Sylvester’s Coin Theorem

Geoff Smith

University of Bath

March 2019
Suppose that you have plenty of 2p coins and 5p coins. What is the largest sum of money which you cannot pay exactly?

Answer: 3p
Suppose that you have plenty of 2p coins and 5p coins. What is the largest sum of money which you cannot pay exactly?

Answer: 3p
Suppose that you have plenty of 2p coins and 10p coins. What is the largest sum of money which you cannot pay exactly?

Answer: there is no largest sum that you cannot pay. You can never pay an odd number of pence.
Suppose that you have plenty of 2p coins and 10p coins. What is the largest sum of money which you cannot pay exactly?

Answer: there is no largest sum that you cannot pay. You can never pay an odd number of pence.
Suppose that you have plenty of 5p coins and 7p coins. What is the largest sum of money which you cannot pay exactly?
Suppose that you have plenty of 5p coins and 7p coins. What is the largest sum of money which you cannot pay exactly?

Answer: 23p
5p, 10p, 15p, 20p, 25p, 30p, ...
7p, 12p, 17p, 22p, 27p, 32p, ...
14p, 19p, 24p, 29p, 34p, 39p, ...
21p, 26p, 31p, 36p, 41p, 46p, ...
28p, 33p, 38p, 43p, 48p, 53p, ...
5p, 10p, 15p, 20p, 25p, 30p, ...  
7p, 12p, 17p, 22p, 27p, 32p, ...  
14p, 19p, 24p, 29p, 34p, 39p, ...  
21p, 26p, 31p, 36p, 41p, 46p, ...  
28p, 33p, 38p, 43p, 48p, 53p, ...
Suppose that you have plenty of $xp$ coins and $yp$ coins, and $hcf(x, y) = gcd(x, y) = 1$. What is the largest sum of money which you cannot pay exactly?
Suppose that you have plenty of \( xp \) coins and \( yp \) coins, and \( \text{hcf}(x, y) = \gcd(x, y) = 1 \). What is the largest sum of money which you cannot pay exactly?

Answer \( xy - x - y \) pence
<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$x + 0$</td>
<td>$2x + 0$</td>
<td>$3x + 0$</td>
<td>...</td>
</tr>
<tr>
<td>$y$</td>
<td>$x + y$</td>
<td>$2x + y$</td>
<td>$3x + y$</td>
<td>...</td>
</tr>
<tr>
<td>$2y$</td>
<td>$x + 2y$</td>
<td>$2x + 2y$</td>
<td>$3x + 2y$</td>
<td>...</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$(x - 1)y$</td>
<td>$x + (x - 1)y$</td>
<td>$2x + (x - 1)y$</td>
<td>$3x + (x - 1)y$</td>
<td>...</td>
</tr>
</tbody>
</table>

Claim: each row consists of numbers which give the same remainder on division by $x$. Different rows give rise to different remainders on division by $x$. If this is right, the largest inaccessible sum is $(x - 1)y - x$, i.e. $xy - x - y$. 

Geoff Smith
\[
\begin{array}{cccc}
0 & x + 0 & 2x + 0 & 3x + 0 \\
y & x + y & 2x + y & 3x + y \\
2y & x + 2y & 2x + 2y & 3x + 2y \\
\vdots & \vdots & \vdots & \vdots \\
(x - 1)y & x + (x - 1)y & 2x + (x - 1)y & 3x + (x - 1)y \\
\end{array}
\]

Claim: each row consists of numbers which give the same remainder on division by \(x\). Different rows give rise to different remainders on division by \(x\). If this is right, the largest inaccessible sum is \((x - 1)y - x\) i.e. \(xy - x - y\).
\[
\begin{array}{cccc}
0 & x + 0 & 2x + 0 & 3x + 0 \\
y & x + y & 2x + y & 3x + 3y \\
2y & x + 2y & 2x + 2y & 3x + 2y \\
\ldots & \ldots & \ldots & \ldots \\
(x - 1)y & x + (x - 1)y & 2x + (x - 1)y & 3x + (x - 1)y \\
\end{array}
\]

The numbers in the first column are:

\[
0 \cdot y, y, 2y, \ldots, iy, \ldots jy, \ldots, (x - 1)y
\]

with \(0 \leq i < j \leq x - 1\). We want them to have different remainders on division by \(x\).
The numbers in the first column are:

\[
\begin{array}{cccc}
0 & x + 0 & 2x + 0 & 3x + 0 \\
y & x + y & 2x + y & 3x + 3y \\
2y & x + 2y & 2x + 2y & 3x + 2y \\
\vdots & \vdots & \vdots & \vdots \\
(x - 1)y & x + (x - 1)y & 2x + (x - 1)y & 3x + (x - 1)y \\
\end{array}
\]

The numbers in the first column are:

\[
0 \cdot y, y, 2y, \ldots, iy, \ldots jy, \ldots, (x - 1)y
\]

with \(0 \leq i < j \leq x - 1\). We want them to have different remainders on division by \(x\). What could go wrong?
If $iy$ and $jy$ leave the same remainder on division by $x$, then $x$ divides $jy - iy$ so $x$ divides $(j - i)y$ (and $0 < j - i < x$). Now

$$\frac{(j - i)y}{x}$$

is an integer and $x$ does not cancel with $y$ (their hcf is 1) so $x$ divides into $j - i$ but $0 < j - i < x$. This is absurd, so $iy$ and $jy$ do NOT leave the same remainder on division by $x$. 

Geoff Smith
You have plenty of 9p and 11p coins. What is the largest sum which you cannot pay exactly?

Answer: 99 - 20 = 79 pence.
You have plenty of 9p and 11p coins. What is the largest sum which you cannot pay exactly?

Answer: $99 - 20 = 79$ pence.
Advice for young mathematicians who are interested in problem solving . . .

http://people.bath.ac.uk/masgcs/advice.html

or type Geoff Smith Bath advice into a search engine.
Advice for young mathematicians who are interested in problem solving . . .

http://people.bath.ac.uk/masgcs/advice.html

or type Geoff Smith Bath advice into a search engine.

Search on: IMO 2019 (and enjoy the video).