Section A

1. In this question, G is a group and X is a set.

   (a) Define what is meant by an *action* of G on X. [2]

   (b) Given an action of G on X and an element $x \in X$, define the *stabiliser* $\mathrm{Stab}_G(x)$ and show that it is a subgroup of G. [3]

   (c) Give an example of an action of a group G on a set X and an $x \in X$ such that $\mathrm{Stab}_G(x)$ is not a normal subgroup of G. [1]

   (d) Show that if y is in the orbit of x then $\mathrm{Stab}_G(y)$ is conjugate to $\mathrm{Stab}_G(x)$: that is, $g\,\mathrm{Stab}_G(y)g^{-1} = \mathrm{Stab}_G(x)$ for some $g \in G$. [2]

2. In this question, R is a commutative ring.

   (a) Define what it means for a subset M of R to be a *maximal ideal*. [2]

   (b) Suppose I is an ideal in R. Show that I is a maximal ideal if and only if $I + aR = R$ for every $a \notin I$. [3]

   (c) Now let $R = K[x, y]$, where K is a field. Let a, $b \in K$. Show that the ideal I given by $I = \{f \in K[x, y] \mid f(a, b) = 0\}$ is a maximal ideal of R. [3]

3. In this question, R is a commutative ring.

   (a) For I and J ideals of R, define $I + J$ and IJ. [3]

   (b) State the Chinese Remainder Theorem for commutative rings. [1]

   (c) Define the *characteristic* $\mathrm{char}\, R$ of a commutative ring R. If S is also a commutative ring and $\mathrm{char}\, R = m$ and $\mathrm{char}\, S = n$, what can you say about $\mathrm{char}(R \times S)$? [2]

   (d) If I and J are ideals in R, show that $\mathrm{char}\left(\frac{R}{I} \times \frac{R}{J}\right) = \mathrm{char}\left(\frac{R}{I \cap J}\right)$. [2]

Section B

4. (a) Define what it means for a commutative ring R to be a *unique factorisation domain* (abbreviated *UFD*). [2]

   (b) Explain briefly why $K[x_1, \ldots, x_n]$ is a UFD for any field K. [2]

   (c) State and prove Eisenstein's criterion for irreducibility of polynomials with integer coefficients. [6]

   (d) Show that each of the following polynomials with integer coefficients is irreducible.

   (i) $x^4 + 14x^3 - 49x^2 + 84x - 14$. [2]

   (ii) $x^4 + 4x^2 - 7$. [3]

   (iii) $x^4 + 7x^3 - 49x^2 + 73x - 21$. [3]

5. In this question, R is an integral domain.

   (a) Define what is meant by a *valuation* on R, and what is meant by a *Euclidean valuation*. [4]

   (b) Define what it means for R to be a *principal ideal domain* (abbreviated *PID*), and what it means for R to be a *Euclidean domain*. [2]

   (c) Show that if R is a Euclidean domain then R is a PID. [4]

   (d) Suppose that R is a PID and S is a subring of R containing 1_R. Is S necessarily a PID? Give a proof or counterexample. [2]

   (e) Suppose that R is a PID and I is a prime ideal of R, with $I \neq R$. Is the quotient ring R/I necessarily a PID? Give a proof or counterexample. [3]

   (f) By considering the ring $\mathbb{R}[x, y]$, or otherwise, give an example of a valuation that is not a Euclidean valuation. [3]

6. In this question, G is a group.

(a) Suppose that $S \subseteq G$ is a subset. Say, in terms of elements, what is meant by the *subgroup* $\langle S \rangle$ *generated by* S. [2]

(b) Show that $\langle S \rangle$ is equal to the intersection of all subgroups of G that contain S. [3]

(c) Is it true that if S_1 and S_2 are subsets of G then $\langle S_1 \cap S_2 \rangle = \langle S_1 \rangle \cap \langle S_2 \rangle$? Give a proof or a counterexample. [2]

(d) Suppose that $S \subseteq G$ and $gSg^{-1} \subseteq S$ for all $g \in G$. Does it follow that $\langle S \rangle$ is a normal subgroup of G? Give a proof or a counterexample. [3]

(e) If a, $b \in G$ then the *commutator* of a and b, denoted $[a, b]$, is the element

$$[a, b] = aba^{-1}b^{-1} \in G.$$

Let $C \subseteq G$ be the set of all commutators, i.e. $C = \{[a, b] \mid a, b \in G\}$. Show that the *derived subgroup* $G' = \langle C \rangle$ is a normal subgroup of G and that G/G' is abelian. [4]

(f) What is the derived subgroup of the symmetric group S_n, for $n \geq 3$? Justify your answer briefly. *[Hint: compute some commutators, and remember that every even permutation is a product of cycles of length 3.]* [4]