MATHEMATICS 2 (MA10193)
EXAMPLES SHEET 4: SOLUTIONS

1. The auxiliary equation is $\lambda^2-8\lambda+41=0$, which has solutions $\lambda = 4\pm 5i$. So the general solution is $y = Ae^{4x}\cos 5x + B4x\sin 5x$. (We could write this in terms of exponentials if we preferred.) The boundary condition at $x = 0$ gives $A = 1$ and the boundary condition at $x = 5\pi/2$ gives (remember that $\cos 5\pi/2 = 0$, $\sin 5\pi/2 = 1$) gives $B = 0$. So the solution is $y = e^{4x}\cos 5x$.

2. The auxiliary equation is $\lambda^2 - 3\lambda + 10 = 0$, which has solutions $\lambda = (3 \pm i\sqrt{31})/2$. A particular integral is of the form $y = \alpha\sin x + \beta\cos x$ for suitable α and β, and the condition is

$$-\alpha\sin x - \beta\cos x - 3\alpha\cos x + 3\beta\sin x + 10\alpha\sin x + 10\beta\cos x = 24\sin x,$$

i.e. $(9\alpha + 3\beta)\sin x + (-3\alpha + 9\beta)\cos x = 24\sin x$. Therefore

$$9\alpha + 3\beta = 24$$
$$-3\alpha + 9\beta = 0$$

which gives $\alpha = 12/5$, $\beta = 4/5$.
So the general solution is

$$y = Ae^{3x/2}\cos\sqrt{31}x/2 + Be^{3x/2}\sin\sqrt{31}x/2 + \frac{12}{5}\sin x + \frac{4}{5}\cos x.$$

The boundary condition at $x = 0$ gives

$$0 = A + \frac{4}{5}$$

so $A = -4/5$, and the condition at $x = \pi/4$ gives

$$1 = -\frac{4}{5}e^{3\pi/4}\cos\sqrt{31}\pi/4 + Be^{3\pi/4}\sin\sqrt{31}\pi/4 + \frac{8\sqrt{2}}{5},$$

since $\cos\pi/2 = \sin\pi/2 = \sqrt{2}/2$. Therefore

$$B = (1 + \frac{4}{5}e^{3\pi/4}\cos\sqrt{31}\pi/4 - \frac{8\sqrt{2}}{5})/(e^{3\pi/4}\sin\sqrt{31}\pi/4) \approx 0.409.$$

3. From the first equation, $3y = -7\frac{dx}{dt} + x$. Putting this into the second equation gives

$$-7\frac{d^x}{dt^2} + \frac{dx}{dt} - 4x = 0.$$

The auxiliary equation for this is $-7\lambda^2 + \lambda - 4 = 0$, so $\lambda = (1 \pm i\sqrt{111})/14$. Call these λ_1 and λ_2. Then $x = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$ and therefore $y = \frac{-7\lambda_1+1}{3}Ae^{\lambda_1 t} + \frac{-7\lambda_2+1}{3}Be^{\lambda_2 t}$.

4. The best way to do this is to eliminate $\frac{dx}{dt}$. If we multiply the first equation by 2 and subtract the first equation from it we get

$$5\frac{dy}{dt} + 10x - 13y = 6t^2 + 2t - 2,$$

so

$$x = -\frac{1}{2}\frac{dy}{dt} + \frac{13}{10}y + \frac{3}{5}t^2 + \frac{1}{5}t - \frac{1}{5}.$$

Substituting this in the second equation we get

$$-\frac{d^y}{dt^2} + \frac{13}{5}\frac{dy}{dt} - \frac{12}{5}t + \frac{2}{5} + \frac{dy}{dt} - 2\frac{dy}{dt} + \frac{26}{5}y + \frac{12}{5}t^2 + \frac{4}{5}t - \frac{4}{5} + 5y = -2t + 2,$$

that is (collecting the terms and multiplying by 5)

$$5\frac{d^y}{dt^2} - 8\frac{dy}{dt} - 51y = 60t^2 + 130t - 60.$$

The CF for this is $Ae^{\lambda_1 t} + Be^{\lambda_2 t}$ where $\lambda_1 = (4 + \sqrt{271})/5$, $\lambda_2 = (4 - \sqrt{271})/5$ A PI is of the form $at^2 + bt + c$ and we get

$$60t^2 + 130t - 60 = 10a - 16at - 8b - 51at^2 - 51bt - 51c$$

so $a = -60/51$, $b = -630/289$ and $c = 18980/14739$.
Hence $y = Ae^{(4+\sqrt{271})t/5} + Be^{(4-\sqrt{271})t/5} - 60t^2/51 - 630t/289 + 18980/14739$, and x can be calculated from the formula above: in floating point

$$x = -2.79Ae^{4.09t} + 3.79Be^{-2.49t} - 0.93t^2 - 0.28t + 3.65.$$

GKS, 9/05/05