ALGEBRA 2B (MA20217)

ALGEBRA 2B MOCK EXAM

This is a mock exam which has not been checked as carefully as a real exam would be. In the real exam you will be asked to attempt all of Section A and two questions out of three from Question B (if you attempt them all, your best answers will count): here I have just indicated those sections with a letter after the question number. The numbers in square brackets are a general indication of how many marks that part would be worth if there were any marks: full marks would be 60.

1A. Define what is meant by a *normal subgroup* of a group G. [2]
Show that if $\phi\colon G \to H$ is a group homomorphism then the kernel $\operatorname{Ker}\phi$ is a normal subgroup of G. [2]
Suppose that $\psi\colon G \to H$ is a map such that $\{g \in G \mid \phi(g) = 1\}$ is a normal subgroup of G. Is it true that ϕ is necessarily a homomorphism? Give a proof or a counterexample. [2]
Show that there cannot exist a surjective homomorphism from S_5 to S_4 (the symmetric groups). [2]

2A. Define what it means for an ideal P to be a *prime ideal* of a commutative ring R. [2]
Give an example of a ring in which every nonzero prime ideal is maximal. [1]
Consider the ring $Q = \{\frac{a}{b} \in \mathbb{Q} \mid \operatorname{hcf}(b, 3) = 1\}$. Show that $\langle 3\rangle$ is a prime ideal in Q. Are there any other nonzero prime ideals in Q? Justify your answer briefly. [5]

3A. State and prove the Chinese Remainder Theorem. [5]
If the characteristic of R is n (possibly $n = 0$) and I is an ideal of R, show that $\operatorname{char} R/I$ divides $\operatorname{char} R$. [3]

4B. Define the terms *Euclidean domain*, *principal ideal domain* (PID) and *unique factorisation domain* (UFD). [5]
Show that every Euclidean domain is a PID. [5]
State Eisenstein's criterion for irreducibility of an element of $\mathbb{Q}[x]$. [2]
For each of the following polynomials, say with brief reasons whether or not it is irreducible in $\mathbb{Q}[x]$.
(a) $x^4 - 2x^3 - 4x^2 - 17x - 20$
(b) $2x^4 + 5x^3 + 25x^2 - 10x - 10$
(c) $2x^4 + 13x^3 + 2x^2 + 3x + 2$

5B. Define what it means for a group G to *act on a set* X. [2]
If G acts on X, say what is meant by the *orbit* of an element of $x \in X$, and what is meant by the *stabiliser* of an element $x \in X$. [4]
Show that the rule $g(h) = ghg^{-1}$ defines an action of G on G, for any group G, called the conjugation action. [2]

Take $G = S_n$ and consider the conjugation action of S_n on S_n.
Suppose that $\sigma = (12 \dots k)$, for some $k \leq n$. What is the orbit of σ under this action? What is its stabiliser? [4]
The alternating group A_5 is a subgroup of S_5 so it also acts on S_5 by conjugation. Consider the elements (12345) and (21345). Do they have the same orbit under the action of A_5? Justify your answer carefully. [6]

6B. Suppose that G is a group and $S \subseteq G$ is a subset. Say what is meant by the *subgroup* $\langle S \rangle$ *generated by* S. [2]
Show that $\langle S \rangle$ is equal to the intersection of all subgroups of G that contain S. [2]
If H_1 and H_2 are both subgroups of a group G, the *product subgroup* $H_1 H_2$ is defined to be $\langle R \rangle$, where $R = \{h_1 h_2 \mid h_1 \in H_1,\, h_2 \in H_2\}$
(a) Explain why $H_1 H_2 = H_2 H_1$, even if G is not abelian. [2]
(b) If S_1 and S_2 are subsets of G, show that $\langle S_1 \cup S_2 \rangle = \langle S_1 \rangle \langle S_2 \rangle$. [2]
(c) Is it true that $\langle S_1 \cap S_2 \rangle = \langle S_1 \rangle \cap \langle S_2 \rangle$? Give a proof or a counterexample. [2]
(d) Suppose that $S \subseteq G$ and $gSg^{-1} \subseteq S$ for all $g \in G$. Does it follow that $\langle S \rangle$ is a normal subgroup of G? Give a proof or a counterexample. [2]
(e) If $a,\, b \in G$ then the *commutator* of a and b, denoted $[a, b]$, is the element $[a, b] = aba^{-1}b^{-1} \in G$. Let $C \subset G$ be the set of all commutators, i.e. $C = \{[a, b] \mid a,\, b \in G\}$, and let $G' = \langle C \rangle$. Show that G' is a normal subgroup of G, that G/G' is abelian, and that if H is a normal subgroup of G such that G/H is abelian then there is a surjective group homomorphism $G/H \to G/G'$. [6]

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