbb{Z}$ by Example V.39, because $\langle p\rangle\langle q\rangle = \langle pq\rangle$ so the same counterexample as in (a) works.

(c) This fails in $\mathbb{Z}$ for the trivial reason that if p and q are different primes then they are coprime so $\langle p\rangle + \langle q\rangle = \mathbb{Z}$ which is explicitly excluded from being a prime ideal by the definition.

The last three may be done by simply pointing out that in $\mathbb{Z}$, all non-zero prime ideals are maximal, so all the counterexamples above still work. There is plenty of room for discussion (can we ever make any of these be prime or maximal? can we make $I + J$ be a proper non-prime ideal?) if anybody is in the mood for that.

3. Show directly that if R is a PID and $p \in R$ is irreducible then the ideal pR is a maximal ideal. Deduce that R/pR is a field (Theorem VI.22).
Solution: *I sort of did this in the lectures. Suppose $a \notin pR$ and look at $pR + aR$. This is equal to dR for some $d \in R$, and $d|p$ and $d|a$. Since $d|p$ and p is irreducible, either d is a unit or $p = de$ with e a unit. But if e is a unit then $dR = pR$ so $p|a$, contrary to the assumption: therefore d is a unit so $aR + pR = R$. This shows that pR is maximal: any attempt to make it bigger immediately generates the whole of R. (The proof in the notes is indirect in that it shows that R/pR is a field.)*

4. What is the content of each of the following polynomials?

(a) $3x^3 - 12x^2 - 9 \in \mathbb{Z}[x]$.

(b) $3x^3 - 12x^2 - 9 \in \mathbb{Q}[x]$.

(c) $3wx^3 - 12wx^2 - 9w \in R[x]$, where $R = \mathbb{Z}[w]$.

(d) $3wx^3 - 12wx^2 - 9w \in S[w]$, where $S = \mathbb{Z}[x]$.

(e) $3wx^3 + 3x^3 - 12w^2x^2 - 12wx^2 - 9w - 9 \in R[x]$, where $R = \mathbb{Z}[w]$.

Solution:

(a) 3.

(b) 1 *(although 3 is not wrong, as it is a unit too).*

(c) $3w$.

(d) $3x^3 - 12x^2 - 9$, *since the expression is equal to* $(3x^3 - 12x^2 - 0)w$.

(e) $3(w + 1)$.

5. Say whether each of the following polynomials is reducible or irreducible, giving reasons.

(a) $3x^3 - 12x^2 - 9 \in \mathbb{Z}[x]$.

(b) $3x^3 - 12x^2 - 9 \in \mathbb{Q}[x]$.

(c) $x^2 + 5x - 3 \in \mathbb{F}_{11}[x]$.

(d) $x^2 + 5x - 3 \in \mathbb{F}_{13}[x]$.

(e) $x^2 + 5x - 3 \in \mathbb{F}_{37}[x]$.

(f) $x^3 + 5x - 3 \in \mathbb{F}_{13}[x]$.

(g) $x^3 + 5x - 3 \in \mathbb{F}_{11}[x]$.

Solution:

(a) $3x^3 - 12x^2 - 9 \in \mathbb{Z}[x]$ *is reducible because it has a factor of* 3.

(b) $3x^3 - 12x^2 - 9 \in \mathbb{Q}[x]$ *is irreducible because* 3 *is now a unit so we are interested in factorising* $x^3 - 4x^2 - 3$*: we know that if we can do that in* $\mathbb{Q}[x]$ *then we can do it in* $\mathbb{Z}[x]$ *and then we would have to have a monic linear factor, i.e. an integer root. There aren't any of those: they would have to be odd,* ± 1 *don't work and neither does* ± 3 *(no need to compute actual values, they won't be divisible by* 9*) and* 5 *is too big.*

(c) $x^2 + 5x - 3 \in \mathbb{F}_{11}[x]$ *has discriminant* $25 + 12 = 37 = 4$ *which is a square so this is reducible.*

(d) $x^2 + 5x - 3 \in \mathbb{F}_{13}[x]$ *has discriminant* $37 = -2$ *which is not a square (because 13 is 5 mod 8, but it is not hard to check the squares mod 13 by hand).*

(e) $x^2 + 5x - 3 \in \mathbb{F}_{37}[x]$ *has discriminant* $37 = 0$ *so this is in fact a square,* $(x + 21)^2$ *to be precise.*

(f) $x^3 + 5x - 3 \in \mathbb{F}_{13}[x]$ *has the root* 3 *so has a factor of* $x - 3$ *so it is reducible.*

(g) $x^3 + 5x - 3 \in \mathbb{F}_{11}[x]$ *has no root: you just try all eleven possibilities. So it is irreducible.*

6. Say whether each of the following polynomials is reducible or irreducible in $\mathbb{Q}[x]$, giving reasons.

(a) $x^4 - 10x^3 - 15 \in \mathbb{Z}[x]$.

(b) $x^4 - 10x^3 - 15 \in \mathbb{Q}[x]$.

(c) $x^4 - x^3 - 10x^2 + 7x + 3 \in \mathbb{Q}[x]$.

(d) $x^4 - 14x^3 + 36x^2 - 34x - 4 \in \mathbb{Q}[x]$.

(e) $x^3 + 5x - 3 \in \mathbb{Q}[x]$.

Solution:

(a) $x^4 - 10x^3 - 15 \in \mathbb{Z}[x]$ *is irreducible by Eisenstein's criterion with* $p = 5$.

(b) $x^4 - 10x^3 - 15 \in \mathbb{Q}[x]$ *is irreducible for the same reason.*

(c) $x^4 - x^3 - 10x^2 + 7x + 3 \in \mathbb{Q}[x]$ *is reducible: it has a factor of* $x - 1$. *More generally, if a quartic doesn't have linear factors and doesn't yield to Eisenstein's criterion quickly, then it may well have two quadratic factors and those can often be guessed by factorising the constant term and trying to choose the coefficients of* x *so as to make everything else work out.*

(d) $x^4 - 14x^3 + 36x^2 - 34x - 4 \in \mathbb{Q}[x]$ *is irreducible. It is not Eisenstein for* $p = 2$ *because of the constant term* 4*, but putting* $x = y + 1$ *gives* $y^4 - 10y^3 - 15$ *which is Eisenstein with* $p = 5$. *Generally, with small integer coefficients, if Eisenstein doesn't do the job immediately, it is worth trying* $x = y \pm 1$ *but if those don't work either it's time to start looking for factors.*

(e) $x^3 + 5x - 3 \in \mathbb{Q}[x]$ *is irreducible because we have already seen that it is irreducible mod 11 (unfortunately it is reducible mod 5 because* $2^3 = 3$*).*

GKS, 19/4/24