

ALGEBRA 2B (MA20217)

SEMESTER 2 MATHEMATICS: PROBLEM SHEET 8

Arrange with your tutors about marking for this sheet. A copy of this sheet is on Moodle and at

<http://people.bath.ac.uk/masgks/Algebra2B/sheet8.algebra2b.pdf>

1. Verify that the relation defined on T in the construction of $\mathcal{Q}(R)$, in Lemma V.20, is in fact an equivalence relation.

2. In this question R is a commutative ring and I and J are ideals in R . Say whether each of the following statements is true or not: give a proof or a counterexample.

- (a) If I and J are both prime ideals then $I \cap J$ is a prime ideal.
- (b) If I and J are both prime ideals then IJ is a prime ideal.
- (c) If I and J are both prime ideals then $I + J$ is a prime ideal.
- (d) If I and J are both maximal ideals then $I \cap J$ is a maximal ideal.
- (e) If I and J are both maximal ideals then IJ is a maximal ideal.
- (f) If I and J are both maximal ideals then $I + J$ is a maximal ideal.

3. Show directly that if R is a PID and $p \in R$ is irreducible then the ideal pR is a maximal ideal. Deduce that R/pR is a field (Theorem VI.22).

4. What is the content of each of the following polynomials?

- (a) $3x^3 - 12x^2 - 9 \in \mathbb{Z}[x]$.
- (b) $3x^3 - 12x^2 - 9 \in \mathbb{Q}[x]$.
- (c) $3wx^3 - 12wx^2 - 9w \in R[x]$, where $R = \mathbb{Z}[w]$.
- (d) $3wx^3 - 12wx^2 - 9w \in S[w]$, where $S = \mathbb{Z}[x]$.
- (e) $3wx^3 + 3x^3 - 12w^2x^2 - 12wx^2 - 9w - 9 \in R[x]$, where $R = \mathbb{Z}[w]$.

5. Say whether each of the following polynomials is reducible or irreducible, giving reasons.

(a) $3x^3 - 12x^2 - 9 \in \mathbb{Z}[x]$.

(b) $3x^3 - 12x^2 - 9 \in \mathbb{Q}[x]$.

(c) $x^2 + 5x - 3 \in \mathbb{F}_{11}[x]$.

(d) $x^2 + 5x - 3 \in \mathbb{F}_{13}[x]$.

(e) $x^2 + 5x - 3 \in \mathbb{F}_{37}[x]$.

(f) $x^3 + 5x - 3 \in \mathbb{F}_{13}[x]$.

(g) $x^3 + 5x - 3 \in \mathbb{F}_{11}[x]$.

6. Say whether each of the following polynomials is reducible or irreducible in $\mathbb{Q}[x]$, giving reasons.

(a) $x^4 - 10x^3 - 15 \in \mathbb{Z}[x]$.

(b) $x^4 - 10x^3 - 15 \in \mathbb{Q}[x]$.

(c) $x^4 - x^3 - 10x^2 + 7x + 3 \in \mathbb{Q}[x]$.

(d) $x^4 - 14x^3 + 36x^2 - 34x - 4 \in \mathbb{Q}[x]$.

(e) $x^3 + 5x - 3 \in \mathbb{Q}[x]$.

GKS, 19/4/24