Summary of the article
Das Titsgebäude von Siegelschen Modulgruppen vom Geschlecht 2

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If Γ is an arithmetic subgroup of $\mathrm{Sp}(4,\mathbb{Q})$, the *Tits Building* $\mathcal{T}(\Gamma)$ is a graph whose vertices are in 1-to-1 correspondence with Γ-orbits of nontrivial isotropic subspaces of $\mathbb{Q}^4$. If $U_1, U_2 \subseteq \mathbb{Q}^4$ are isotropic subspaces then the edges corresponding to the orbits $\Gamma \cdot U_1$ and $\Gamma \cdot U_2$ are joined by an edge in $\mathcal{T}(\Gamma)$ if and only if there exists $\gamma \in \Gamma$ such that $\gamma U_1 \subset U_2$ or $U_2 \subset \gamma U_1$.

An arithmetic subgroup of $\mathrm{Sp}(4,\mathbb{Q})$ corresponds to a moduli space $\mathcal{A}_\Gamma$ of abelian surfaces of some polarization type with some level structure, and it is well known that the Tits building describes the configuration of the boundary components in the Satake compactification $\bar{\mathcal{A}}_\Gamma$ of $\mathcal{A}_\Gamma$. Because of this, detailed analysis of specific moduli spaces of abelian surfaces often necessitates calculation of the associated Tits building. This is an elementary but not entirely trivial procedure. In this paper we calculate the Tits buildings for some of the more commonly occurring cases.

We set

$$\Lambda = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}, \text{ where } \qquad E = \mathrm{diag}(1,t)$$

and we assume that $t \in \mathbb{N}$ is squarefree. We consider the following arithmetic subgroups of $\mathrm{Sp}(4,\mathbb{Q})$:

$$\begin{aligned} \tilde{\Gamma}^\circ_{1,t} &= \mathrm{Sp}(\Lambda,\mathbb{Z}) = \left\{\gamma \in \mathrm{GL}(4,\mathbb{Z}) \mid \gamma\Lambda^t\gamma = \Lambda\right\} \\ \tilde{\Gamma}_{1,t} &= \left\{g \in \tilde{\Gamma}^\circ_{1,t} \mid vg \equiv v \bmod \mathbb{L} \text{ for all } v \in \mathbb{L}^\vee\right\} \\ \tilde{\Gamma}_{1,t}(n) &= \left\{g \in \tilde{\Gamma}^\circ_{1,t} \mid vg \equiv v \bmod \mathbb{L} \text{ for all } v \in \mathbb{L}^\vee_n\right\} \end{aligned}$$

where $\mathbb{L} = \mathbb{Z}^4$, $\mathbb{L}^\vee$ is the dual of $\mathbb{L}$ relative to Λ, and $\mathbb{L}^\vee_n$ is the dual of $\mathbb{L}$ relative to nJ ($n \in \mathbb{N}$, where J is the standard symplectic form $\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$. Thus $\mathbb{L}^\vee = \mathbb{Z} \oplus \frac{1}{t}\mathbb{Z} \oplus \mathbb{Z} \oplus \frac{1}{t}\mathbb{Z}$ and $\mathbb{L}^\vee_n = \frac{1}{n}\mathbb{Z} \oplus \frac{1}{n}\mathbb{Z} \oplus \frac{1}{n}\mathbb{Z} \oplus \frac{1}{n}\mathbb{Z}$.

These groups correspond to the moduli of abelian surfaces with, respectively, a $(1,t)$ polarization, a $(1,t)$ polarization and a canonical level structure, and a a $(1,t)$ polarization and a full level-n structure.

For each natural number n we define the set of non-torsion elements in $\mathbb{Z}_n^2$:

$$\mathcal{N}(n) = \left\{v \in \mathbb{Z}_n^2 \mid \lambda v \neq 0 \text{ if } 0 \neq \lambda \in \mathbb{Z}_n\right\}$$

and we put

$$\mathcal{M}(n) = \mathcal{N}(n)/\pm 1 \qquad \text{and} \qquad \mathcal{O}(n) = \mathcal{N}(n)/\mathbb{Z}_n^\times$$

where $\mathbb{Z}_n^\times$ is the group of units in $\mathbb{Z}_n$.

Theorem A. *Under the action of $\tilde{\Gamma}^\circ_{1,t}$ (t squarefree) the isotropic lines in $\mathbb{Q}^4$ fall into $\mu(t)$ orbits, where $\mu(t)$ is the number of divisors of t. The isotropic planes are all equivalent to one another. Thus the Tits building $\mathcal{T}(\tilde{\Gamma}^\circ_{1,t})$ has one central vertex and $\mu(t)$ peripheral vertices, and an edge joining the central vertex to each peripheral one.*

In other words, $\mathcal{T}(\tilde{\Gamma}^\circ_{1,t})$ is a $(\mu(t)_1, 1_{\mu(t)})$-configuration.

Theorem B. *The orbits* $H\left([w_2,w_4]_{\mathcal{O}(t)}\right)$ *of isotropic planes in* $\mathbb{Q}^4$ *under the action of* $\tilde{\Gamma}_{1,t}$ *(*t *squarefree) are indexed by* $\mathcal{O}(t)$*. The orbits* $L\left([x_2,x_4]_{\mathcal{M}(r)}\right)$ *of isotropic lines in* $\mathbb{Q}^4$ *under the action of* $\tilde{\Gamma}_{1,t}$ *are indexed by pairs* $(r,[x_2,x_4]_{\mathcal{M}(r)})$ *where* r *is a divisor of* t *and* $[x_2,x_4]_{\mathcal{M}(r)}\in\mathcal{M}(r)$*. The vertices* $H\left([w_2,w_4]_{\mathcal{O}(t)}\right)$ *and* $L\left([x_2,x_4]_{\mathcal{M}(r)}\right)$ *in* $\mathcal{T}(\tilde{\Gamma}_{1,t})$ *are joined by an edge if* (w_2,w_4) *and* (x_2,x_4) *give the same residue class in* $\mathcal{O}(r)$*.*

In the remaining case we restrict ourselves for simplicity to the case where $t=p$ and $n=q$ are prime, and $p\neq 2$.

Theorem C. *There are* q^4-1 *if* $q\neq p,2$ *(respectively* q^4-q^2 *if* $q=p$*,* 30 *if* $q=2$*) orbits of isotropic lines in* $\mathbb{Q}^4$ *under the action of* $\tilde{\Gamma}_{1,p}(q)$ *and* $(q^4-1)/2$ *(respectively* $(q^2-1)(q^2+q)/2$ *and* 15*) orbits of isotropic planes. The Tits building* $\mathcal{T}(\tilde{\Gamma}_{1,p}(q))$ *has* $(q^2-1)(q^4-1)/2$ *(respectively* $(q^2-1)(q^4-q^2)/2$ *and* 90*) edges.*

More precisely, $\mathcal{T}(\tilde{\Gamma}_{1,p}(q))$ is an (a_b,c_d)-configuration, where $a=q^4-1$, q^4-q^2, 30; $b=(q^2-1)/2$, $(q^2-1)/2$, 3; $c=(q^4-1)/2$, $(q^2-1)(q^2+q)/2$, 15 and $d=q^2-1$, q^2-q, 6 in the three cases $q\neq p,2$, $q=p$ and $q=2$ respectively.