ALGEBRA 2B (MA20217)

SEMESTER 3 MATHEMATICS: PROBLEM SHEET 3

Homework questions, marked H, should be handed in by the time agreed with your tutor. A copy of this sheet is on Moodle and at

`http://people.bath.ac.uk/masgks/Algebra2B/sheet3.algebra2b.pdf`

1 W Consider the map $\varphi\colon \mathbb{R} \to \mathbb{C}^*$ given by $\varphi(x) = e^{2\pi i x}$. (Remember what the group operations on $\mathbb{R}$ and $\mathbb{C}^*$ are.) Verify that φ is a group homomorphism. What is its kernel? Describe the three maps π, $\bar{\varphi}$ and ι from the factorisation in Corollary II.26.

2 H In each of the following cases say what the kernel and image of the group homomorphism φ are and describe π, $\bar{\varphi}$ and ι briefly.

(a) $\varphi\colon S_n \to \mathbb{Z}/2$ where $\varphi(\sigma)$ is the signature of σ.

(b) $G = \mathrm{SL}(2, \mathbb{Z})$ and $\varphi(M)$ is M mod p for some prime p. The hard part is to determine the image of φ: you may want to use the Chinese Remainder Theorem.

3 W In I.40 we mentioned "the smallest subgroup that contains S" (a subset of G) as another way to describing $\langle S\rangle$. Let G be a group, suppose $S \subset G$ and let H be the intersection of all (not necessarily proper) subgroups of G that contain S. Show that H is a subgroup, and that any subgroup that contains S also contains H. Deduce that $H = \langle S\rangle$.

4 H

(a) Let G be a group and suppose $S \subseteq G$ is a subset. Is there a smallest normal subgroup of G that contains S? If so, can you describe what the elements look like?

(b) If $H < G$, define the normaliser $N_G(H)$ to be the largest subgroup of G such that H is normal in $N_G(H)$, Make this definition precise, and show that $N_G(H)$ is a subgroup of G. Is $N_G(H)$ a normal subgroup of G?

GKS, 1/3/24