ALGEBRA 2B (MA20217)

SOLUTIONS TO PROBLEM SHEET 0

Before you begin, look up, if you need to, the definition of a *group* and of a *group homomorphism*.
The notation $\mathbb{Z}$ means the group of integers where the operation is addition. (Why not multiplication?) $\mathbb{Z}/r$ means the integers mod r: you may have called this $\mathbb{Z}_r$ or $\mathbb{Z}/r\mathbb{Z}$ or C_r, which are common alternatives.

1. Which of the following are groups and which are not? If not, what is wrong?

(a) $\mathbb{N}$ with addition.

(b) $\mathbb{N}$ with multiplication.

(c) $\mathbb{R}^3$ with vector cross product.

(d) $\mathbb{R}^3$ with vector dot product.

(e) For a fixed non-empty set S, the set of functions $f\colon S \to \mathbb{Z}/2$, with addition: that is, we define $(f_1 + f_2)(x) := f_1(x) + f_2(x)$, where the $+$ on the right-hand side is addition in $\mathbb{Z}/2$.

(f) $H \subset S_5$, where $\sigma \in H$ if $\sigma = \rho$ or $\sigma = (45)\rho$ for some $\rho \in S_3$.

(g) For a fixed non-empty set S, the power set of S (the set of all subsets of S) with the operation of symmetric difference: $A \cdot B = (A \cup B) \smallsetminus (A \cap B)$.

Solution:

(a) Not a group: for example, 1 *has no inverse.*

(b) Not a group (regardless of whether you think $0 \in \mathbb{N}$*): for example,* 2 *has no inverse.*

(c) Not a group for many reasons, but the simplest is that vector cross product is not associative.

(d) This isn't even a binary operation. The output of dot product isn't in $\mathbb{R}^3$ *at all.*

(e) Yes: the identity is the constant function 0 *and all the axioms hold because they hold in* $\mathbb{Z}/2$.

(f) Yes. You can check this by hand: writing $\tau = (45)$ *we have*

$$(\tau\rho_1)(\tau\rho_2)^{-1} = \tau\rho_1\rho_2\tau = \rho_1\rho_2$$

and similarly for the other cases. But you can also say that it is the symmetries of three red balls and two blue ones.

(g) *Yes. You can check it by hand but a much better way is to look at the indicator functions and notice that it's the same group as in part (e). If $A \subseteq S$, define $\chi_A \colon S \to \mathbb{Z}/2$ by $\chi_A(s) = 1$ if and only if $s \in A$. Then $\chi_{A \cdot B} = \chi_A + \chi_B$, and $\chi \colon A \mapsto \chi_A$ gives a bijection between the set of functions $S \to \mathbb{Z}/2$ and the power set of A (the inverse is $\chi \mapsto \{s \in S \mid \chi(s) = 1\}$. So χ is an isomorphism between the group in (e) and the alleged group here, which therefore is indeed a group. Remember that Boolean + is XOR!*

2. Here is a list of maps between groups. Say which ones are group homomorphisms. Don't guess: try to find a proof.

(a) $\phi \colon \mathbb{Z} \to \mathbb{Z}$ given by $\phi(x) = 0$.

(b) $\phi \colon \mathbb{Z} \to \mathbb{Z}$ given by $\phi(x) = 1$.

(c) $\phi \colon \mathbb{Z} \to \mathbb{Z}$ given by $\phi(x) = -x$.

(d) $\phi \colon \mathbb{Z} \to \mathbb{Z}$ given by $\phi(x) = x^2$.

(e) $\varepsilon \colon S_n \to \mathbb{Z}/2$, where ε is the sign of σ, i.e. 0 if σ is an even permutation and 1 if it is odd.

(f) $\eta \colon S_n \to C_2$ where $\eta(\sigma) = 0$ if σ is the product of an even number of disjoint cycles (possibly of length 1, i.e. fixed points) and 1 if that number is odd.

Solution:

(a) *Yes. $\phi(0) = 0$ and $\phi(-a) = 0 = -0 = -\phi(a)$ and $\phi(a+b) = 0 = 0 + 0 = \phi(a) + \phi(b)$.*

(b) *No. $\phi(0) = 1 \neq 0$.*

(c) *Yes. $\phi(0) = -0 = 0$ and $\phi(-a) = a = -(-a) = -\phi(a)$ and $\phi(a+b) = -(a+b) = (-a) + (-b) = \phi(a) + \phi(b)$*

(d) *No. $\phi(-1) = 1$ but $-\phi(1) = -1$.*

(e) *Yes. If σ_1 is even and σ_2 is even then so is $\sigma_1\sigma_2^{-1}$, and the same for the other cases.*

(f) *Yes if and only if n is even. If n is odd it fails immediately because $\eta(1) = 1$ whereas it should be 0. If n is even, suppose σ has a odd cycles and b even cycles. Then a must be even because $a \equiv n \bmod 2$. So $\eta(\sigma) = b \bmod 2$, but the signature of σ is also $b \bmod 2$ because odd cycles are even permutations, so in this case $\eta = \varepsilon$.*

GKS, 9/2/24