ALGEBRA 2B (MA20217)

SEMESTER 3 MATHEMATICS: PROBLEM SHEET 6

Homework questions, marked H, should be handed in by the time agreed with your tutor. A copy of this sheet is on Moodle and at

`http://people.bath.ac.uk/masgks/Algebra2B/sheet6.algebra2b.pdf`

1 W Let R and S be rings. Show that $R \times S = \{(r, s) \mid r \in R,\ s \in S\}$ becomes a ring if we define

$$(a, b) + (c, d) = (a + c, b + d) \quad \text{and} \quad (a, b) \cdot (c, d) = (ac, bd)$$

for a, $c \in R$ and b, $d \in S$. (This ring is called the direct product of R and S).

2 H Let R be a commutative ring, and let $a \in R$. Show that if R is an integral domain then the equation $x^2 = a$ has at most two solutions in R. Find a commutative ring R and an element $a \in R$ such that $x^2 = a$ has more than two solutions.

3 H Consider the evaluation homomorphism $\varphi\colon \mathbb{R}[t] \to \mathbb{C}$ defined by setting $\phi(f) = f(i)$. Identify $\mathrm{Ker}(\phi)$: using the division algorithm, prove carefully that your answer is correct.
What does the First Isomorphism Theorem tell us in this case?

4 W Prove that if I and J are ideals in a ring R, then $I + J$, IJ and $I \cap J$ are ideals in R and $IJ \subseteq I \cap J \subseteq I + J$.

5 A Let R be a finite ring, i.e. the number $|R|$ of elements of R is finite. Show that $|R|$ is divisible by $\operatorname{char} R$. Deduce that if $|R| = p$ is prime, then $R \cong \mathbb{Z}/p\mathbb{Z}$.
By considering the map $m_a\colon R \to R$ given by $m_a(b) = ab$, or otherwise, show that a finite integral domain is a field.

GKS, 22/3/24