ALGEBRA 2B (MA20217)

PROBLEM SHEET 4 WITH SOLUTIONS

1 W Prove the assertions in III.17(v) in the notes: that in the action of $\mathrm{SL}(2,\mathbb{Z})$ on the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$, the stabiliser of most $z \in \mathbb{H}$ is $\pm I$, but the stabiliser of $i \in \mathbb{H}$ is a group of order 4 generated by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and the stabiliser of $\omega = e^{2\pi i/3}$ is of order 6, generated by $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$.

Solution: *It is important to show both inclusions. Clearly* $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}(i) = \frac{1}{-i} = i$ *and since* $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ *we have* $1 + \omega = \frac{1}{2} + i\frac{\sqrt{3}}{2} = e^{\pi i/3}$. *Thus* $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}(\omega) = \frac{-1}{1+\omega} = -e^{-\pi i/3} = e^{\pi i - \pi i/3} = e^{2\pi i/3} = \omega$. *But we also need to show that there is nothing else.*
If $\frac{ai+b}{ci+d} = i$ *then* $ai + b = -c + di$ *so* $d = a$ *and* $b = -c$ *so the only elements that stabilise* i *are* $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ *with* $a^2 + b^2 = 1$, *and the only way to satisfy* $a^2 + b^2 = 1$ *in integers is* $a = 0$ *and* $b = \pm 1$ *or* $b = 0$ *and* $a = \pm 1$, *as required.*
Similarly, if $\frac{a\omega+b}{c\omega+d} = \omega$ *then* $a\omega + b = c\omega^2 + d\omega$. *They will now probably use* $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ *again, which is fine, but I prefer* $\omega^2 = -1 - \omega$ *so* $a\omega + b = -c - c\omega + d\omega$ *which (since* 1 *and* ω *are linearly independent over* $\mathbb{Q}$*) gives* $b = -c$ *and* $a + c = d$. *The determinant is* 1 *so* $ad + c^2 = 1$ *so* $a^2 + ac + c^2 = 1$. *Let's try to find solutions, treating it as a quadratic in* a. *There are real solutions only if the discriminant* $c^2 - 4(c^2 - 1)$ *is non-negative, so we must have* $4 \geq 3c^2$ *so* $c = \pm 1$ *or* $c = 0$, *and similarly for* a. *Of these, only* $(a, c) = (\pm 1, 0)$, $(a, c) = (0, \pm 1)$ *and* $(a, c) = (\pm 1, \mp 1)$ *actually give solutions, and those give the six matrices required.*

2 H Prove the assertion in the proof of Proposition III.18, that left multiplication by G on $X = \{gH \mid g \in G\}$ defines a group action and that the stabiliser of $1_G H$ is H.

Solution: *We need to check that if* $g_1, g_2 \in G$ *and* $gH \in X$ *then* $g_1(g_2 gH) = (g_1 g_2)gH$, *and that* $1(gH) = gH$, *according to Definition III.2. But the first two are both equal to* $g_1 g_2 gH$ *and the second is trivial. For the stabiliser, this is the statement that* $gH = H$ *if and only if* $g \in H$, *which is a case of Corollary II.6.*

3 W Is it true that if a finite G acts on a set X and the orbits $\operatorname{orb}_G(x)$ and $\operatorname{orb}_G(y)$ are the same size, then $\operatorname{Stab}_G(x) \cong \operatorname{Stab}_G(y)$? Give a proof or a counterexample.

Solution: *False in general. You need two different subgroups of the same order, so let's take* $H_1 = \mathbb{Z}/2 \times \mathbb{Z}/2$ *and* $H_2 = \mathbb{Z}/4$ *(but any other pair, such as* $\mathbb{Z}/6$ *and* S_3*, would do just as well). Let* $G = H_1 \times H_2$ *and let* G *act on* $X = H_1 \sqcup H_2$ *(recall that* $\sqcup$ *means disjoint union) by* $(h_1, h_2)(x) = h_1 x$ *if* $x \in H_1$ *and* $(h_1, h_2)(x) = h_2 x$ *if* $x \in H_2$*. It is immediate that this is an action. Now* $\mathrm{Stab}_G(1_{H_1}) = \{(1_{H_1}, h_2) \mid h_2 \in H_2\} \cong H_2$ *and* $\mathrm{Stab}_G(1_{H_2}) = \{(h_1, 1_{H_2}) \mid h_1 \in H_1\} \cong H_1$*: these two have the same order, so the orbits are the same size, but they are not isomorphic by the choice we made.*

4 H There are fifteen ways of organising six objects into pairs (unordered). Show this directly by counting. Then use the action of the symmetric group S_6 to give a different proof using the orbit-stabiliser theorem.
How many ways are there of organising $2k$ objects into k pairs?
Solution: *Directly: we need to know what to pair with object* **1** *(so five choices) and then what to pair with the next free object (either* **2**, *or* **3** *if we've paired* **2** *with* **1**)*, and that's three choices. So* $5 \times 3 = 15$ *ways. There are other ways you can do this count: your correct method may not be exactly the same.*
By stabilisers: the permutation action of S_6 *on the objects also acts on the pairings. This action is transitive, because if I can get to the pairing* **a** *&* **b**, **c** *&* **d**, **e** *&* **f** *from the standard pairing* **1** *&* **2**, **3** *&* **4**, **5** *&* **6** *by using the permutation that send* **1** *to* **a**, *etc. So we can look at the stabiliser of any pairing: let's just use the one we've just called the standard one. Any of the transpositions* (12), (34) *and* (56) *will do that, and they generate a group of order eight, isomorphic to* $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$*. So will permutations of the pairs themselves, which is a copy of* S_3 *of order six (strictly,* $\sigma \in S_3$ *sends* **2n-1** *&* **2n** *to* **2m-1** *&* **2m** *if* $\sigma(n) = m$*). That's a stabiliser of order* $8 \times 6 = 48$ *so the orbit is of size* $6!/48 = 15$.
If instead we have $2k$ *objects, we can take either approach. The direct approach gives* $(2k-1)(2k-3)\ldots 1$*; the approach by stabilisers gives a stabiliser of order* 2^k *(for the transposition of each pair) times* $k!$ *(permuting the pairs) and hence orbit of size* $\frac{(2k)!}{2^k k!}$.

GKS, 8/3/24