Group Actions

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1 Group Actions

Let $G$ be a group and $\Omega$ be a non-empty set. An action of $G$ on $\Omega$ is a map $\Omega \times G \rightarrow \Omega$ usually denoted by an infix symbol $\cdot$, or simply by juxtaposition if this is unambiguous, which satisfies two axioms.

(i) $\omega \cdot 1_G = \omega \ \forall \omega \in \Omega$.

(ii) $\omega \cdot (gh) = (\omega \cdot g) \cdot h \ \forall \omega \in \Omega, \forall g, h \in G$.

Where there is a group operation under discussion, we reserve juxtaposition for that, and use the dot to denote the group action.

Example 1.1

(a) $G = \Omega$, and we define

$$\omega \cdot g = \omega g \ \forall \omega \in \Omega, \forall g \in G.$$  

(b) $G = \Omega$, and we define

$$\omega \cdot g = g^{-1} \omega \ \forall \omega \in \Omega, \forall g \in G.$$  

(c) $G = \Omega$, and we define

$$\omega \cdot g = g^{-1} \omega g \ \forall \omega \in \Omega, \forall g \in G.$$  

(d) $H \leq G, \Omega = H\backslash G = \{Hx \mid x \in G\}$. We define

$$Hy \cdot g = H(yg) \ \forall x, y \in G.$$  

(e) $G = \text{Sym}(\Omega)$ where $\Omega$ is a non-empty set. Here $G$ consists of all the bijections from $\Omega$ to $\Omega$, and for the purposes of this course, if $f, g \in \text{Sym}(\Omega)$, then $fg \in \text{Sym}(\Omega)$ is defined by $fg : \omega \mapsto ((\omega)f)g$. Thus maps are written on the right. Now $G$ acts on $\Omega$ via

$$\omega \cdot f = (\omega)f \ \forall f \in \text{Sym}(\Omega), \forall \omega \in \Omega.$$  

(f) Let $k$ be a field, and suppose that $n \in \mathbb{N}$. Let $G = \text{GL}(n, k)$ denote the set of invertible $n$ by $n$ matrices with entries in $k$. This $G$ is a group under matrix multiplication. Let $V = k^n$ be the set of row vectors of length $n$ with entries in $k$. Now $G$ acts on $V$ via matrix multiplication.
Definition 1.2 If \(G\) acts on \(\Omega\) and \(\omega \in \Omega\), then we define two important concepts.

(i) \(\omega G = \{\omega \cdot g \mid g \in G\}\) is called the \(G\)-orbit of \(\omega\), or just the orbit of \(\omega\) where there no confusion.

(ii) \(G_\omega = \{g \mid g \in G, \omega \cdot g = \omega\}\). It is easy to verify that \(G_\omega\) is a subgroup of \(G\). This group is called the isotropy group of \(\omega\) or the stabilizer of \(\omega\).

Lemma 1.3 Let \(G\) act on \(\Omega\). Write \(\omega_1 \sim \omega_2\) if and only if there is \(g \in G\) with \(\omega_1 \cdot g = \omega_2\). It follows that \(\sim\) is an equivalence relation on \(\Omega\) and the equivalence classes are the orbits.

Proof For every \(\omega \in \Omega\) we have \(\omega \cdot 1 = \omega\) by the first group action axiom, so \(\sim\) is reflexive. Now suppose that \(\omega_1, \omega_2 \in \Omega\) and \(\omega_1 \sim \omega_2\). Thus there is \(g \in G\) such that \(\omega_1 \cdot g = \omega_2\). Thus \((\omega_1 \cdot g) \cdot g^{-1} = \omega_2 \cdot g^{-1}\) and so \(\omega_1 \cdot (gg^{-1}) = \omega_2 \cdot g^{-1}\) by the second group action axiom. Thus \(\omega_1 \cdot 1_G = \omega_1 = \omega_2 \cdot g^{-1}\) by the first group action axiom. Thus \(\sim\) is symmetric. Now for transitivity: suppose that \(\omega_1 \sim \omega_2\) and \(\omega_2 \sim \omega_3\). There are \(x, y \in G\) with \(\omega_1 \cdot x = \omega_2\) and \(\omega_2 \cdot y = \omega_3\). Now

\[\omega_1 \cdot (xy) = (\omega_1 \cdot x) \cdot y = \omega_2 \cdot y = \omega_3.\]

Thus \(\sim\) is transitive and so is an equivalence relation.

The equivalence class of \(\omega \in \Omega\) is \(\{\omega g \mid g \in G\}\) and this is just the orbit \(\omega G\).

\(\square\)

Lemma 1.4 There is a natural bijection \(\beta : G_\omega \backslash G \rightarrow \omega G\) defined by \(\beta : G_\omega x \mapsto \omega x\) for all \(x \in G\).

Proof The notation \(G_\omega \backslash G\) denotes \(\{G_\omega x \mid x \in G\}\), the set of right cosets of \(G_\omega\) in \(G\). We must first check that the map is well defined, so we assume we have rival descriptions of the same coset: \(G_\omega x_1 = G_\omega x_2\) for \(x_1, x_2 \in G\). Thus \(x_1 x_2^{-1} \in G_\omega\) so \(\omega \cdot (x_1 x_2^{-1}) = \omega\). Act via \(x_2\) to deduce that \(\omega \cdot x_1 = \omega \cdot x_2\), and \(\beta\) is well defined.

Define \(\gamma : \omega G \rightarrow G_\omega \backslash G\) via \(\omega \cdot x \mapsto G_\omega x\). Again there is the issue as to whether or not \(\gamma\) is well defined. Suppose that \(\omega \cdot x_1 = \omega \cdot x_2\) for \(x_1, x_2 \in G\). Now \(x_1 x_2^{-1} \in G_\omega\) so \(G_\omega x_1 = G_\omega x_2\), and \(\gamma\) is well defined. Now note that \(\beta\) and \(\gamma\) are mutually inverse, and are therefore both bijections. Thus \(\beta\) is a bijection.

\(\square\)

The cardinality of \(G_\omega \backslash G\) is denoted \(|G : G_\omega|\). If \(G\) happens to be finite this quantity is \(|G|/|G_\omega|\).

Corollary 1.5 The cardinality of the orbit \(\omega G\) is \(|G : G_\omega|\). If \(G\) is finite and \(\theta \in \omega G\), then \(|G : G_\omega| = |G : G_\theta|\), so \(|G_\omega| = |G_\theta|\).

However, it is not just a matter of size, as the next proposition shows.

Proposition 1.6 Suppose that \(G\) acts on the non-empty set \(\Omega\), that \(\omega_1, \omega_2 \in \Omega\). If \(h \in G\), and \(\omega_1 \cdot h = \omega_2\), then \(G_{\omega_2} = h^{-1}G_{\omega_1}h\).
Proof We suppose that \( \omega_1 \cdot h = \omega_2 \). Choose \( x \in h^{-1}G \omega_1 h \), then \( x = h^{-1}y \) for some \( y \in G \omega_1 \). Now

\[
\omega_2 \cdot x = (\omega_1 \cdot h) \cdot (h^{-1}y) = \omega_1 \cdot (yh) = (\omega_1 \cdot y) \cdot h = \omega_1.
\]

Thus \( x \in G \omega_2 \). Next suppose that \( p \in G \omega_2 \), so \( \omega_2 \cdot p = \omega_2 \) and therefore \( (\omega_1 \cdot h) \cdot p = \omega_1 \cdot h \). Thus \( \omega_1 \cdot (hph^{-1}) = \omega_1 \) and so \( hph^{-1} \in G \omega_1 \). Premultiply by \( h^{-1} \) and postmultiply by \( h \) to obtain \( p \in h^{-1}G \omega_1 h \).

\[\square \]

**Theorem 1.7 (not Burnside)** Let \( G \) be a finite group acting on a non-empty finite set \( \Omega \). The number of orbits of \( G \) on \( \Omega \) is

\[
\frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)|.
\]

**Proof** Let \( \Gamma = \{ (\omega, g) \mid \omega \in \Omega, g \in G, \omega \cdot g = \omega \} \subseteq \Omega \times G \). We count \( \Gamma \) in two ways, and equate the answers.

(i) \( |\Gamma| = \sum_{g \in G} |\text{Fix}(g)| \).
(ii) \( |\Gamma| = \sum_{\omega \in \Omega} |G\omega| \). Now let the distinct \( G \)-orbits be \( \omega_1 G, \omega_2 G, \ldots, \omega_t G \). Observe that for all \( \alpha \in \omega_j G \) we have \( |G\alpha| = |G\omega_j| \). Thus

\[
\Gamma = \sum_{i=1}^{t} \left( \sum_{\alpha \in \omega_i G} |G\alpha| \right) = \sum_{i=1}^{t} \left( \sum_{\alpha \in \omega_i G} |G\omega_i| \right) = \sum_{i=1}^{t} |\omega_i G||G\omega_i| = \sum_{i=1}^{t} |G : G\omega_i||G\omega_i| = \sum_{i=1}^{t} |G| = t|G|.
\]

Equate these answers, solve for \( t \) and we are done. \[\square \]

**Example 1.8** \( G = \text{Sym}(n) \) acts on \( \Omega = \{1, 2, \ldots, n\} \) in natural fashion via \( i \cdot f = (i)f \) where \( f : \Omega \rightarrow \Omega \) is a bijection. Notice that there \( 1 \cdot G = \Omega \) so there is a single orbit. Applying our theorem we have

\[
1 = \frac{1}{n!} \sum_{g \in \text{Sym}(n)} |\text{Fix}(g)|,
\]

so an average permutation of \( n \) letters fixes 1 of them!

**Example 1.9** Next the action of \( G \) on \( \Omega \) induces an action of \( G \) on \( \Omega^2 = \Omega \times \Omega \) via \( (\omega_1, \omega_2) \cdot g = (\omega_1 \cdot g, \omega_2 \cdot g) \). There are two orbits of \( G \) on \( \Omega^2 \): the diagonal \( \{ (\omega, \omega) \mid \omega \in \Omega \} \) and the rest. Thus we have

\[
2 = \frac{1}{n!} |\text{Fix}(g)|^2
\]

where \( \text{Fix}(g) \) still denotes the fixed point set of \( G \) acting on \( \Omega \). Thus the average value of the square of the size of the fixed point set of a permutation \( n \) letters is 2.
We verify this for $G = \text{Sym}(3)$. The six permutations are

$$\text{id, (12), (13), (23), (123), (132)}$$

and our example predicts that

$$\frac{3^2 + 1^2 + 1^2 + 1^2 + 0^2 + 0^2}{6} = 2,$$

which (happily) is true.

Similarly there are 5 orbits of $G = \text{Sym}(3)$ on $\Omega^3$ (representatives $1,1,1), (1,2,2), (1,1,2), (1,2,1),$ and $(1,2,3)$. This squares with

$$\frac{3^3 + 1^3 + 1^3 + 1^3 + 0^3 + 0^3}{6} = 5.$$

**Example 1.10** Now we discuss how many essentially different ways there are to colour the faces of a cube using $c$ colours. Two colourings are deemed to be *not essentially different* if the cube may be rotated from one etc the other. Here it is understood that each face must be painted monochromatically. Let $\Omega$ be the set of all painted cubes. Thus $|\Omega| = c^6$. Let $G$ be the group of rotational motions of the cube. Thus $|G| = 24$. We classify the elements of $G$ geometrically.

(a) There is the identity, which does nothing.

(b) There are motions which are a rotation through $2\pi/3$ about a long diagonal. There are 8 of these.

(c) There are motions which are a rotation through $\pi/2$ about a straight line joining the centres of two opposite edges. There are 6 of these.

(d) There are motions which are a rotation through $\pm\pi/4$ about a straight line joining the centres of two opposite faces. There are 6 of these.

(e) There are motions which are a rotation through $\pm\pi/2$ about a straight line joining the centres of two opposite faces. There are 3 of these.

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</table>

The number of colourings is therefore

$$\frac{c^6 + 3c^4 + 12c^3 + 8c^2}{24}$$
When $c = 1$ this is $24/24 = 1$. When $c = 2$ this is $\frac{64+48+72+32}{24} = \frac{216}{24} = 9$ which is easy enough to verify in your head. When $c = 2$ this is

$$\frac{729 + 243 + 12 \times 27 + 72}{24} = 57.$$ 

Thus there are 57 essentially different face colourings of a cube using three different colours.