ALGEBRA 2B (MA20217)

SEMESTER 3 MATHEMATICS: PROBLEM SHEET 7

This sheet will be discussed in the first tutorial after Easter. A copy of this sheet is on Moodle and at

`http://people.bath.ac.uk/masgks/Algebra2B/sheet7.algebra2b.pdf`

1. Give proofs or counterexamples to each of the following statements.

(a) If G is a group and G acts on X, then for any normal subgroup $K \lhd G$ the rule $(Kg)(x) := g(x)$ defines an action of G/K on X.

(b) If G acts on X and $H \leq G$ then $\mathrm{Stab}_H(x) \leq \mathrm{Stab}_G(x)$.

(c) If G acts on X and $K \lhd G$ then $\mathrm{Stab}_K(x) \lhd \mathrm{Stab}_G(x)$.

(d) If G_1 and G_2 act on X and $G_1 \cong G_2$. then $\mathrm{Stab}_{G_1}(x) \cong \mathrm{Stab}_{G_2}(x)$ for every $x \in X$.

(e) If G_1 and G_2 act on X and $G_1 \cong G_2$. then $\mathrm{Stab}_{G_1}(x) \cong \mathrm{Stab}_{G_2}(x)$ for every $x \in X$.

(f) The action of $\mathrm{SL}(2\mathbb{Z})$ on the upper half-plane $\mathbb{H} \subset \mathbb{C}$ by Möbius transformations is transitive.

(g) The action of $\mathrm{SL}(2\mathbb{R})$ on the upper half-plane $\mathbb{H} \subset \mathbb{C}$ by Möbius transformations is transitive.

(h) If R is a ring of characteristic m and I is an ideal of R then $\mathrm{char}\, R/I = m$.

(i) If R is a ring of characteristic m and I is an ideal of R then $\mathrm{char}\, R/I$ divides m.

2. In each of the following cases, say whether φ is a ring homomorphism or not. Give reasons. If it is, say what the kernel and image of φ are.

(a) $\varphi\colon \mathbb{Z}[i] \to \mathbb{Z}$ given by $\varphi(z) = \mathrm{Re}(z)$, the real part of z.

(b) $\varphi\colon M_{3\times 3}(\mathbb{C}) \to \mathbb{C}$ given by $\varphi(A) = \det(A)$.

(c) $\varphi\colon \mathbb{R}[t] \to \mathbb{R}[t]$ given by $\varphi(f(t)) = f(t^2)$.

(d) $\varphi(M_{2\times 2}(\mathbb{Z}) \to M_{2\times 2}(\mathbb{F}_2))$ by $\varphi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}$ where for $n \in \mathbb{Z}$ we define $\bar{n} = 1$ if n is odd and $\bar{n} = 0$ if n is even.

3.

(a) Which of the following rings is an integral domain? Give reasons.

$$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}; \qquad \mathbb{Z}/10\mathbb{Z}; \qquad \mathbb{Z}[\sqrt{10}].$$

(b) What is the characteristic of each of the following rings?

$$\mathbb{Z}; \qquad \mathbb{Z}/15\mathbb{Z}; \qquad R/3R, \text{ where } R = \mathbb{Z}/15\mathbb{Z}.$$

(c) If $R = \mathbb{Z}[\pi]$, what is the field of fractions $\mathcal{Q}(R)$?

(d) Suppose that R is an integral domain, $F = \mathcal{Q}(R)$ and S is a proper subring of F with $1_R \in S$, such that $\mathcal{Q}(S) = F$. Does this necessarily imply that $S = R$? You must give a proof or a counterexample.

(e) Suppose that R is an integral domain. For each of the following rings A, say whether A is always a domain; never a domain; or possibly a domain, depending on what R is. Give brief proofs or counterexamples.

  (a) A is the direct product $R \times R$.

  (b) A is the ring of formal power series $R[[t]]$ (this is ring whose elements are power series $\sum_{i=0}^{\infty} a_i t^i$ with $a_i \in R$: "formal" means we don't worry about whether they converge, even if it makes sense to ask that question).

  (c) $R/6R$, where 6 means $1_R + 1_R + 1_R + 1_R + 1_R + 1_R$.

GKS, 4/4/24