MATHEMATICS 2 (MA10193)
EXAMPLES SHEET 2: SOLUTIONS

1.

$$\begin{pmatrix} 1 & 2 & -8 \\ 2 & -3 & 5 \\ 3 & 2 & -12 \end{pmatrix} \xrightarrow{R_2-2R_1} \begin{pmatrix} 1 & 2 & -8 \\ 0 & -7 & -11 \\ 3 & 2 & -12 \end{pmatrix}$$

$$\xrightarrow{R_3-3R_1} \begin{pmatrix} 1 & 2 & -8 \\ 0 & -7 & -11 \\ 0 & -4 & -36 \end{pmatrix}$$

$$\xrightarrow{R_3-\frac{4}{7}R_1} \begin{pmatrix} 1 & 2 & -8 \\ 0 & -7 & -11 \\ 0 & 0 & -208/7 \end{pmatrix}$$

and the bottom row gives $\frac{-208}{7}z =$ so $z = 0$. Then the next row gives $-7y - 11z = 0$ so $y = 0$ and the top row gives $x + 2y + 8z = 0$ so $x = 0$ also.

2.

$$\begin{pmatrix} 9 & 4 & 3 & \mathbf{-1} \\ 5 & 1 & 2 & \mathbf{1} \\ 7 & 3 & 4 & \mathbf{1} \end{pmatrix} \xrightarrow{R_2-\frac{5}{9}R_1} \begin{pmatrix} 9 & 4 & 3 & \mathbf{-1} \\ 0 & \frac{-11}{9} & \frac{-3}{9} & \mathbf{\frac{14}{9}} \\ 7 & 3 & 4 & \mathbf{1} \end{pmatrix}$$

$$\xrightarrow{R_2\times 9} \begin{pmatrix} 9 & 4 & 3 & \mathbf{-1} \\ 0 & -11 & 3 & \mathbf{14} \\ 7 & 3 & 4 & \mathbf{1} \end{pmatrix}$$

$$\xrightarrow{R_3-\frac{7}{9}R_1} \begin{pmatrix} 9 & 4 & 3 & \mathbf{-1} \\ 0 & -11 & 3 & \mathbf{14} \\ 0 & \frac{-1}{9} & \frac{15}{9} & \mathbf{\frac{16}{9}} \end{pmatrix}$$

$$\xrightarrow{R_3\times 9} \begin{pmatrix} 9 & 4 & 3 & \mathbf{-1} \\ 0 & -11 & 3 & \mathbf{14} \\ 0 & -1 & 15 & \mathbf{16} \end{pmatrix}$$

$$\xrightarrow{R_3-\frac{1}{11}R_2} \begin{pmatrix} 9 & 4 & 3 & \mathbf{-1} \\ 0 & -11 & 3 & \mathbf{14} \\ 0 & 0 & \frac{162}{11} & \mathbf{\frac{162}{11}} \end{pmatrix}$$

$$\xrightarrow{R_3\times \frac{11}{162}} \begin{pmatrix} 9 & 4 & 3 & \mathbf{-1} \\ 0 & -11 & 3 & \mathbf{14} \\ 0 & 0 & 1 & \mathbf{1} \end{pmatrix}$$

$$\xrightarrow{R_2-3R_3} \begin{pmatrix} 9 & 4 & 3 & \mathbf{-1} \\ 0 & -11 & 0 & \mathbf{11} \\ 0 & 0 & 1 & \mathbf{1} \end{pmatrix}$$

$$\xrightarrow{R_1-3R_3}\begin{pmatrix}9 & 4 & 0 & \mathbf{-4}\\ 0 & -11 & 0 & \mathbf{11}\\ 0 & 0 & 1 & \mathbf{1}\end{pmatrix}$$

$$\xrightarrow{R_2\times\frac{1}{11}}\begin{pmatrix}9 & 4 & 0 & \mathbf{-4}\\ 0 & -1 & 0 & \mathbf{1}\\ 0 & 0 & 1 & \mathbf{1}\end{pmatrix}$$

$$\xrightarrow{R_1+4R_2}\begin{pmatrix}9 & 0 & 0 & \mathbf{0}\\ 0 & -1 & 0 & \mathbf{1}\\ 0 & 0 & 1 & \mathbf{1}\end{pmatrix}$$

so the solution is $z = 1$, $y = -1$ and $x = 0$.

3. We may assume that $a_{11} \neq 0$ (otherwise, swap the rows or columns round).

$$\begin{pmatrix}a_{11} & a_{12} & \mathbf{b_1}\\ a_{21} & a_{22} & \mathbf{b_2}\end{pmatrix} \xrightarrow{R_2-\frac{a_{21}}{a_{11}}R_1} \begin{pmatrix}a_{11} & a_{12} & \mathbf{b_1}\\ 0 & a_{22}-\frac{a_{12}a_{21}}{a_{11}} & \mathbf{b_2}-\frac{\mathbf{b_1a_{21}}}{\mathbf{a_1 1}}\end{pmatrix}$$

$$\xrightarrow{R_2\times a_{11}} \begin{pmatrix}a_{11} & a_{12} & \mathbf{b_1}\\ 0 & a_{11}a_{22}-a_{12}a_{21} & \mathbf{b_2a_{11}}-\mathbf{b_1a_{21}}\end{pmatrix}$$

so $y = \frac{b_2a_{11}-b_1a_{21}}{a_{11}a_{22}-a_{12}a_{21}}$ and hence $y = \frac{b_2a_{12}-b_1a_{22}}{a_{11}a_{22}-a_{12}a_{21}}$, which is Cramer's Rule.

4. Taking twice the second row away from the third gives

$$\begin{pmatrix}0 & 2 & -1\\ 1 & 0 & 3\\ 0 & -4 & 4\end{pmatrix}$$

which has determinant -4. This is not zero, so the rank of A is 3 and the equations therefore have exactly one solution.

5. The determinant is $(x+1)(1\times 1+5\times 4)-x(2\times 1+3\times 4)+(x-4)(2\times 5-3\times 1) = 21x+21-14x+7x-28 = 14x-7$, which is zero when $x=\frac{1}{2}$.

6. $\det A = -2x^2-5x+4$ so if $\det A = 0$ then $-2x^2-5x+4=0$. In that case $x = (-5\pm\sqrt{57})/4$, i.e. $x \approx -3.14$ or $x \approx 0.64$. If $x = 2$ then $\det A \neq 0$ so $\operatorname{rank} A = 3$.

GKS, 9/05/05