Euclidean Geometry for Maths Competitions

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In many cultures, the ancient Greek notion of organizing geometry into a deductive system was taught using Euclid's *Elements*, and the cultural consequences of this persist to this day.

Euclid is not a model of perfection

Euclid organized a body of knowledge concerning plane geometry very well, and set up an axiom system. He was sufficiently clear sighted to realise that he had no way of deducing the 'parallel postulate' from the other axioms.

However, by the standards of modern mathematics, Euclid's system looks very shaky. We now use the language of sets and maps to express mathematics with exquisite precision. Euclid had no such luxury, and is very vague about the meaning of "point" and "line".

Geometry is not uniquely suited to deductive reasoning

In the modern era, every branch of pure mathematics is a formal deductive system, and plane geometry has no special place, except for the historical accident that the first attempts to use the axiomatic method were made in that context.

Where does one start?

Well, it is possible to develop Euclidean Geometry in a very formal way, starting with the axioms. Some people advocate this as being a necessary part of education.

My **personal opinion** is that, for most people, this is not the sensible thing to do. Now, this is only a personal opinion, and some people would disagree strongly, but I will explain my attitude.

I think that the foundations of Euclidean Geometry, studied axiomatically, are rather tricky, and the need to make sure that the "House of Cards" is being built correctly actually distracts from the enjoyment of the subject.

I prefer that one regards it as intuitively clear what one means by "point", "line", "angle", "length" and "area". I think that it is also psychologically useful to accept basic angle facts without question: vertically opposite angles are equal, corresponding angles associated to a line transverse to a pair of parallel lines are equal. Simply accept that SSS, SAS, ASA and AAS(corresponding) are legitimate justifications for triangles to be congruent. Also accept that AAA and other related conditions are enough to justify the similarity of triangles (for example, two triangles involve the same angle, and the adjacent sides are in the same ratio).

These ideas could be analyzed carefully of course, and chased back to more primitive geometric notions. It is important for some people to do this, because they cannot abide informal foundations. However, from this position, I think that most people prefer to build palaces rather than dig to check that the foundations are solid. I am happy for other people to check the foundations, and to enjoy myself playing with the fancy architecture.

What should I learn?

Well, the best way to learn geometry is to do it. This means solving geometry problems. However, there are a range of standard theorems which are appropriate to different levels of mathematics competition. Of course you will need to know the basic "circle theorems" (angle in the alternate segment, angle subtended by an arc in a circle is half the angle subtended at the centre, angle in the same segment etc) and the theorem of Pythagoras. You should also understand the intersecting chords (and intersecting secants) theorem (also known as "power of a point"). On top of that, you should know at least a little trigonometry, certainly the sine rule ($a/\sin A = 2R$) and the cosine rule.

You will need to learn results about isosceles triangles, equilateral triangles, parallelograms, trapezia (trapezoids), cyclic quadrilaterals.

Then there some more advanced theorems which are not necessary for very elementary competitions, but are essential for international competitions. These results certainly include the theory of the Simson line and the Euler line, the theory of excircles and Ptolemy's theorem.

After that, it depends on your level of enthusiasm. You are likely to enjoy the famous theorems of projective geometry: Pascal's hexagram theorem, the theorem of Desargues, the theorem of Brianchon. If your interest is strictly practical (I want an IMO medal) then you do not have to know the proofs of these fantastic results. You can simply regard them as extra axioms of geometry and don't tell people your guilty secret that you have no idea why they are true.

You may want to supplement your Euclidean geometry techniques by becoming confident in one or more of the various algebraic systems which can be used to resolve geometric questions: trigonometry, complex numbers, vectors, areal co-ordinates (or the almost equivalent trilinear co-ordinates). You might get interested in tiling methods, or proofs by the statics of mechanical systems, or proofs by origami. There are also many formulas in the spirit of Heron's formula which may be usefully learned.

You should definitely know Ceva's theorem and the strange theorem of Menelaus. This last result often enables us to show by calculation that three points are collinear. Incidentally, I regard the use of Menelaus as involving an implicit reproach because it seems that when you find yourself using Menelaus, it means that you have missed a better way to prove the collinearity. However, if the only way you can see to prove a collinearity is to use Menelaus, then it is certainly the best method. It makes one feel dirty of course, but it is much better than being stuck and unable to prove the collinearity.

However, the algebra should be regarded as a bonus method. Most geometry questions found in maths competitions have (intended) solutions based on classic Euclidean techniques. Being confident at algebraic methods gives you more lines of attack if you don't see a Euclidean proof. However, simply learning the algebraic machines in order to have an exhaustive knowledge of methods of attack is like purchasing lots of expensive sports cars and keeping them in your garage. Most of the time, the fast solution is by Euclidean methods, and algebra plays only a supplementary role, if any. The truth is, similar triangles and ingenuity are enough to solve most maths competition problems.

British Books

You should look for good material written in your language. By chance I happen to be British. My maths enrichment organization is UKMT. It has a publishing arm which currently produces three geometry books which are very relevant to competition geometry. See `http://www.ukmt.org.uk/publications`

Bradley and Gardiner's *Plane Euclidean Geometry* has seven chapters. I recommend starting with Chapter 3. The first two chapters concern digging for the foundations. Of course, that might appeal to you, but it is not necessary for competition mathematics.

The two books by Gerry Leversha are "Crossing the Bridge" (a not very scrutable reference to the *Pons Asinorum*), and the more sensibly titled "Geometry of the Triangle".

If you come from a developed country, it is quite possible that you will have a national mathematics enrichment organization, and it may publish materials.

Local books and illegal books

If you come from a developing country, then the cost of books priced in hard currency may be prohibitive, and it is less likely (but not impossible) that you will have a good national mathematics enrichment organization. Look to see if there are good materials available produced in your own country. Also Kiran Kedlaya's wonderful "Geometry Unbound" has been made freely available by the author (find the PDF using a search engine).

Of course there are illegal copies of classic geometry books on the internet. If you come from a developed country, I hope that you realise that it is illegal, immoral and corrosive to steal intellectual property. I make no judgement on people from poor countries who download such materials, for their situation is very different.

How should I study?

Solving geometry problems is easily the best way to become a good geometer. Simply reading the theory will not do it. You have to engage directly with problems.

It is really, really important to draw good diagrams. This is because a clean, accurately drawn diagram is very likely to give you clues as to what is true. It will reveal apparent collinearities of points, concurrencies of lines, and sometimes the fact that four points lie on a circle. Such help is invaluable. Time spent drawing a good diagram (using a sharp pencil, ruler and compasses) is not wasted. It is simply an investment which will often quickly reward you with clues about how to solve the problem. It is a very good idea to practise drawing diagrams accurately (and eventually, quickly).