ALGEBRA 2B (MA20217)

SOLUTIONS TO PROBLEM SHEET 2

1 W In each of the following cases say what the order $|G|$ of G is.

(a) $G = S_5$, the symmetric group on 5 letters.

(b) The alternating group A_5.

(c) $\mathbb{Z}/n$.

(d) The subgroup $n\mathbb{Z}$ of $\mathbb{Z}$

Solution:

(a) $5!$, *which is* 120.

(b) 60, *namely* $\frac{5!}{2}$ *because* $S_5 = A_5 \cup (12)A_5$ *and the union must be disjoint because different cosets don't meet.*

(c) n. *Note that if* p *is a prime then* $(\mathbb{Z}/p)^*$ *is also a group, under multiplication this time, and is of order* $p-1$. *Don't confuse these two!*

(d) ∞.

2 H In each of the following cases say what the order $o(g)$ of g is. Verify that $o(g)$ divides $|G|$.

(a) $G = S_5$ and $g = (13)(245)$.

(b) $G = \mathrm{SL}(2, \mathbb{F}_2)$ and $g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. (By $\mathbb{F}_2$ we mean $\mathbb{Z}/2$ considered as a field, i.e. with both addition and multiplication mod 2.)

Solution:

(a) g *has order* 6*: either because you know that the order of a permutation is the lowest common multiple of its cycle lengths or just by computing* g^2, g^3 *and so on, and* 6 *does indeed divide* 120.

(b) We can simply compute that $g^2 = \mathrm{id}$ *so* $o(g) = 2$. *The harder part is finding the order of* $\mathrm{SL}(2, \mathbb{F}_2)$, *which is most simply done by simply writing down the six elements. They are the identity along with*

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

In other words: $\mathrm{SL}(2, \mathbb{F}_2) = \mathrm{GL}(2, \mathbb{F}_2)$ *(if the determinant is not* 0*, it is* 1*) and the columns form an ordered basis, so the first column may be any nonzero vector in* $\mathbb{F}_2$ *(three possibilities) and the second column may be either of the other two.*

3 W Suppose that G is a cyclic group of finite order n and that g generates G (in this case, we say that g is a generator). Is g^2 a generator? Which other elements of G are generators? How many of them are there?

Solution: *g^2 is a generator if n is odd but not if n is even. More generally, g^k generates G if and only if $\mathrm{hcf}(k, n) = 1$. So the number of generators is $\phi(n) = \prod_{p|n} n(1 - \frac{1}{p})$ (where p is prime), the Euler ϕ function.*

4 W By considering left and right cosets, show that if $H < G$ and $|G : H| = 2$ then $H \lhd G$. Give an example of a group G and a non-normal subgroup of index 3.

Solution: *There are two left cosets of H and one of them is H: call the other gH. Then $Hg \subset G = H \cup gH$, but $g \notin H$ since otherwise $gH = H$ (see II.2 in the lecture notes) so $Hg \cap H = \varnothing$ (otherwise $gh = h'$ for some $h\, h' \in H$ and then $g = h'h^{-1} \in H$). So $Hg \subseteq gH$. Similarly, $gH \subseteq Hg$ so $gH = Hg$ so $H \lhd G$. The subgroup H generated by (12) is non-normal of index 3 in S_3, since $(132)(12)(123) = (13) \notin H$.*

5 H Suppose that G is a group and H is a subgroup.

(a) Verify that there is no such thing as "left index" and "right index": the number of left cosets of H in G is equal to the number of right cosets. You may wish to consider the map $gH \mapsto Hg^{-1}$.

(b) Suppose that every non-identity element of G has order 2. Show that G is abelian.

(c) Show that the alternating group A_4, which is of order 12, does not have a subgroup of order 6: hence the converse of Lagrange's Theorem is false in general.

Solution:

(a) The map given is a bijection between the set of left cosets and the set of right cosets. To show this it is only necessary to check that it is well defined. If $g_1H = gH$ then $g_1 = gh$ for some $h \in H$ and then $Hg_1^{-1} = Hh^{-1}g^{-1} = hg^{-1}$, which is all that we need. Once we know that the map is defined, we can see that it is a bijection because its inverse is $Hg \mapsto g^{-1}H$.

(b) Suppose a, $b \in G$. Then we claim that $ab = ba$, i.e. that $aba^{-1}b^{-1} = 1$. But $a^{-1} = a$ and $b^{-1} = b$ since $a^2 = b^2 = 1$, s $aba^{-1}b^{-1} = abab = (ab)(ab) = 1$ because ab also has order 2 (or order 1).

(c) One way to see this is as follows: A_4 contains the identity, three elements of order 2 such as $(12)(34)$ and eight elements of order 3 such as (123). A subgroup H of order 6 would have to contain the identity and if it contains an element of order 3 it must also contain its inverse. So it

contains an odd number of elements of order 2*, either just one or all of them. If it contains all of them then it contains* $(12)(34)$ *and a* 3*-cycle, say* (123)*, and its inverse, but no more; but* $(12)(34)(123) = (243)$*. If it contains only one, say* $(12)(34)$*, then it also contains a* 3*-cycle, say* (123)*, and its inverse: but* $(321)(12)(34)(123) = (13)(24)$*.*

GKS, 26/2/24