Group Actions

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1 Group Actions

Let G be a group and Ω be a non-empty set. An *action* of G on Ω is a map $\Omega \times G \to \Omega$ usually denoted by an infix symbol $\cdot$, or simply by juxtaposition if this is unambiguous, which satisfies two axioms.

(i) $\omega \cdot 1_G = \omega \ \forall \omega \in \Omega$.

(ii) $\omega \cdot (gh) = (\omega \cdot g) \cdot h \ \forall \omega \in \Omega, \ \forall g, h \in G$.

Where there is a group operation under discussion, we reserve juxtaposition for that, and use the dot to denote the group action.

Example 1.1

(a) $G = \Omega$, and we define

$$\omega \cdot g = \omega g \ \forall \omega \in \Omega, \forall g \in G.$$

(b) $G = \Omega$, and we define

$$\omega \cdot g = g^{-1}\omega \ \forall \omega \in \Omega, \forall g \in G.$$

(c) $G = \Omega$, and we define

$$\omega \cdot g = g^{-1}\omega g \ \forall \omega \in \Omega, \forall g \in G.$$

(d) $H \leq G$, $\Omega = H\backslash G = \{Hx \mid x \in G\}$. We define

$$Hy \cdot g = H(yg) \ \forall x, y \in G.$$

(e) $G = \mathrm{Sym}(\Omega)$ where Ω is a non-empty set. Here G consists of all the bijections from Ω to Ω, and for the purposes of this course, if $f, g \in \mathrm{Sym}(\Omega)$, then $fg \in \mathrm{Sym}(\Omega)$ is defined by $fg : \omega \mapsto ((\omega)f)g$. Thus maps are written on the right. Now G acts on Ω via

$$\omega \cdot f = (\omega)f \ \forall f \in \mathrm{Sym}(\Omega), \ \forall \omega \in \Omega.$$

(f) Let k be a field, and suppose that $n \in \mathbb{N}$. Let $G = \mathrm{GL}(n, k)$ denote the set of invertible n by n matrices with entries in k. This G is a group under matrix multiplication. Let $V = k^n$ be the set of row vectors of length n with entries in k. Now G acts on V via matrix multiplication.

Definition 1.2 If G acts on Ω and $\omega \in \Omega$, then we define two important concepts.

(i) $\omega G = \{\omega \cdot g \mid g \in G\}$ is called the G-orbit of ω, or just the orbit of ω where there no confusion.

(ii) $G_\omega = \{g \mid g \in G, \omega \cdot g = \omega\}$. It is easy to verify that G_ω is a subgroup of G. This group is called the *isotropy group* of ω or the *stabilizer* of ω.

Lemma 1.3 *Let G act on Ω. Write $\omega_1 \sim \omega_2$ if and only if there is $g \in G$ with $\omega_1 \cdot g = \omega_2$. It follows that $\sim$ is an equivalence relation on Ω and the equivalence classes are the orbits.*

Proof For every $\omega \in \Omega$ we have $\omega \cdot 1 = \omega$ by the first group action axiom, so $\sim$ is reflexive. Now suppose that $\omega_1, \omega_2 \in \Omega$ and $\omega_1 \sim \omega_2$. Thus there is $g \in G$ such that $\omega_1 \cdot g = \omega_2$. Thus $(\omega_1 \cdot g) \cdot g^{-1} = \omega_2 \cdot g^{-1}$ and so $\omega_1 \cdot (gg^{-1}) = \omega_2 \cdot g^{-1}$ by the second group action axiom. Thus $\omega_1 \cdot 1_G = \omega_1 = \omega_2 \cdot g^{-1}$ by the first group action axiom. Thus $\sim$ is symmetric. Now for transitivity: suppose that $\omega_1 \sim \omega_2$ and $\omega_2 \sim \omega_3$. There are $x, y \in G$ with $\omega_1 \cdot x = \omega_2$ and $\omega_2 \cdot y = \omega_3$. Now

$$\omega_1 \cdot (xy) = (\omega_1 \cdot x) \cdot y = \omega_2 \cdot y = \omega_3.$$

Thus $\sim$ is transitive and so is an equivalence relation.

The equivalence class of $\omega \in \Omega$ is $\{\omega g \mid g \in G\}$ and this is just the orbit ωG. □

Lemma 1.4 *There is a natural bijection $\beta : G_\omega \backslash G \to \omega G$ defined by $\beta : G_\omega x \mapsto \omega x$ for all $x \in G$.*

Proof The notation $G_\omega \backslash G$ denotes $\{G_\omega x \mid x \in G\}$, the set of right cosets of G_ω in G. We must first check that the map is well defined, so we assume we have rival descriptions of the same coset: $G_\omega x_1 = G_\omega x_2$ for $x_1, x_2 \in G$. Thus $x_1 x_2^{-1} \in G_\omega$ so $\omega \cdot (x_1 x_2^{-1}) = \omega$. Act via x_2 to deduce that $\omega \cdot x_1 = \omega \cdot x_2$, and β is well defined.

Define $\gamma : \omega G \to G_\omega \backslash G$ via $\omega \cdot x \mapsto G_\omega x$. Again there is the issue as to whether or not γ is well defined. Suppose that $\omega \cdot x_1 = \omega \cdot x_2$ for $x_1, x_2 \in G$. Now $x_1 x_2^{-1} \in G_\omega$ so $G_\omega x_1 = G_\omega x_2$, and γ is well defined. Now note that β and γ are mutually inverse, and are therefore both bijections. Thus β is a bijection. □

The cardinality of $G_\omega \backslash G$ is denoted $|G : G_\omega|$. If G happens to be finite this quantity is $|G|/|G_\omega|$.

Corollary 1.5 *The cardinality of the orbit ωG is $|G : G_\omega|$. If G is finite and $\theta \in \omega G$, then $|G : G_\omega| = |G : G_\theta|$, so $|G_\omega| = |G_\theta|$.*

However, it is not just a matter of size, as the next proposition shows.

Proposition 1.6 *Suppose that G acts on the non-empty set Ω, that $\omega_1, \omega_2 \in \Omega$. If $h \in G$, and $\omega_1 \cdot h = \omega_2$, then $G_{\omega_2} = h^{-1} G_{\omega_1} h$.*

Proof We suppose that $\omega_1 \cdot h = \omega_2$. Choose $x \in h^{-1}G_{\omega_1}h$, then $x = h^{-1}yh$ for some $y \in G_{\omega_1}$. Now

$$\omega_2 \cdot x = (\omega_1 \cdot h) \cdot (h^{-1}yh) = \omega_1 \cdot (yh) = (\omega_1 \cdot y) \cdot h = omega_1 \cdot h = \omega_1.$$

Thus $x \in G_{\omega_2}$. Next suppose that $p \in G_{\omega_2}$, so $\omega_2 \cdot p = \omega_2$ and therefore $(\omega_1 \cdot h) \cdot p = \omega_1 \cdot h$. Thus $\omega_1 \cdot (hph^{-1}) = \omega_1$ and so $hph^{-1} \in G_{\omega_1}$. Premultiply by h^{-1} and postmultiply by h to obtain $p \in h^{-1}G_{\omega_1}h$. □

Theorem 1.7 (not Burnside) *Let G be a finite group acting on a non-empty finite set Ω. The number of orbits of G on Ω is*

$$\frac{1}{|G|}\sum_{g\in G}|Fix(g)|\,.$$

Proof Let $\Gamma = \{(\omega, g) \mid \omega \in \Omega, g \in G, \omega \cdot g = \omega\} \subseteq \Omega \times G$. We count Γ in two ways, and equate the answers.

(i) $|\Gamma| = \sum_{g\in G} |\mathrm{Fix}(g)|$.

(ii) $|\Gamma| = \sum_{\omega\in\Omega} |G_\omega|$. Now let the distinct G-orbits be $\omega_1G, \omega_2G, \ldots, \omega_tG$. Observe that for all $\alpha \in \omega_jG$ we have $|G_\alpha| = |G_{\omega_j}|$. Thus

$$\Gamma = \sum_{i=1}^{t}\left(\sum_{\alpha\in\omega_iG}|G_\alpha|\right) = \sum_{i=1}^{t}\left(\sum_{\alpha\in\omega_iG}|G_{\omega_i}|\right)$$

$$= \sum_{i=1}^{t}|\omega_iG||G_{\omega_i}| = \sum_{i=1}^{t}|G : G_{\omega_i}||G_{\omega_i}| = \sum_{i=1}^{t}|G| = t|G|.$$

Equate these answers, solve for t and we are done. □

Example 1.8 $G = \mathrm{Sym}(n)$ acts on $\Omega = \{1, 2, \ldots, n\}$ in natural fashion via $i \cdot f = (i)f$ where $f : \Omega \to \Omega$ is a bijection. Notice that there $1 \cdot G = \Omega$ so there is a single orbit. Applying our theorem we have

$$1 = \frac{1}{n!}\sum|\mathrm{Fix}(g)|,$$

so an average permutation of n letters fixes 1 of them!

Example 1.9 Next the action of G on Ω induces an action of G on $\Omega^2 = \Omega\times\Omega$ via $(\omega_1, \omega_2) \cdot g = (\omega_1 \cdot g, \omega_2 \cdot g)$. There are two orbits of G on Ω^2: the diagonal $\{(\omega, \omega) \mid \omega \in \Omega\}$ and the rest. Thus we have

$$2 = \frac{1}{n!}|\mathrm{Fix}(g)|^2$$

where Fix(g) still denotes the fixed point set of G acting on Ω. Thus the average value of the square of the size of the fixed point set of a permutation n letters is 2.

We verify this for $G = \mathrm{Sym}(3)$. The six permutations are

$$\mathrm{id}, (12), (13), (23), (123), (132)$$

and our example predicts that

$$\frac{3^2 + 1^2 + 1^2 + 1^2 + 0^2 + 0^2}{6} = 2,$$

which (happily) is true.

Similarly there are 5 orbits of $G = \mathrm{Sym}(3)$ on Ω^3 (representatives $(1, 1, 1)$, $(1, 2, 2)$, $(1, 1, 2)$, $(1, 2, 1)$, and $(1, 2, 3)$. This squares with

$$\frac{3^3 + 1^3 + 1^3 + 1^3 + 0^3 + 0^3}{6} = 5.$$

Example 1.10 Now we discuss how many essentially different ways there are to colour the faces of a cube using c colours. Two colourings are deemed to be *not essentially different* if the cube may be rotated from on etc the other. Here it is understood that each face must be painted monochromatically. Let Ω be the set of all painted cubes. Thus $|\Omega| = c^6$. Let G be the group of rotational motions of the cube. Thus $|G| = 24$. We classify the elements of G geometrically.

(a) There is the identity, which does nothing.

(b) There are motions which are a rotation through $2\pi/3$ about a long diagonal. There are 8 of these.

(c) There are motions which are a rotation through $\pi/2$ about a straight line joining the centres of two opposite edges. There are 6 of these.

(d) There are motions which are a rotation through $\pm\pi/4$ about a straight line joining the centres of two opposite faces. There are 6 of these.

(e) There are motions which are a rotation through $\pm\pi/2$ about a straight line joining the centres of two opposite faces. There are 3 of these.

Element type	number of this type	size of Fix	total size of fix
a	1	c^6	c^6
b	8	c^2	$8c^2$
c	6	c^3	$6c^3$
d	6	c^3	$6c^3$
e	3	c^4	$3c^4$

The number of colourings is therefore

$$\frac{c^6 + 3c^4 + 12c^3 + 8c^2}{24}$$

When $c = 1$ this is $24/24 = 1$. When $c = 2$ this is $\frac{64+48+72+32}{24} = 216/24 = 9$ which is easy enough to verify in your head. When $c = 2$ this is

$$\frac{729 + 243 + 12 * 27 + 72}{24} = 57.$$

Thus there are 57 essentially different face colourings of a cube using three different colours.