

## ALGEBRA 2B (MA20217)

### SOLUTIONS TO PROBLEM SHEET 1

**1 W.** In each of the following cases say whether  $H$  is a subgroup of  $G$ . If it is, say whether it is a normal subgroup.

- (a)  $G = \mathbb{Z}$  and  $H = \{2^k \mid k \in \mathbb{N}\}$
- (b)  $G = \mathbb{Q}^*$  (with multiplication) and  $H = \{2^k \mid k \in \mathbb{Z}\}$ .
- (c)  $G = \text{SL}(2, \mathbb{R})$  and  $H = \text{SL}(2, \mathbb{Z})$ .

**Solution:**

- (a) *This not a subgroup: for example, 2 has no inverse. This is because  $-2 \notin H$ , not because  $\frac{1}{2} \notin H$ : the alleged group operation is addition, the same as in  $\mathbb{Z}$ , not multiplication.*
- (b) *This is a subgroup because  $H \neq \emptyset$  and  $2^k(2^l)^{-1} = 2^{k-l} \in H$ , which verifies the condition for  $H$  to be a subgroup in Lemma I.14. Notice that here the group operation is multiplication: that's what it has to be to make  $\mathbb{Q}^*$  into a group, so we don't have to specify it explicitly.  $H$  is a normal subgroup, because  $G$  is abelian.*
- (c) *This is a subgroup, because the condition of being in  $\text{SL}$  is that the determinant is 1 so the inverse of an integral matrix is again integral, and of course the product of integral matrices is integral. But it is not normal: e.g.*

$$\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{5}{4} \\ 1 & -\frac{1}{2} \end{pmatrix}$$

*so we have found an element  $g \in \text{SL}(2, \mathbb{R})$  and an element  $h \in \text{SL}(2, \mathbb{Z})$  such that  $g^{-1}hg \notin \text{SL}(2, \mathbb{Z})$ .*

**2 H** Prove or disprove the following statements.

- (a) If  $G$  is a group and  $H$  and  $K$  are subgroups of  $G$ , then  $H \cap K$  is always a subgroup of  $G$ .
- (b) If  $G$  is a group and  $H$  and  $K$  are subgroups of  $G$ , then  $H \cup K$  is always a subgroup of  $G$ .
- (c) If  $G$  is a group and  $H$  and  $K$  are subgroups of  $G$ , and  $K \triangleleft G$  then  $H \cap K \triangleleft G$ .

- (d) If  $G$  is a group and  $H$  and  $K$  are subgroups of  $G$ , and  $K \triangleleft G$  then  $H \cap K \triangleleft H$ .

**Solution:**

- (a) Yes. It is enough to check that  $g_1g_2^{-1} \in H \cap K$  if  $g_1, g_2 \in H \cap K$ ; but in that case,  $g_1g_2^{-1} \in H$  because  $g_1, g_2 \in H$  and similarly for  $K$ .
- (b) No. For example, take  $G = \mathbb{Z}$ , and  $H = 2\mathbb{Z}$  and  $K = 3\mathbb{Z}$ . Then  $H \cup K \not\cong 5 = 2 + 3$  so  $H \cup K$  is not closed under the group operation (which is  $+$  here).
- (c) No. It is not enough to say “it is not necessarily the case that  $ghg^{-1} \in H \cap K$ ”: you must give a counterexample, not just a failed attempt at a proof. For example, we know that  $A_n \triangleleft S_n$  (check it!), so let us take  $G = S_4$  and  $K = A_4$ . Then we could take  $H$  to be the cyclic subgroup of order 3 generated by  $(123)$ : that is contained in  $A_3$  so  $K \cap H = H$ , but  $(14)(123)(14) = (423) \notin H$  so  $H$  is not a normal subgroup of  $G = S_4$ .
- (d) Yes. If  $h \in H$  and  $k \in H \cap K$  then  $h^{-1}kh \in K$  because  $K \triangleleft G$  and  $h \in G$ , and  $h^{-1}kh \in H$  because  $h, k \in H$ . With (iii), this gives an example of a subgroup that is normal in  $H$  but not in  $G > H$ ; a simpler example is to take any non-normal subgroup of  $H$  and observe that  $H \triangleleft H$ .

**3 W** Discuss the question in 1.17: is  $\mathbb{Z}/2$  a subgroup of  $\mathbb{Z}/6$ ? Is  $\mathbb{Z}/3$  a subgroup of  $\mathbb{Z}/6$ ? Also: is  $\mathbb{Z}/2$  a subgroup of  $\mathbb{Z}$ ?

**Solution:** The last of these is clear: no, because  $\mathbb{Z}/2$  contains an element of order 2 and  $\mathbb{Z}$  does not. But the first at least is genuinely debatable. At one level, no: residue classes mod 2 are not residue classes mod 6. On the other hand, there is a unique subgroup of  $\mathbb{Z}/6$ , uniquely isomorphic to  $\mathbb{Z}/2$ , generated by 3. What about  $\mathbb{Z}/3$ ? Again there is a unique subgroup isomorphic to  $\mathbb{Z}/3$  but is there really a preferred isomorphism? Do you care if there isn't?

**4 W/H** Prove:

- (a) **(W)** Lemma 1.27 (the image of a group homomorphism is a group);
- (b) **(H)** Lemma 1.29 (a group homomorphism is injective if and only if its kernel is trivial);
- (c) **(W)** 1.34 (isomorphism is an equivalence relation on groups: first formulate a precise statement of this).

**Solution:**

- (a) If  $\varphi: G \rightarrow H$  is a group homomorphism and  $h_1, h_2 \in H_0 := \text{Im } \varphi$  then  $h_1 = \varphi(g_1)$  and  $h_2 = \varphi(g_2)$  for some  $g_1, g_2 \in G$  then

$$h_1 h_2^{-1} = \varphi(g_1) \varphi(g_2)^{-1} = \varphi(g_1) \varphi(g_2^{-1}) \in \text{Im } \varphi = H_0$$

So  $H_0$  is a subgroup as long as it is not empty: but  $H_0 \ni \varphi(1_G) = 1_H$ .

- (b) If  $\text{Ker } \varphi$  (notation as above) is not trivial then it contains at least two elements ( $1_G$  and something else) both of which are mapped to  $1_H$  by  $\varphi$ , so  $\varphi$  is not injective. Conversely, suppose that  $\text{Ker } \varphi = \{1_G\}$  and suppose that  $\varphi(g_1) = \varphi(g_2)$ , for  $g_1, g_2 \in G$ . Then  $1_H = \varphi(g_1) \varphi(g_2)^{-1} = \varphi(g_1 g_2^{-1})$ , so  $g_1 g_2^{-1} \in \text{Ker } \varphi$  so  $g_1 g_2^{-1} = 1_H$ . Therefore  $g_1 = g_2$ , i.e.  $\varphi$  is injective.
- (c) The precise statement is this: suppose  $G, H$ , and  $K$  are groups. Then  $G \cong G$ ; if  $G \cong H$  then  $H \cong G$ ; and if  $G \cong H$  and  $H \cong K$  then  $G \cong K$ . You have to say it this way: you can't talk about "the set of all groups".

For the first, observe that  $\text{id}: G \rightarrow G$  is an isomorphism, with inverse again  $\text{id}$ .

For the second, suppose that  $\varphi: G \rightarrow H$  is an isomorphism, with inverse  $\psi: H \rightarrow G$ . Then  $\psi$  is an isomorphism, with inverse  $\varphi$ : there is nothing to check.

For the third, suppose that  $\varphi: G \rightarrow H$  is an isomorphism with inverse  $\psi$  and  $\varphi': H \rightarrow K$  is an isomorphism with inverse  $\psi'$ . Then clearly  $\varphi' \varphi$  is an isomorphism with inverse  $\psi \psi'$ , provided those maps are homomorphisms. But the composite of two homomorphisms is a homomorphism. This has been implicit but not stated directly so we should check it: if  $\theta: G \rightarrow H$  and  $\theta': H \rightarrow K$  are group homomorphisms and  $a, b \in G$  then

$$(\theta' \theta)(ab) = \theta'(\theta(ab)) = \theta'(\theta(a)\theta(b)) = \theta'(\theta(a))\theta'(\theta(b)) = (\theta' \theta)(a)(\theta' \theta)(b)$$

so  $\theta' \theta$  is indeed always a homomorphism.