ALGEBRA 2B (MA20217)

PROBLEM SHEET 7 WITH SOLUTIONS

1. Give proofs or counterexamples to each of the following statements.

(a) If G is a group and G acts on X, then for any normal subgroup $K \lhd G$ the rule $(Kg)(x) := g(x)$ defines an action of G/K on X.

(b) If G acts on X and $H \leq G$ then $\mathrm{Stab}_H(x) \leq \mathrm{Stab}_G(x)$.

(c) If G acts on X and $K \lhd G$ then $\mathrm{Stab}_K(x) \lhd \mathrm{Stab}_G(x)$.

(d) If G_1 and G_2 act on X and $G_1 \cong G_2$, then $\mathrm{Stab}_{G_1}(x) \cong \mathrm{Stab}_{G_2}(x)$ for every $x \in X$.

(e) If G_1 and G_2 act on X and $G_1 \cong G_2$, then $|\,\mathrm{orb}_{G_1}(x)| = |\,\mathrm{orb}_{G_2}(x)|$ for every $x \in X$.

(f) The action of $\mathrm{SL}(2, \mathbb{Z})$ on the upper half-plane $\mathbb{H} \subset \mathbb{C}$ by Möbius transformations is transitive.

(g) The action of $\mathrm{SL}(2, \mathbb{R})$ on the upper half-plane $\mathbb{H} \subset \mathbb{C}$ by Möbius transformations is transitive.

(h) If R is a ring of characteristic m and I is an ideal of R then $\mathrm{char}\, R/I = m$.

(i) If R is a ring of characteristic m and I is an ideal of R then $\mathrm{char}\, R/I$ divides m.

Solution:

(a) No, this is not well defined (unless K acts trivially). For example, $G = S_n$ and $K = A_n$ with the usual action on $\{1, \dots, n\}$ then $K(12) = K(13)$ so $K(12)(x)$ should be equal to $K(13)(x)$ for any x, but $(12)(3) = 3 \neq 1 = (13)(3)$.

(b) Yes: $\mathrm{Stab}_H(x) = \{g \in H \mid gx = x\} \leq< \mathrm{Stab}_G(x) = \{g \in G \mid gx = x\}$. In fact $\mathrm{Stab}_H(x) = H \cap \mathrm{Stab}_G(x)$.

(c) Yes. If $k \in \mathrm{Stab}_K(x)$ and $g \in \mathrm{Stab}_G(x)$ then $g^{-1}kg(x) = g^{-1}kx = g^{-1}x = x$ so $g^{-1}kg \in \mathrm{Stab}_G(x)$, but also $g^{-1}kg \in K$ because K is normal in G.

(d) *No. E.g.* $\mathbb{Z}/2$ *acts on* $\{0,1,2\}$ *by interchanging* 1 *and* 2 *and also by interchanging* 0 *and* 1*: in the first case the stabiliser of* 0 *is* $\mathbb{Z}/2$ *but in the second case it is trivial. Or the action might not be faithful, e.g.* $\mathbb{Z}/2$ *acts on* $\{0,1\}$ *by interchanging them but there is another action that simply does nothing, and again the stabilisers are different. Even if the action is faithful and we allow ourselves to permute* X *as well we still can't be sure: compare the* $\mathbb{Z}/2$ *actions on* $\{1,2,3,4\}$ *by* (12) *and by* $(12)(34)$*. In the second case all stabilisers are trivial: in the first case, not so.*

(e) *Again no: for instance, if we take the first counterexample to (d) above then* G_1 *is finite and the stabilisers are of different orders so the orbits must be of different siezes too, by the orbit-stabiliser theorem.*

(f) *No. For example it is enough to observe that* $\mathrm{SL}(2,\mathbb{Z})$ *is countable and* $\mathbb{H}$ *isn't, so* $\mathbb{H}$ *can't be an orbit of* $\mathrm{SL}(2,\mathbb{Z})$.

(g) *Yes. Suppose* $\tau = \alpha i + \beta$*. Then* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *moves* τ *to* $\frac{a\alpha i + (a\beta + b)}{ci+d}$ *and we can make this be* i *by taking* $c = 0$*: then we need* $a\alpha = d$ *and* $a\beta + b = 0$*, and to ensure that the determinant is* 1 *we also need* $ad = 1$*. This gives* $d = a\alpha$ *so* $a^2\alpha = 1$ *so we should take* $a = \frac{1}{\sqrt{\alpha}}$ *(remember that* $\tau \in \mathbb{H}$ *so* $\alpha > 0$*) and* $b = -a\beta = \frac{-\beta}{\sqrt{\alpha}}$*. So* i *is in the orbit of* τ*, so there is only one orbit.*

(h) *No.* $\mathbb{Z}/2$ *is not of characteristic zero, even though* $\mathbb{Z}$ *is.*

(i) *Yes. If* $\mathrm{char}\, R = 0$ *there is nothing to prove. Otherwise the prime subring of* R/I *is a quotient of the prime subring of* R*, and we can forget about multiplication and just think about just the additive groups. Now we have a finite group of order* $\mathrm{char}\, R$ *and a quotient of it of order* $\mathrm{char}\, R/I$*. But if* G *is a group and* K *a normal subgroup then* $|G| = |G/K||K|$ *by Lagrange.*

2. In each of the following cases, say whether φ is a ring homomorphism or not. Give reasons. If it is, say what the kernel and image of φ are.

(a) $\varphi\colon \mathbb{Z}[i] \to \mathbb{Z}$ given by $\varphi(z) = \mathrm{Re}(z)$, the real part of z.

(b) $\varphi\colon M_{3\times 3}(\mathbb{C}) \to \mathbb{C}$ given by $\varphi(A) = \det(A)$.

(c) $\varphi\colon \mathbb{R}[t] \to \mathbb{R}[t]$ given by $\varphi(f(t)) = f(t^2)$.

(d) $\varphi(M_{2\times 2}(\mathbb{Z}) \to M_{2\times 2}(\mathbb{F}_2))$ by $\varphi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}$ where for $n \in \mathbb{Z}$ we define $\bar{n} = 1$ if n is odd and $\bar{n} = 0$ if n is even.

Solution:

(a) No: e.g. $\varphi(i^2) = -1$ *but* $\varphi(i)^2 = 0$.

(b) No: e.g. $\det(I - \delta_{11}) = 0$ *but* $\det(I) - \det(\delta_{11}) = 1$ *(where* δ_{11} *has a* 1 *in the* $(1, 1)$ *place and* 0 *everywhere else).*

(c) Yes, by the definitions of fg *and* $f + g$*. The kernel is* 0 *and the image is* $\mathbb{R}[t^2] = \{a_0 + a_1t + \ldots + a_dt^d \mid a_i = 0$ *if* i *is odd*$\}$.

(d) Yes, because matrix multiplication is by ring operations on the entries. The kernel is the set of matrices whose entries are all even.

3.

(a) Which of the following rings is an integral domain? Give reasons.

$$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}; \qquad \mathbb{Z}/10\mathbb{Z}; \qquad \mathbb{Z}[\sqrt{10}].$$

(b) What is the characteristic of each of the following rings?

$$\mathbb{Z}; \qquad \mathbb{Z}/15\mathbb{Z}; \qquad R/3R, \text{ where } R = \mathbb{Z}/15\mathbb{Z}.$$

(c) If $R = \mathbb{Z}[\pi]$, what is the field of fractions $\mathcal{Q}(R)$?

(d) Suppose that R is an integral domain, $F = \mathcal{Q}(R)$ and S is a proper subring of F with $1_R \in S$, such that $\mathcal{Q}(S) = F$. Does this necessarily imply that $S = R$? You must give a proof or a counterexample.

(e) Suppose that R is an integral domain. For each of the following rings A, say whether A is always a domain; never a domain; or possibly a domain, depending on what R is. Give brief proofs or counterexamples.

  (a) A is the direct product $R \times R$.

  (b) A is the ring of formal power series $R[[t]]$ (this is ring whose elements are power series $\sum_{i=0}^{\infty} a_it^i$ with $a_i \in R$: "formal" means we don't worry about whether they converge, even if it makes sense to ask that question).

  (c) $R/6R$, where 6 means $1_R + 1_R + 1_R + 1_R + 1_R + 1_R$.

Solution:

(a) The first isn't a domain because $(0, 1) \cdot (1, 0) = (0, 0)$*; the second is the same ring as the first anyway by the Chinese Remainder Theorem; the third is a domain, because it is a subring of* $\mathbb{R}$ *which is an integral domain.*

(b) 0*, since no amount of adding* 1*s gets to the integer* 0*;* 15*;* 3*, since* $1 + 1 + 1 = 3 \in 3R$.

(c) $\mathbb{Q}(\pi)$*, the set of rational functions of* π. *Not* $\mathbb{R}$*, still less* $\mathbb{C}$.

(d) It's not true in general: for example $\mathbb{Q} = \mathcal{Q}(\mathbb{Z}[1/2]) = \mathcal{Q}(\mathbb{Z})$.

(e)
  - *(i) Never a domain, because* $(1, 0)(0, 1) = (0, 0) = 0_{R\times R}$ *but* $1_R \neq 0_R$.
  - *(ii) Always a domain, because we may write two nonzero elements of* $R[[t]]$ *as* $at^i + h.o.t.$ *and* $bt^j + h.o.t.$ *with* a*,* $b \neq 0$*, and then their product is* $abt^{i+j} + h.o.t$ *which is also not zero.*
  - *(iii) This depends on* R*. For example if we take* $R = \mathbb{Z}$ *we get* $A = \mathbb{Z}/6\mathbb{Z}$ *which is not a domain but if we take* $R = \mathbb{Z}/3\mathbb{Z}$ *then* $6 = 0$ *so we get* $A = R/6R = \mathbb{Z}/3\mathbb{Z}$ *again, which is a domain.*

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