MATHEMATICS 2 (MA10193)
EXAMPLES SHEET 1: SOLUTIONS

1. We have

$$AA^T = \begin{pmatrix} \cos w & p \\ \sin w & q \end{pmatrix} \begin{pmatrix} \cos w & \sin w \\ p & q \end{pmatrix} = \begin{pmatrix} \cos^2 w + p^2 & \cos w \sin w + pq \\ \sin w \cos w + qp & \sin^2 w + q^2 \end{pmatrix}.$$

This is supposed to be equal to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so $\cos^2 w + p^2 = 1 = \cos^2 w + \sin^2 w$, $\sin^2 w + q^2 = 1$, and $\cos w \sin w + pq = 0$. Therefore $p = \pm \sin w$, $q = \pm \cos w$ and the signs have to be opposite. So we get two possible solutions: $p = \sin w$, $q = -\cos w$ or $p = -\sin w$, $q = \cos w$.

2. We have

$$A = \begin{pmatrix} 1 & 1 & 2 & -1 \\ 3 & 2 & 1 & -2 \\ 0 & 2 & 1 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & 3 \\ -1 & 2 \\ 3 & 0 \\ 1 & 1 \end{pmatrix}.$$

so

$$A^T = \begin{pmatrix} 1 & 3 & 0 \\ 1 & 2 & 2 \\ 2 & 1 & 1 \\ -1 & -2 & 1 \end{pmatrix} \text{ and } B^T = \begin{pmatrix} 2 & -1 & 3 & 1 \\ 3 & 2 & 0 & 1 \end{pmatrix}.$$

This gives

$$AB = \begin{pmatrix} 6 & 4 \\ 5 & 11 \\ 2 & 5 \end{pmatrix} \text{ and } B^T A^T = \begin{pmatrix} 6 & 5 & 2 \\ 4 & 11 & 5 \end{pmatrix}.$$

3. $(A+B)^2 = A^2 + AB + BA + B^2$ (not $A^2 + 2AB + B^2$). If

$$A = \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix},$$

then

$$A + B = \begin{pmatrix} 4 & 2 \\ 4 & 3 \end{pmatrix}, \qquad (A+B)^2 = \begin{pmatrix} 4 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 4 & 2 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 24 & 14 \\ 28 & 17 \end{pmatrix},$$

and

$$A^2 = \begin{pmatrix} 1 & -4 \\ 12 & 1 \end{pmatrix}, \quad AB = \begin{pmatrix} 3 & 5 \\ 8 & 11 \end{pmatrix}, \quad BA = \begin{pmatrix} 13 & 4 \\ 5 & 1 \end{pmatrix}, \qquad B^2 = \begin{pmatrix} 7 & 0 \\ 3 & 4 \end{pmatrix}$$

and adding up all of these gives $\begin{pmatrix} 24 & 14 \\ 28 & 17 \end{pmatrix}$.

4. In this case, A and B are symmetric, so $A^T = A$ and $B^T = B$. If AB is symmetric as well, then we have $BA = B^T A^T = (AB)^T = AB$, so it is true that $AB = BA$ if A, B and AB are all symmetric.
In this particular case we multiply the matrices and get

$$AB = \begin{pmatrix} 3-3x+z & -6+2x & z-3x+1 \\ 3x-6+yz & 4-3x-3y & xz-6+y \\ 3-3y+3z & -12+2y & z-3y+3 \end{pmatrix},$$

so if this is symmetric we have

$$\begin{aligned} 3x-6+yz &= -6+2x \\ z-3x+1 &= 3-3y+3z \\ xz-6+y &= -12+2y. \end{aligned}$$

So $x = -yz$ from the first equation, and $y - xz = 6$ from the third equation so $y + yz^2 = 6$. Hence $y = \frac{6}{z^2+1}$ and $x = \frac{-6z}{z^2+1}$. The second equation says $3x - 3y + 2z = -2$, so we substitute the expressions we have fouund for x and y and then multiply by $z^2 + 1$:

$$3\left(\frac{-6z}{z^2+1}\right) - 3\left(\frac{6}{z^2+1}\right) + 2z = -2$$

so

$$-18z - 18 + 2z(z^2+1) = -2(z^2+1),$$

i.e. $2z^3 + 2z^2 - 16z - 16 = 0$. One solution to this is $z = -1$: in fact it says $(2(z+1)(z^2-8) = 0$ so the solutions are $z = -1$ or $z = \pm\sqrt{8} = \pm 2\sqrt{2}$. So there are three possible solutions: $z = -1$, $x = 3$ and $y = 3$; or $z = 2\sqrt{2}$, $x = 4\sqrt{2}/3$ and $y = 2/3$; or $z = -2\sqrt{2}$, $x = -4\sqrt{2}/3$ and $y = 2/3$.

5. We can write $B = (B + B^T)/2 + (B - B^T)/2$, i.e.

$$B = \begin{pmatrix} 4 & \frac{3}{2} & \frac{1}{2} \\ \frac{3}{2} & -1 & \frac{5}{2} \\ \frac{1}{2} & \frac{5}{2} & 1 \end{pmatrix} + \begin{pmatrix} 0 & \frac{-1}{2} & \frac{-5}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{5}{2} & \frac{-1}{2} & 0 \end{pmatrix}.$$

6. B^2, C^2, $C^T AB$ and $(AB)^T C$ are undefined. The others are

$$A^2 = \begin{pmatrix} 9 & 7 & 15 \\ 1 & 10 & 5 \\ 2 & 4 & 13 \end{pmatrix}, \qquad AA^T = \begin{pmatrix} 30 & -1 & 13 \\ -1 & 6 & 4 \\ 13 & 4 & 13 \end{pmatrix}, \qquad BB^T = \begin{pmatrix} 23 & 12 \\ 12 & 21 \end{pmatrix},$$

and

$$CC^T = \begin{pmatrix} 4 & 2 & 6 \\ 2 & 1 & 3 \\ 6 & 3 & 9 \end{pmatrix} = C^T C.$$

7. $AB = BA = I$ by multiplying them out. $A^T B^T = (BA)^T = I^T = I$.
The first set of equations is

$$B \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

so the solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = B^{-1} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = A \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 9 \\ 30 \end{pmatrix}$$

since $B^{-1} = A$. The second set is

$$B^T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

and since ${B^T}^{-1} = A^T$ the solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = {B^T}^{-1} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = A^T \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 17 \\ 1 \\ 32 \end{pmatrix}$$

GKS, 9/05/05