1. Suppose that G is a group.

   (a) Define a relation $\sim$ on G by $x \sim y$ if and only if there exists $g \in G$ such that $g^{-1}xg = y$. Prove that this is an equivalence relation on G (see section 1.18 in the book). The equivalence classes of this equivalence relation are called the conjugacy classes of G.

   (b) Prove that G is an abelian group if and only if each of its conjugacy classes has size 1.

   (c) Calculate the conjugacy classes of S_3, the symmetric group on three letters (see section 5.3 in the book).

   (d) Give an example of a group which has infinitely many distinct conjugacy classes.

2. Suppose that G is a group. A subgroup N of G is said to be *normal* in G if and only if whenever $x \in G$, then $xN = Nx$.

   (a) Show that the group S_3 has a normal subgroup of size 3.

   (b) Prove that the subgroup M of G is normal in G if and only if $x^{-1}Nx = N$ for every $x \in G$.

   (c) Suppose that N is a normal subgroup of G, that C is a conjugacy class of G (see question 1), and that $N \cap C \neq \emptyset$. Prove that $C \subseteq N$.

3. Suppose that G is a group, that $x \in G$ and that C is the conjugacy class of G which contains x. Let $C_G(x) = \{g \mid g \in G,\ g^{-1}xg = x\}$ (this set is called the *centralizer of x in G*).

   (a) Show that $C_G(x)$ is a subgroup of G.

   (b) Define Let $S = \{C_G(x)a \mid a \in G\}$ be the set of right cosets of $C_G(x)$ in G. Attempt to define a map $\theta : S \to C$ by $C_G(x)z \mapsto z^{-1}xz$. *You need to be careful here. You need to verify that if $C_G(x)z_1 = C_G(x)z_2$, then $z_1^{-1}xz_1 = z_2^{-1}xz_2$, otherwise this definition makes no sense. Think about this remark carefully, and make sure you understand its significance. This is a subtle and important point.*

   (c) Show that the map θ of part (b) is bijective.

   (d) Show that if G is a finite group, then $|C||C_G(x)| = |G|$.