A comparison of models and methods for simulating the microwave heating of moist foodstuffs

C.J. Budd and A.D.C. Hill
Centre for Nonlinear Mechanics, University of Bath, Bath, BA2 7AY, UK

Abstract

We study the problem of heating a one-dimensional approximation to a slb-sided moist foodstuff in a microwave oven, allowing for a phase change and drying. We initially investigate the accuracy of the Lambert law of exponential decay of the applied electric field into the foodstuff and derive an approximation for the field comprising the exponential decay term and an oscillatory component. We then show that the temperature of the foodstuff is given, to a good approximation, by only considering the heating effects of the exponentially decaying field. We then study the effects of drying. This process changes the dielectric properties of the material, which leads to changes in the field. Hoever, these lead to smaller changes in the moisture content. A fast and accurate numerical method is derived which relies on smoothing the phase transition.

1 Introduction

Microwave ovens are frequently used in domestic situations for the heating of chilled foodstuffs which are generally approximately 80% by weight water. Rapid, internal heating are some of the key benefits over conventional ovens. The food industry utilises these attributes in a number of ways most in particular with the introduction of microwave ready meals and convenience food. It is important that the food properly heated to ensure that is is micro-biologically safe for consumption. One of the principle modes of heating in microwave cooking is through dipole orientation. In the case of foodstuffs there is a large concentration of polar water molecules, when exposed to an electromagnetic field these molecules attempt to align themselves with the field. Domestic microwave ovens typically use an EM frequency of 2.45GHz and so the water attempts to line up with a rapidly changing field. Internal reflections within the oven cavity can lead to standing wave patterns forming in the EM field, and these troughs in the field can lead to so called "cold spots" in the food. It is in these regions of low temperature that harmful organisms can propagate and survive.

One of the main difficulties in modelling microwave cooking mathematically is that of determining the electromagnetic field both inside and outside the foodstuff. Maxwell's equations can be difficult to solve and the continuously changing field patterns require a new field solution to be calculated frequently. Full three dimensional electromagnetic calculations of the solution to Maxwell's equations must then be coupled to a suitable model for heat and moisture transport within a foodstuff, taking into account the phase changes within the foodstuff and also the change in the various dielectric constants. Such calculations can take many hours [4] and it is consequently difficult to use these approaches to consider the effects of parameter variations in the design of microwave heating devices. An effective approach to speeding up these computations is to derive simplified, yet accurate, approximations to both the electromagnetic field and to the resulting heating patterns in the food. This can result in very significant speed ups of the calculations, although this is at the expense of a degree of accuracy in the calculations. The purpose of this paper is to study certain of these approximate models, to derive estimates on their accuracy and suitability for modelling and to determine efficient numerical methods to find useful approximations to their solution. A common approximation to the field within a heated sample, known as Lambert's law, is to assume an exponential decay of the field intensity with depth from the surface of the foodstuff. Lambert's law is derived from Maxwell's equations in one dimension [8], and the model is valid for semi infinite domains. The Lambert law model can be used to model the moisture changes within a heated sample [14] and [9]. The work carried out by Ni [9] results in an extensive moisture transport model for microwave heating in one dimension. Significantly the dielectric properties of a sample are found to be highly dependent on the moisture content. When the electric field is calculated using Lambert's law, the dielectric properties must be recalculated to reflect the changing moisture content of the sample. It is found that the field was able to penetrate further into a dry sample than a wet one, because as the moisture content decreases, the materials ability to absorb microwave energy also decreases. This leads to a change in the electric field and the power absorbed by the sample. This will have an effect on the temperature and moisture content of the load. However, the Lambert law approximation to the field does not take into account the internal reflections which can occur in shorter sample lengths where the internal reflected waves interfere with the incident waves resulting in a standing wave pattern. The resulting fields take the form of an oscillating electric field intensity centred around an exponential decay [7]. A comparison of Lambert's law and the exact solution to Maxwell's equations is given in the paper by Ayappa et al. [1]. Basak [2] extends the work of Ayappa to conduct an analysis on a multilayered material which undergoes a phase change from frozen to liquid. The numerical studies find spatial resonance patterns in the multilayered slabs. In this paper we study a simple model of the microwave heating of a chilled and moist foodstuff, with the main example being mashed potato which comprises a mixture of starch and water. When heating such a foodstuff the moisture content remains nearly constant until a temperature of $T_b = 100°C$ is reached at which point the starch starts to dry out, leading to a change in the moisture content and a consequent change in the dielectric properties of the foodstuff (which also depend weakly upon temperature) [11],[10]. The purpose of the analysis presented in this paper is twofold. Firstly we compare and contrast the foodstuff temperature profiles that result from using Maxwell's equations for the field, from those given by the Lambert law. Secondly we consider the differences between the temperature and moisture profiles that arise when using constant dielectric properties with those that arise with dielectric properties which depend upon temperature and moisture content. We consider both an analytic approach and also a numerical approach related to a smoothed form of the enthalpy method. For the purposes of this investigation, we investigate the oscillations in the field and temperature inside a one dimensional sample of food. This is a reasonable first approximation to a slab-sided foodstuff. In a related paper [3] the results of this paper are applied to study the temperature and moisture content of a slab-sided foodstuff with a more realistic geometry, and the results are compared with experiment.

The main conclusions of this paper are that, for the typical wavelength of microwaves used in cooking, the difference in the temperature profile calculated from using the Lambert law approximation for the field from that calculated by solving Maxwell's equations, are small provided that the foodstuff is more than 2cm in extent. The difference is manifest as an oscillation about a decaying solution and the relative amplitude of this temperature oscillation is significantly smaller than the oscillations of the field strength around the exponential decay profile. We also conclude that there is not a particularly significant difference in the temperature and moisture content profiles that arise when using constant values for the dielectric parameters from those given by variable parameters. A further conclusion is that the smoothed numerical method we employ gives a fast and accurate way of calculating the temperature and moisture profiles provided that the smoothing parameter is chosen carefully.
The layout of this paper is as follows. In Section 2 we outline the basic theory for the field equations interior to the foodstuff and derive both the Lambert law approximation and the oscillatory correction to this. In Section 3 we determine the resulting temperature distribution of a two-phase moist foodstuff. In Section 4 we determine the effects of dielectric variation under changes in moisture and temperature and make a numerical calculation of the resulting temperature by using a version of the smoothed Enthalpy Method. Finally in Section 5 we draw some further conclusions from this work.

2 The Field Distribution

2.1 Lambert's law

We will consider a one-dimensional foodstuff with microwave radiation incident from the left and with the boundary of the foodstuff at the position $x = 0$. (This is a not unreasonable approximation to the geometry of a slab-sided food in a microwave oven). Initially we assume that the foodstuff occupies the whole region $0 \leq x$ with x the distance into the food. The electric field intensity E of the electromagnetic field with frequency, ω, obeys Maxwell's equations, which in a one-dimensional medium reduce to the Helmholtz equation [1].

$$E_{xx} + \lambda^2 E = 0 \tag{1}$$

where

$$\lambda^2 = \omega^2 \mu \epsilon_0 \kappa^*(x). \tag{2}$$

and $\mu(x)$ is the magnetic permeability of the propagating medium, ϵ_0 is the permittivity of free space and $\kappa^*(x) = \kappa'(x) + i\kappa''(x)$ is the complex dielectric. Setting

$$\lambda = \alpha + i\beta$$

yields the attenuation coefficient

$$\beta = \omega\sqrt{\mu\epsilon_0}\sqrt{\frac{\kappa'(\sqrt{1+\tan^2(\delta)}-1)}{2}}, \qquad \tan(\delta) = \frac{\kappa''}{\kappa'}. \tag{3}$$

We now compare two solutions to the above Helmholtz equation making the initial assumption that the dielectric properties of the foodstuff remain constant in space and time throughout the heating process. The general solution of (1) for constant λ is

$$E = A_0 e^{i\lambda x} + B_0 e^{-i\lambda x} \tag{4}$$
$$= A_0 e^{i\alpha x} e^{-\beta x} + B_0 e^{-i\alpha x} e^{\beta x} \tag{5}$$

When considering a semi-infinite domain $0 \leq x < \infty$ we must impose the condition $B_0 = 0$ to prevent $|E| \to \infty$ as $x \to \infty$. The power P absorbed by a sample per unit volume is given by [8]

$$P = \frac{1}{2}\omega\varepsilon_0\kappa''|E|^2 \tag{6}$$

so that in this case

$$P = Q_0 e^{-2\beta x} \tag{7}$$

where Q_0 is the power density at the surface of the material exposed to the electro-magnetic field. This description of the exponentially decaying power is precisely the Lambert law. For a typical moist starchy foodstuff

$$\alpha \approx 450m^{-1} \quad \text{and} \quad \beta \approx 60m^{-1}.$$

2.2 The power absorbed by a finite section of foodstuff

We will now summarise some of the work of Ayappa [1] who derived an expression for the power absorbed by a *finite section* of foodstuff occupying the region $0 \leq x \leq L$ and will compare this with the predictions of Lambert's law for varying lengths of domain. This formulation will take into account internal reflections within the foodstuff. We examine a section of foodstuff as in the diagram below, Figure 1.

We denote the region to the left of the foodstuff with the subscript 1, the region within the foodstuff by 2 and the region to the right of the foodstuff as 3, and use subscripts on all coefficients to represent the appropriate region . In each of

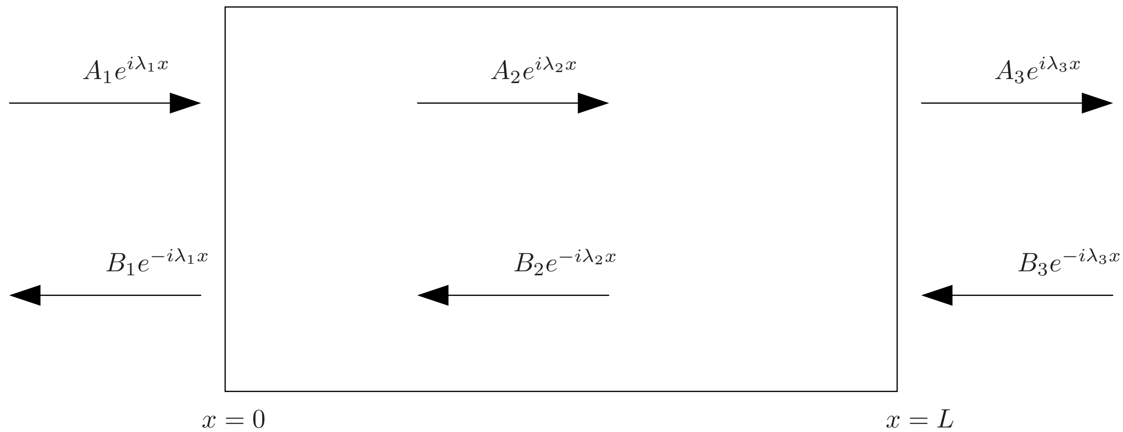

Figure 1: Finite one-dimensional sample of a foodstuff.

these regions the Helmholtz equation for the electric field intensity, (1), holds and so the electric field is

$$E_n = A_n e^{i\lambda_n x} + B_n e^{-i\lambda_n x} \tag{8}$$

for regions $1 \leq n \leq 3$. The first term in the expression describes the wave traveling from left to right, the second describes the wave traveling from right to left. We impose the following continuity boundary conditions at the interfaces of the three regions at $x = 0$ and $x = L$

$$E_1(0) = E_2(0), \tag{9}$$
$$E_2(L) = E_3(L), \tag{10}$$

which ensure the continuity of tangential electric field intensities across the boundaries of the regions. Applying these boundary conditions to the general solution of the electric field, (8), at $x = 0$ and $x = L$ yields

$$A_1(T_{1,2} - R_{1,2}) + B_1 = A_2 + B_2(T_{1,2} - R_{1,2}), \qquad x = 0 \tag{11}$$
$$A_2(T_{2,3} - R_{2,3})e^{i\lambda_2 L} + B_2 e^{-i\lambda_2 L} = A_3 e^{i\lambda_3 L} + B_3(T_{2,3} - R_{2,3})e^{-i\lambda_3 L}, \qquad x = L, \tag{12}$$

where $T_{l,l+1}$ is the transmission coefficient between regions l and $l+1$, $R_{l,l+1}$ is the reflection coefficient between regions l and $l+1$ and $T_{l,l+1} + R_{l,l+1} = 1$. For an incident wave $A_l e^{i\lambda_l x}$ at a boundary between regions l and $l+1$ at x_l the wave transmitted into region $l+1$ is given by $A_l T_{l,l+1} e^{i\lambda_l x_l}$ and the wave reflected back into region l is given by $A_l R_{l,l+1} e^{i\lambda_l x_l}$. Balancing the waves traveling from left to right in (12) yields

$$A_1 T_{1,2} = -B_2 R_{1,2} + A_2, \quad x = 0, \tag{13}$$
$$A_2 T_{2,3} e^{i\lambda_2 L} = -B_3 R_{2,3} e^{-i\lambda_3 L} + A_3 e^{i\lambda_3 L}, \quad x = L. \tag{14}$$

Balancing the amplitude of the waves traveling from right to left results in the identities

$$B_1 - A_1 R_{1,2} = B_2 T_{1,2}, \quad x = 0 \tag{15}$$
$$-A_2 R_{2,3} e^{i\lambda_2 L} + B_2 e^{-i\lambda_2 L} = B_3 T_{2,3} e^{-i\lambda_3 L}, \quad x = L. \tag{16}$$

Here $T_{l,l+1} = |T_{l,l+1}| e^{i\tau_{l,l+1}}$ and $R_{l,l+1} = |R_{l,l+1}| e^{i\delta_{l,l+1}}$ are the complex valued transmission and reflection coefficients respectively between regions l and $l+1$. (These can be determined [1] by considering the relative impedence of the materials.) To investigate the validity of the two solutions to the field we consider a section of foodstuff with a microwave field of intensity, E_0, applied at $x = 0$ and no microwave field applied at $x = L$. This results in a wave at $x = 0$ traveling from left to right, corresponding to $A_1 = E_0$, and the condition of no field at the face $x = L$ implies $B_3 = 0$. Solving the above system of equations and substituting into the expression for the power absorbed by a material we obtain [1]

$$P = \frac{1}{2}\omega\epsilon_0\kappa_2'' E_0^2 |T_{1,2}| \left[\frac{e^{-2\beta_2 x} - 2|R_{2,3}| e^{-2\beta_2 L}\cos(2\alpha_2(x-L) - \delta_{2,3}) + |R_{2,3}|^2 e^{-4\beta_2 L} e^{2\beta_2 x}}{1 - 2|R_{1,2}||R_{2,3}| e^{-2\beta_2 L}\cos(\delta_{1,2} + \delta_{2,3} + 2\alpha_2 L) + |R_{1,2}|^2 |R_{2,3}|^2 e^{-4\beta_2 L}} \right] \tag{17}$$

which is the exact solution for the power absorbed by a one dimensional material, with constant dielectric properties, irradiated from one side. The equation contains the exponential decay terms given Lambert's law perturbed by an additional oscillatory component and a small but increasing exponential component. It is clear that as L tends to infinity the decaying exponential term dominates to give Lambert's law in the limit.

2.3 The surface power density

To examine the absorbed power profiles given by we must estimate the surface power density, Q_0. This value will depend on the power rating of the oven, the dielectric properties of the foodstuff and also the dimensions of the load. We equate the power rating of the oven, P_r to the integral of the power absorbed from Lambert's law, (2.2), over the volume of the domain.

$$Q_0 = \frac{2\beta P_r}{L_y L_z [1 - e^{-2\beta L}]} \tag{18}$$

where L_y and L_z are the lengths of foodstuff in the y and z direction respectively (which we assume are sufficiently large so that the one-dimensional approximation is a good one). From the surface power density, Q_0, we then calculate a value for the surface field, E_0, needed in the exact solution, (17) and find the surface field by equating the power absorbed from Lambert's law (2.2) with the exact solution (17) applied to a semi-infinite domain. This gives [1]

$$E_0 = \sqrt{\frac{2Q_0}{\omega\epsilon_0\kappa_2'' \left|T_{1,2}\right|^2}} \tag{19}$$

Varying the length and plotting results in the profiles for the absorbed power presented in Figure 2. It is clear from these figures that, for large lengths

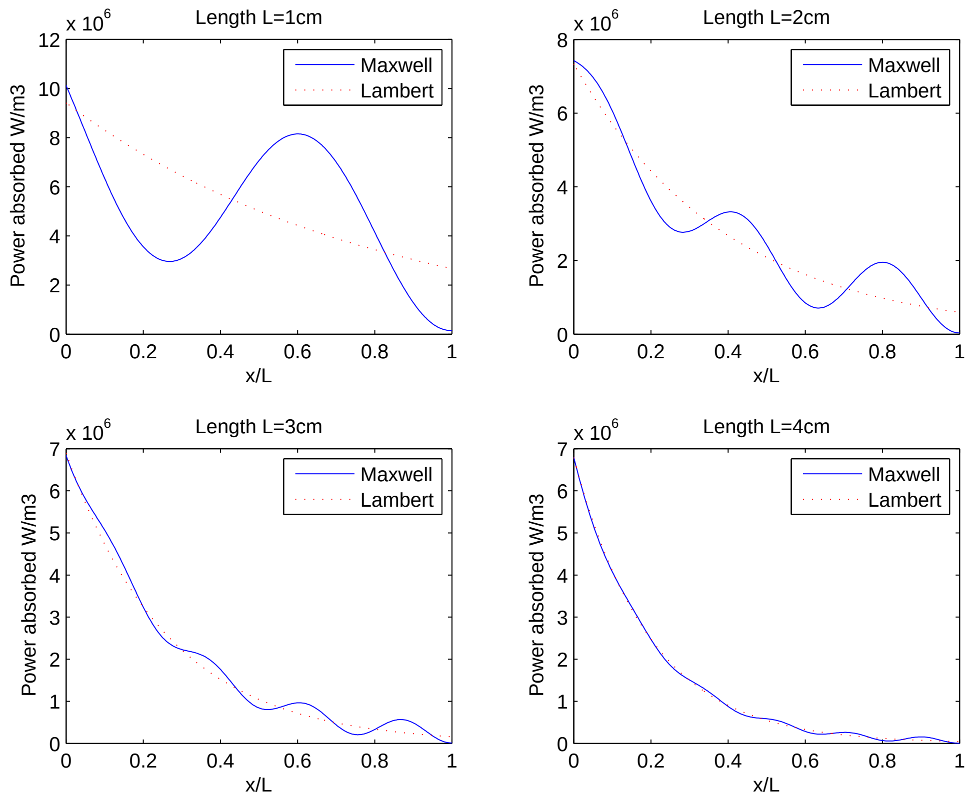

Figure 2: A comparison of the Lambert law approximation of the power absorbed derived from solutions to Maxwell's equations

$L \gg 1$ Lambert's law is a reasonable approximation to the exact solution of Maxwell's equations. For smaller lengths the solutions to Maxwell's equations oscillates about the exponential decay curve, with the size of the oscillations increasing as $L \to 0$. The oscillations in the exact solution are the results of internal reflections inside the foodstuff. These reflected waves interfere with the transmitted wave resulting in the standing wave patterns observed above. As the length of domain increases the amplitude of the oscillations decreases due to the decay of the transmitted wave inside the material which results in a reflected wave of reduced intensity. Interestingly, typical lengths of sample foodstuffs commonly used in domestic applications have a smallest length of around 2cm which is on the division point between a good and a poor approximation of the field by Lambert's law.

The deviation of the absorbed power derived from Maxwell's equations from the Lambert law approximation is given by

$$\frac{a_0 e^{-2\beta_2 L} + a_1 e^{-2\beta_2(L+x)} + a_2 e^{-4\beta_2 L}}{1 - 2\left|R_{1,2}\right|\left|R_{2,3}\right| e^{-2\beta_2 L}\cos(\delta_{1,2}+\delta_{2,3}+2\alpha_2 L) + \left|R_{1,2}\right|^2\left|R_{2,3}\right|^2 e^{-4\beta_2 L}} \quad (20)$$

$$\begin{aligned} a_0 &= -2Q_0\left|R_{2,3}\right|\cos(2\alpha_2(x-L)-\delta_{2,3}) && (21)\\ a_1 &= 2Q_0\left|R_{1,2}\right|\left|R_{2,3}\right|\cos(\delta_{1,2}+\delta_{2,3}+2\alpha_2 L) && (22)\\ a_2 &= Q_0(\left|R_{2,3}\right|^2 e^{2\beta_2 x} - \left|R_{1,2}\right|^2\left|R_{2,3}\right|^2 e^{-2\beta_2 x}) && (23) \end{aligned}$$

This expression is dominated by the ocscillatory term

$$\frac{-2Q_0\left|R_{2,3}\right| e^{-2\beta_2 L}\cos(2\alpha_2(x-L)-\delta_{2,3})}{1 - 2\left|R_{1,2}\right|\left|R_{2,3}\right| e^{-2\beta_2 L}\cos(\delta_{1,2}+\delta_{2,3}+2\alpha_2 L) + \left|R_{1,2}\right|^2\left|R_{2,3}\right|^2 e^{-4\beta_2 L}} \quad (24)$$

The amplitude of which is dependent on the surface power density, Q_0, the dielectric properties of the sample and the length of the domain of the sample. Using the dominant term for the amplitude, (24) The variation of the amplitude of oscillations with increasing length is presented in Figure 3.

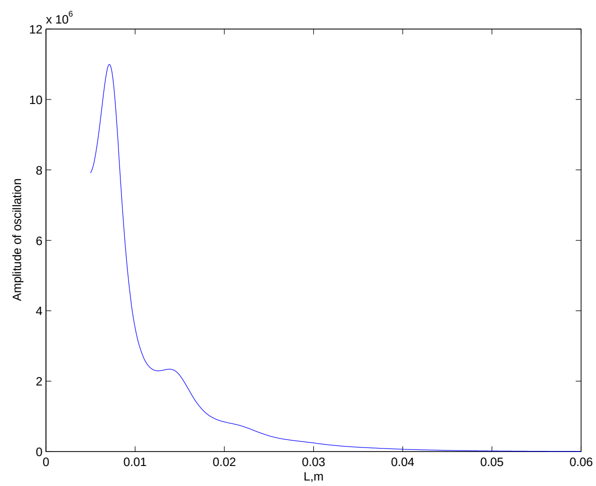

Figure 3: A comparison of the amplitude of the oscillation of the power density around that given by Lambert's law, with increasing length of sample

We see that provided L is not especially small, the amplitude of the oscillations decays exponentially with the length of the sample, and we can closely approximate the source term (17) by the simple expression

$$Q_M(e^{-2\beta x} + \delta e^{-2\beta L}\cos(\phi x + \theta)). \quad (25)$$

Here we have set $\beta \equiv \beta_2 \approx 60m^{-1}$ and take

$$\begin{aligned}
Q_M &= Q_0(1 - 2|R_{1,2}||R_{2,3}|e^{-2\beta_2 L}\cos(\delta_{1,2} + \delta_{2,3} + 2\alpha_2 L) + |R_{1,2}|^2|R_{2,3}|^2 e^{-4\beta_2 L})^{-1} \\
\delta &= 2|R_{2,3}| \\
\phi &= 2\alpha_2 \approx 890m^{-1} \\
\theta &= -2\alpha_2 L - \delta_{2,3}.
\end{aligned}$$

For the case of $L = 2cm$ the above expressions give $\delta\exp(-2\beta L) \approx 0.1$ and $\theta \approx -5\pi/6$ and the resulting approximation (25) is shown in Figure 4 which we compare with Figure 2.

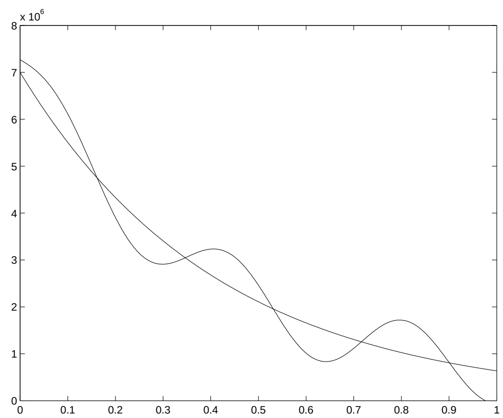

Figure 4: A comparison of the power density found from solving Maxwell's equations when $L = 2cm$ and the analytic approximation (25)

Observe that $2\beta = 120m^{-1}$ and $\phi \approx 890m^{-1}$ so that the lengthscale of the oscillations is approximately 1/7 that of the natural decay length. This estimate will prove important in subsequent the calculation of the temperature profile.

3 The temperature profile prior to any phase change

3.1 The model and numerical calculations of its solution

We now investigate the temperature profile in the foodstuff prior to the phase change at T_b, assuming (at this stage) initially constant dielectric properties and compare the effects of taking the exact and approximate expressions for the heat source derived from the field calculated in the previous section. Experimental data presented in [5], [6] indicates that the primary source of heat transfer within the foodstuff is via thermal conduction and that convective and evaporative effects are negligible for this temperature range, and we will make this assumption in our model. The principle mode of heating in the microwave heating of foodstuff is through dipole alignment of the polar water molecules. To describe the temperature $T(x,t)$ of foodstuff as it is heated in a microwave oven we accordingly solve the heat equation for the temperature T given by

$$c\rho\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} + P, \qquad P = \frac{1}{2}\omega\varepsilon_0\kappa_2''|E|^2 \tag{26}$$

Here P is the absorbed power density from the electromagnetic field, ρ the foodstuff density, c the specific heat capacity and k the thermal conductivity. Typical values for moist mashed potato are: $\rho = 1070kgm^{-3}, c = 4200Jkg^{-1}K^{-1}, k = 0.6Wm^{-1}K^{-1}$. At the boundary the foodstuff will lose heat through convection and radiation and to simplify the following calculations we consider a foodstuff with such heat loss occuring at the boundary exposed to the microwave radiation and with the far boundary acting as a thermal insulator. According at $x = 0$ we set

$$k\frac{\partial T}{\partial x} = h(T - T_a) + \gamma\sigma T^4 \tag{27}$$

and at $x = L$ we have

$$\frac{\partial T}{\partial x} = 0. \tag{28}$$

We also consider a system initially at the ambient temperature so that

$$T(x,0) \equiv T_a \tag{29}$$

Here h is the convective heat transfer coefficient, T_a the ambient temperature, γ is the radiative surface emissivity and σ is Stefan's constant. Typical values of these physical parameters are $h = 10Wm^{-2}, T_a = 5°C \equiv 278K \leq T \leq T_b = 373K \equiv 100°C, \gamma = 0.9, \sigma = 5.67 \times 10^{-8}Wm^{-2}K^{-4}$.

The solution of the heat equation with the power source, together with the boundary conditions, can be approximated numerically by discretising the temperature in space with a finite difference scheme and solving the resulting large nonlinear system of stiff ordinary, algebraic differential equations, using the fast and accurate Gear solver `ode15s` in the general purpose package `MATLAB` [13]. For this calculation we use both the solution of Maxwell's equations to give the expression (17) for P and also the Lambert law approximation (2.2).
In Figure 5 we present the results of these two calculations for varying values of L in which we take the values of the various physical parameters as given above.

It is clear from the results presented in these figures, that at the time given the temperature of the Maxwell based calculation oscillates about a decaying profile in a qualitatively similar manner to that of the electrical field. However the relative magnitude of the oscillations is smaller than the relative magnitude

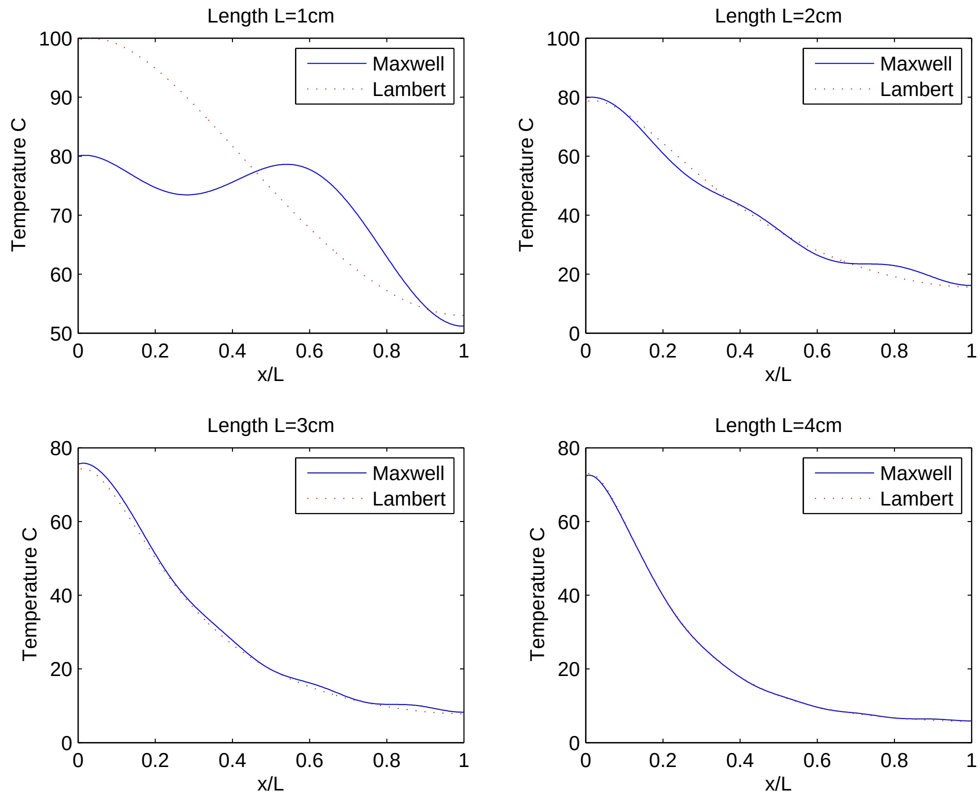

Figure 5: Comparison of the temperature distributions at a time t=60 seconds using the power absorbed derived from both the solutions to Maxwell's equations and the Lambert Law approximations. Note that the temperature variation is smaller than the variation of the field.

of the field variations. This is due to the averaging effect of the conductive terms and we now make an short analytic study of this effect.

3.2 An analytic estimate of the temperature profile

To make an analytic calculation the temperature variation we take the first order approximation (25) of the power density. To this order of accuracy, the equation for the temperature of the foodstuff is given by the heat equation

$$c\rho T_t = kT_{xx} + Q_M(e^{-2\beta x} + \delta e^{-2\beta L}\cos(\phi x + \theta)), \quad 0 \le x \le L \tag{30}$$

with the convective and radiative heat loss boundary condition at the irradiated surface and the insulating boundary condition as given earlier, and with $T = T_a$ initially. It is useful at this stage to estimate the magnitude of the various terms in this calculation. For the maximum value of $T = T_b$ we have

$$h(T - T_a) \approx 950Wm^{-2} \quad \text{and} \quad \gamma\sigma T^4 \approx 987Wm^{-2}$$

so, to a good approximation, the radiative and convective heat losses are very similar. Note further that for a foodstuff for which the temperature varies on a typical length scale of $l = 1cm$ we have $hl/k \approx 1/6$. Thus the heat loss on the exposed boundary is relatively small. This is consistent with the Figures of the temperature given above. Furthermore, observe also that the diffusive time-scale τ for the heat transport by thermal conduction is given by

$$\tau = \frac{l^2 c\rho}{k} \approx 750s \approx 12 \text{ minutes} \tag{31}$$

This implies that in the typical time-scales for microwave heating of one to five minutes, the heat is not significantly transported by conduction, and that the localised heating of the foodstuff by the source term due to the localised microwave power density term will dominate in the calculation of the temperature profile. The surface power density used in the previous sub-section was of the order of $Q_M = 7.5 \times 10^6 Wm^{-2}$ giving an initial heating rate (in the absence of conductive and boundary heat loss) of $T_t = Q_M/c\rho = 1.66Ks^{-1}$. Hence the time to reach boiling temperatures from ambient is around 60s, consistent with both experiment and the calculations of the previous sub-section.

We now non-dimensionalise the above system using the following scalings

$$\hat{x} = \frac{x}{L}, \qquad \hat{T} = \frac{T - T_a}{\Delta T}, \qquad \hat{t} = \frac{tk}{L^2 c\rho}, \qquad \hat{Q}_M = \frac{Q_M L^2}{\Delta T k}, \tag{32}$$

where $\Delta T = T_b - T_a$. Dropping hats yields

$$T_t = T_{xx} + Q_M(e^{-2\beta Lx} + \delta e^{-2\beta L}\cos(\phi Lx + \theta)) \tag{33}$$

with boundary conditions

$$x = 0, \qquad T_x = BiT + \left(\frac{T_a}{\Delta T} + T\right)^4 \frac{L\gamma\sigma\Delta T^3}{k}, \tag{34}$$

$$x = 1, \qquad T_x = 0, \tag{35}$$

$$t = 0, \qquad T = 0. \tag{36}$$

where $Bi = Lh/k$ is the Biot number. It is difficult to make direct analytic progress with the above expression owing to the nonlinearity in the term describing the radiative heat loss at the exposed boundary. However, noting that this is of similar order to the convective heat loss (which is linear) and the observed relatively small effect of the heat loss at the boundary, we will initially perform an analysis considering only this latter term. The resulting partial differential equation together with its boundary conditions is then linear. Whilst it could, in principle, be then solved by using a Laplace Transform, it is convenient for our purposes to represent the solution of the forced heat equation with the convective boundary conditions by using a normal form expansion in terms of the eigenfunctions $\theta_n(x)$ of the second derivative operator. These satisfy the equations

$$\frac{d^2\theta_n}{dx^2} = -\lambda_n^2\theta_n, \quad \frac{d\theta_n(0)}{dx} = Bi\theta_n(0), \quad \frac{d\theta_n(L)}{dx} = 0.$$

The solutions to this ordinary differential equation are given by

$$\theta_n(x) = a_n\left(\frac{\lambda_n}{B}\cos(\lambda_n x) + \sin(\lambda_n x)\right)$$

for those values of λ_n which satisfy the transcendental equation

$$\frac{\lambda_n}{Bi}\sin(\lambda_n L) = \cos(\lambda_n x).$$

It is straightforward to show that the eigenfunctions θ_n are orthogonal over the interval $[0, L]$ and the values of a_n can be chosen so that they form a complete orthonormal set with $\lambda_1 < \lambda_2 < \ldots$ and $\theta_1(x) > 0$. Setting

$$T(x,t) = \sum_{n=1}^{\infty} A_n(t)\theta_n, \quad T(x,0) = 0$$

we find that the coefficients A_n satisfy the ordinary differential equations

$$\frac{dA_n}{dt} = -\lambda_n^2 A_n + P_n^E + P_n^O, \quad A_n(0) = 0 \tag{37}$$

where the exponentially decaying and oscillatory contributions to the source are given respectively by

$$P_n^L = \langle Q_M e^{-2\beta Lx}, \theta_n\rangle \quad \text{and} \quad P_n^O = \langle \delta Q_M e^{-2\beta L}\cos(\phi Lx + \theta), \theta_n\rangle. \tag{38}$$

Here $\langle f, g\rangle$ is the usual inner product $\int_0^L fg\ dx$. The equations (37) have the solution

$$A_n(t) = \left(1 - e^{-\lambda_n t}\right)\left(\frac{P_n^E + P_n^O}{\lambda_n}\right). \tag{39}$$

so that

$$T(x,t) = \sum_{n=1}^{\infty}\left(1 - e^{-\lambda_n t}\right)\left(\frac{P_n^E + P_n^O}{\lambda_n}\right)\theta_n(x) \equiv T^E(x,t) + T^O(x,t). \tag{40}$$

As $t \to \infty$ the solution eventually evolves to a steady state for which

$$T(x,t) = \sum_{n=1}^{\infty}\left(\frac{P_n^E + P_n^O}{\lambda_n}\right)\theta_n \equiv S^E(x) + S^O(x)$$

where the functions $S^E(x) + S^O(x)$ can be evaluated explicitly as

$$S^E(x) = \frac{Q_M}{2\beta L}\left[\frac{-e^{-2\beta L x}}{2\beta L} - e^{-2\beta L}x + \frac{1}{Bi} + \frac{1}{2\beta L} - \frac{e^{-2\beta L}}{Bi}\right]. \qquad (41)$$

and

$$S^O(x) = \frac{Q_M\delta}{\phi L}\left[\frac{\cos(\phi L x + \theta)}{\phi L} + \sin(\phi L + \theta)x + \frac{1}{Bi}\left(\sin(\phi L + \theta) - \sin(\theta)\right) - \frac{\cos(\theta)}{\phi L}\right]. \qquad (42)$$

The analytical solution enables us to estimate the effect on the temperature profile of both the exponentially decaying source term obtained from Lambert's law and the oscillatory component of the solution of Maxwell's equations approximated by (25). A numerical calculation with a representative value of $Bi = 0.33$ shows that for the range of parameters considered $\lambda_1 \approx 0.397$ and $\lambda_2 \approx 3.1937$. Hence the expression (40) for $T(x,t)$ is dominated by the terms involving the *positive* function θ_1 In the calculations of the inner products to determine the source terms for A_n the integrals of the oscillatory terms of the form $\delta\exp(-2\beta L)\cos(\phi L x + \theta)$ multiplied by the more slowly changing function $\theta_1(x)$ involve large cancellations and are approximately in proportion to $\delta\exp(-2\beta L)/(\phi L)$. In comparison, the integrals of the terms involving $\exp(-2\beta L)$ will be proportional to $1/(2\beta L)$. Consequently, the ratio of the effect on the temperature of the exponential and oscillatory sources is proportional to $(2\delta\exp(-2\beta L)\beta)/\phi$. Provided that ϕ is rather larger than 2β this will be significantly smaller than $\delta\exp(-2\beta L)$ and consequently the Lambert law approximation for the power leads a rather better approximation for the temperature than for the field. In the case of a moist starchy foodstuff the previous estimates show that $2\beta/\phi \approx 1/7$, satisfying this condition. Whilst this analytic calculation has used a simple (linear) convective boundary condition, a similar result will hold for a (nonlinear) radiative boundary condition owing to the averaging effect of the diffusion operator on the oscillatory terms in the power source.

To illustrate this result we present, in Figure 6, the results of a numerical calculation of the problem with the convective boundary condition. In this calculation we take $\delta e^{-2L\beta} = 0.1$, $L = 2cm$, $2\beta L = 2.4$, $B = 0.2$, a scaled power of $Q_M = 52.6$. We compare three calculations of $T(x,t)$ with $T^E(x,t)$ taking values of $\phi L = 10, 20, 30$ respectively, at the scaled value of $t = 0.02$ (corresponding to an actual time of 60s). Note that the previous calculation implies that the true scaled value of $\phi L = 17.8$.) It is clear from this figures that, as predicted, the variation in T from T^E is proportionally much smaller than $\delta\exp(-2\beta L)$ and that this variation reduces significantly as ϕL increases.

4 Dielectric sensitivity and phase changes

We now extend the model of the heating of a moist foodstuff, with initially 80% moisture content, to include a phase change at $T = T_b$ leading to localised dry-

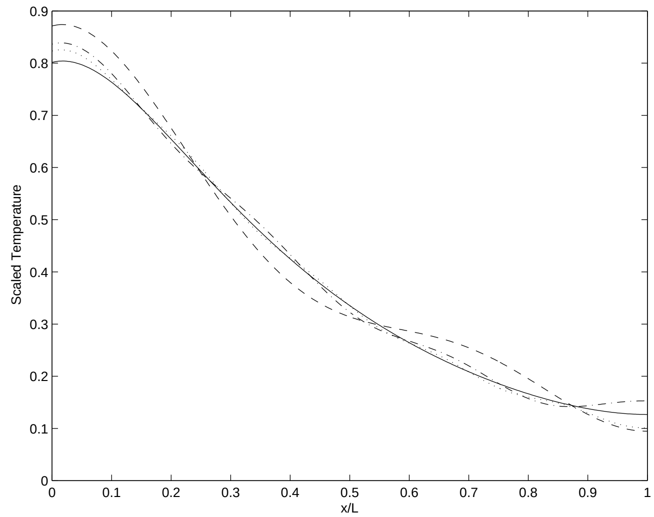

Figure 6: Comparison of the temperature distributions $T(x,t)$ at the normalised times $t = 0.02$ using the expression for the power absorbed derived from the approximation (25) to the solutions to Maxwell's equations with $\delta = 0.1$ and $\phi L = 10$ (dashed) $\phi L = 20$ (dash-dotted), $\phi L = 30$ (dotted) compared with the temperature $T^E(x,t)$ derived from the Lambert Law approximation to the power (solid)

ing. The Stefan number expressing the ratio of Sensible to Latent heat is given by $Ste = \Delta Tc/\Lambda$ where $\Lambda = 2.26 \times 10^6 Jkg^{-1}$ is the Latent Heat of water. This takes the relatively low value of $Ste = 0.2$ which implies that substantially more thermal energy is required to dry the foodstuff than to heat it up. Consequently, what is observed in microwave heating is that parts of the foodstuff reach the boiling temperature T_b relatively quickly, and then remain at this temperature for the remainder of the cooking period. At this temperature the moisture content of the foodstuff changes significantly, and with it the dielectric properties. Hence we must also consider the case of variable dielectric properties and the consequent changes to the temperature due to these variations. A smoothed front tracking method provides an efficient way of capturing the location of the moving phase change boundary numerically and we study its accuracy.

4.1 The dependence of the dielectric properties on the temperature and moisture content of the foodstuff.

To calculate the electric field intensity as moisture and temperature vary, it is important to consider the effect of the variations in the dielectric properties. Experimental literature provides tables of these values for a variety of materials, temperatures and moisture contents. Regier et al. [11], measure the dielectric properties of mashed potato for a variety of temperatures in a 2.45GHz microwave oven using a cavity perturbation technique. A marked change is observed in the dielectric properties as the foodstuff passes from a frozen phase to a liquid phase at 0°C. This is partly because ice absorbs significantly less electromagnetic energy than water. For temperatures above 0°C the dielectric properties remain nearly constant with temperature. As parts of the foodstuff reach 100°C and bound water is boiled off, the changing moisture content again strongly influences the dielectric properties. In [10], Mudgett outlines methods of finding the dielectric properties of a material giving the experimental values for the permittivity, ϵ' , and dielectric loss factor, ϵ'', for mashed potato with varying moisture contents. The values given are for heating at a microwave frequency of 3GHz. Taking values from the data given we construct a fifth order polynomial to approximate the dielectric properties at all temperatures in the range 0-120°C and in Figure 7 give the dielectric properties with changing moisture contents.

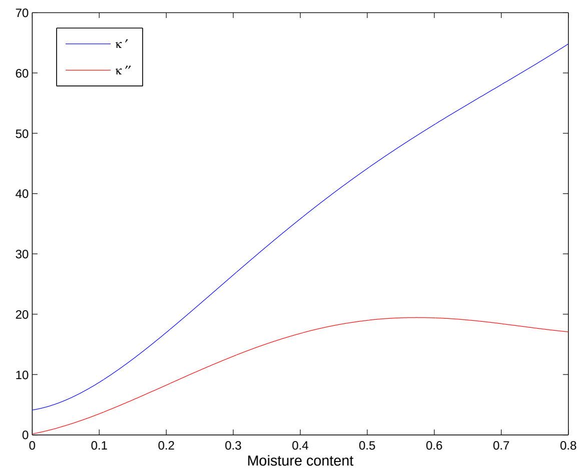

Figure 7: Variation of the dielectric properties of mashed potato at 3GHz as a function of moisture content

The dielectric properties determine the decay rate β of the field inside the foodstuff through the formula (3) and hence the penetration depth, $d = 1/2\beta$, of the power density. This gives a relationship between d and the moisture content which is illustrated in Figure 8.

It is clear that as the moisture content of the foodstuff decreases the penetration depth of the sample increases. In the early stages of heating there is little change in the moisture content of the foodstuff and so we should expect that the field will remain unchanged. In the later stages of heating as the foodstuff dries locally, we observe the moisture content of the foodstuff dropping to 20% in some parts (though never to zero), and it is in this period that we observe the most significant effect on the field.

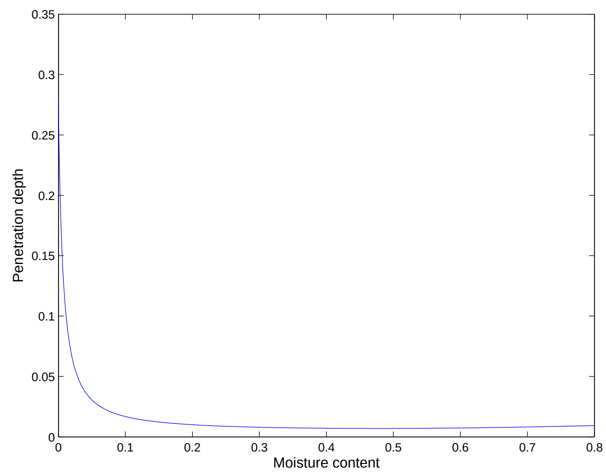

Figure 8: The variation of the penetration depth d of the microwave power density in mashed potato at 3GHz as a function of the moisture content.

4.2 Calculating the changing field

As before, we may calculate the electric field intensity $\mathbf{E}$ for the one-dimensional geometry by again solving the Helmholtz equation (1)

$$\mathbf{E}_{xx} + \lambda^2(x)\mathbf{E} = 0, \tag{43}$$

but we now consider $\lambda^2(x) = \omega^2 \mu \epsilon_0 \kappa^*(x,t)$ to be a spatially varying parameter. As the dielectric properties are slowly varying relative to the rapidly changing electric field we can make use of the WKB approximation [12] to yield an approximate solution in the form

$$\mathbf{E} \approx \frac{A_\pm}{\sqrt{\lambda(x)}} e^{\pm i \int_0^x \lambda(x)dx}, \quad P = \frac{1}{2}\omega \varepsilon_0 \kappa''(x)|\mathbf{E}|^2, \tag{44}$$

where $A_\pm$ are constants. To investigate the effects of the changing dielectric properties we shall assume, following the calculations of the previous two sections, that L is sufficiently large so that the internal reflections of the field in the foodstuff do not have a significant effect on the temperature profile. Thus we neglect the oscillatory component of the solutions obtained earlier, and set $A \equiv A_+$ to give a decaying function for the absorbed power. In the early stages of heating where there is little or no moisture loss λ will remain constant and the field is approximated by Lambert's law. In the latter stages of heating, after the temperature of parts of the foodstuff has reached T_b the changing moisture content will result in a changing field which we approximate by using (44). Here the integrals in this expression must be evaluated by numerical quadrature. Similarly, a second integration is required to calculate the constant A in (44). As in Section 2.3 this parameter can be calculated from the effective power rating, P_r, of the oven by determining the total power absorbed by the foodstuff via the formula

$$P_r = \int_0^{L_z} \int_0^{L_y} \int_0^{L} \frac{1}{2}\omega\epsilon_0\kappa'' \left|\mathbf{E}\right|^2 \ dxdydz \tag{45}$$

Substituting the expression (44) for $\mathbf{E}$ into this identity and equating this with the effective power rating of the oven gives a value for A at any time.

4.3 The numerical method

Essential to all of the calculations is the need for a fast and accurate numerical method for calculating the temperature profile with phase change both accounting for, and leading to, changes in the dielectric properties). There are two widely used methods for doing this, front *tracking* which aims to follow the phase change boundary (often by placing mesh points on this boundary) and front *capturing* by solving (a smoothed set of) equations that allow for the phase change. The front tracking approach formulates the model as a Stefan problem [15]. In this we consider a material which undergoes a phase change at $T = T_b$. The heat equation is given by

$$\rho_i c_i \frac{\partial T_i}{\partial t} = k_i \frac{\partial^2 T_i}{\partial x^2} + P_i \tag{46}$$

where the subscript $i = l, g$ denotes the liquid and gas phases respectively. At the phase change boundary $X(t)$ the temperature of each state must be equal so that $T_l = T_g = T_b$. Balancing the heat flux across the phase change boundary yields the Stefan condition

$$k_l \frac{\partial T_l}{\partial x}|_{x=X(t)} - k_g \frac{\partial T_g}{\partial x}|_{x=X(t)} = \rho\Lambda \frac{dX(t)}{dt}. \tag{47}$$

In the front tracking approach this is considered to be a moving boundary problem where the phase change front, $X(t)$ is tracked and the interface conditions above are applied. However, this method of solution, whilst accurate, can be computationally expensive. In contrast, we consider an approach (related to the enthalpy method) which smooths the transition This method is generally less accurate than front tracking as it is of lower order in the discretisation parameter. However this is not a huge problem due to the uncertainty of many of the model parameters in the cooking process. However, it is fast and easy to implement. This method exploits the fact that significantly more energy is used to dry the foodstuff than to heat it up, and that the heating time is sufficiently short such that the foodstuff does not enter a fully dried phase. (Indeed, if it did enter such a phase then the foodstuff would be likely to burn, and the foodstuff quality would deteriorate significantly.) It also relies on the experimental observation [5], [6] that in the microwave cooking process, when the water bound in the sample is vapourised at $T = T_b$ it is then free to escape into the oven cavity. In this approximation the water vapour is not further heated by the microwave field and so does not increase in temperature or directly influence the field. As a result of these assumptions the temperature of the foodstuff is limited to a maximum value of $T = T_b$ as it dries, and the heat equation equation becomes

$$c\rho T_t = \begin{cases} k\nabla^2 T + P & T < T_b \\ 0 & T = T_b. \end{cases} \tag{48}$$

To capture the location of the phase front numerically, we then solve a smoothed form of (48) in which we regularise the transition from heating to drying over a narrow temperature range of width proportional to the small constant τ. This regularisation increases the computational efficiency immensely without significantly reducing the accuracy of the model. To regularise the transition we consider the smoothed equation given by

$$c\rho T_t = \frac{1}{2}\left(1 - \tanh\left(\frac{T - T_b}{\tau}\right)\right)\left[k\nabla^2 T + P\right] \tag{49}$$

where $\tau > 0$ is a small parameter. As $\tau \to 0$, we obtain the original formulation. As τ increases this increases the range of temperatures over which the phase change occurs. In the case of water and pure substances, the phase change occurs over a narrow temperature range and so for accuracy, we keep τ as low as possible.

The experiments by Hooper [5] also indicate that in microwave cooking, evaporative loss of moisture is relatively small and that the the rate of moisture lost by the foodstuff, and hence of drying, is dominated by the phase change at T_b and is proportional to the power absorbed by the regions of foodstuff at $T = T_b$. As P is the power absorbed per unit volume the mass m of water converted to vapour per unit volume, is given by

$$m_t = -\frac{P}{\Lambda}. \tag{50}$$

Assuming that the volume of the sample remains constant and the density of water is denoted by ρ_w then the volume fraction of moisture, M in the sample obeys the differential equation

$$M_t = -\frac{P}{\rho_w \Lambda}. \tag{51}$$

The temperature and moisture content are found by solving (51),(49), taking the initial, uniform, temperature and moisture distributions to be $T = T_a$ and $M = 0.8$ In this calculation, the power density P is determined at each time-step by evaluating the the moisture content and calculating the new dielectric properties using the empirical data. The electric field, and hence P is then found from the WKB approximation (44), applying numerical quadrature to evaluate the exponential terms. The regularised partial differential equation (49) is then discretised in space using a uniform mesh of size Δx. This leads to a set of ordinary differential equations for the temperature at each point in the mesh, which we again solve by using a stiff ODE solver. Whilst a small value of the regularisation parameter τ is important for accuracy, decreasing the value of τ increases the stiffness of this set of differential equations and consequently increases the computation time of solving the system numerically. The optimal τ is thus obtained as a compromise between speed and accuracy.

As a first calculation we determine the moisture loss and computation time as a function of τ in a problem where we use a 5 minute heating time for a 1000W microwave oven with $L = 2cm$ and taking $\Delta x = L/350$. The same heat transfer parameters are used as in Section 3, and the dielectric prperties as in Section 4.1. In this calculation parts of the foodstuff reach a temperature of T_b at about two minutes, and by five minutes the temperature of a substantial portion of the foodstuff reaches T_b. It is clear that increasing τ decreases the

τ	Time taken	Moisture loss
0	10 Hours	37.8767g
0.01	1 Hour	37.878g
0.02	0.5 Hours	37.879g
0.05	9 minutes	37.8947g
0.1	6 minutes	40.26g
1	5 minutes	44.1g

Table 1: Values of τ and corresponding time taken to calculate solution and moisture lost by the sample after 5 minutes heating in a 1000W microwave oven

computation time but also increases the moisture loss, because the phase change linked to drying occurs at a slightly lower temperature than T_b. However, the very rapid decrease in computational time, and small change in the moisture loss as τ is slowly increased, indicates that it is possible to find values of τ giving good accuracy with a reasonable time. A value of the regularisation parameter $\tau = 0.05$ seems optimal in this case. In Figure 9 we present the overall moisture loss as a function of t for various values of τ we see an excellent agreement with $\tau = 0.05$ and the case with no smoothing and the calculation with smoothing for this value of τ is significantly quicker than the calculation with no smoothing. In a further comparison, in Figure 10 we plot the point temperature at $x = 1cm$ in the sample of the moist foodstuff over the course of the heating for the four values of τ. Again we see that as the width of the transition increases the final temperature attained by the sample increases. (In [6] these results are also compared to experimental values and show good agreement even for foodstuffs with a much more complicated geometry.) The moisture calculation appears to be the most sensitive to changes in τ and again we see good agreement between the value $\tau = 0.05$ and the temperature with no smoothing.

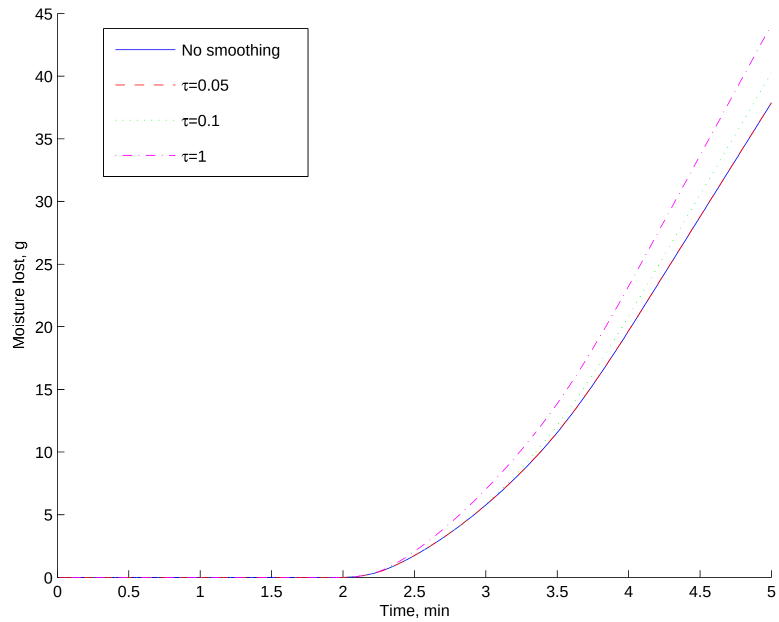

Figure 9: Moisture loss profiles for a tray of mashed potato heated in a 1000W microwave oven using various values of τ

4.4 The temperature and moisture distribution

Using the numerical method and data as described in the previous subsection we may now consider the effects of the varying dielectric properties upon the temperature and moisture content of the foodstuff. A series of numerical computations for the temperature, moisture and field values are presented in Figures (11)-(14). In the early stages of heating there is no change in the moisture content and, as the dielectric properties depend only weakly upon the temperature, the field will not be affected. However, as the calculation proceeds and the foodstuff starts to dry out close to the surface at $x = 0$, the field begins to change. As the local moisture content of the sample decreases the dielectric properties of the foodstuff change, increasing the penetration depth of the field into the foodstuff. We see the calculations with moisture dependent dielectric properties agree closely with those for constant dielectric properties in the early stages of heating. The calculations diverge after approximately 2 minutes heating and continue to do so after 5 minutes heating. The changing properties influence the electric field intensity which in turn influences the rate of moisture loss. However, even after 5 minutes heating we still see a difference of only 5% in these values.

We present in Figure (13) the total moisture lost from the sample with time. As in the previous Figure (12) we see that in the early stages of heating there is no change in the moisture content. Again after 5 minutes, and indeed throughout

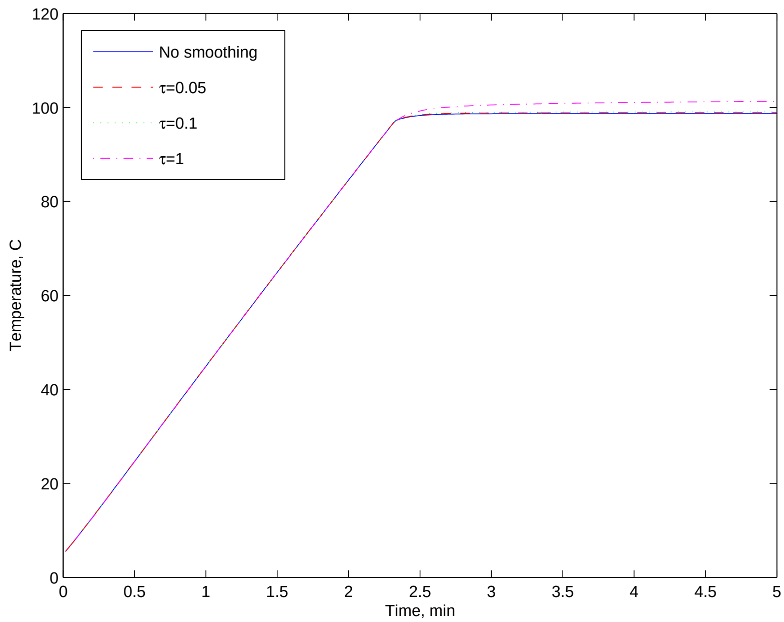

Figure 10: Point temperatures at $x = 1cm$ in a sample of mashed potato heated in a 1000W microwave oven using various values of τ

the heating process, there is little difference between the two curves.
Finally, in Figure (14) we present the change in the absorbed power over the heating process. Initially, we have the exponential decay of the surface power density predicted by Lambert's law. The field remains unchanged for the next minute, after two minutes heating the field decays more rapidly. This corresponds to a decrease in the penetration depth as the moisture content decreases because only in the later stages does the dielectric loss (which has the greater influence on the penetration depth) change. In the later stages of heating where the moisture content is significantly lower than the initial value we observe a marked change in the field. The low moisture content at the surface results in an increase in the penetration depth. This means that the field is less attenuated by the load and so is absorbed less in the surface boundary layer. Thus we see that at 5 minutes heating the absorbed power increases into the sample before decaying as the field enters a region with higher dielectric properties. Observe that the changes in the temperature and moisture loss profiles with varying, as opposed to constant, dielectric properties are significantly smaller than the changes in the power density. This is a similar observation to that made in Section 4 when we showed that the temperature variation was smaller than the field variation due to the averaging effects of conduction. A similar averaging effect is occuring in this case as well.

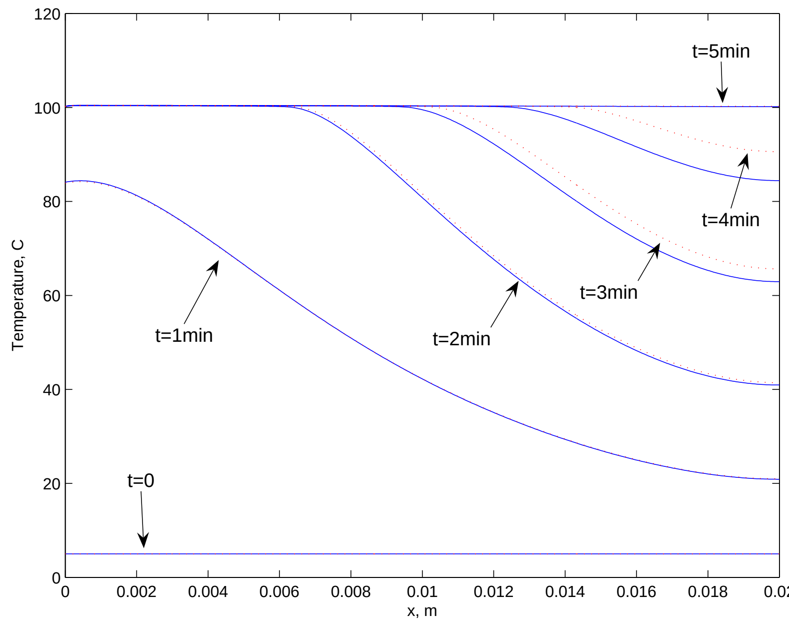

Figure 11: Temperature profiles of a one dimensional sample of mashed potato of length L =2cm, heated from ambient in a microwave oven comparing solutions with both constant and varying dielectric properties. Constant dielectric properties in dash, varying dielectric properties in solid.

5 Conclusions

The analysis of the temperature of the one dimensional foodstuff shows that, for sufficiently large lengths of the foodstuff, the effect on the temperature of the internal reflections of the field inside the foodstuff load are small and that analytical solution of the heat equation indicates that the oscillations of the power density lead, through the averaging effects of conduction, to smaller oscillations in the temperature. The variation in the dielectric properties with respect to the moisture content of the sample has also been investigated. The dielectric properties vary greatly with the changes in the moisture content of the foodstuff that occur at the phase change. As the moisture content decreases it is found that the field is able to penetrate further into the food load due to the change in these properties. However the analysis in Section 4 reveals that this field variation does not have a major effect on the temperature inside the food load, and on the overall moisture loss, for typical heating times below 5 minutes. We also conclude that numerical method that we have used, based on the various modelling assumptions made in Section 4 and the smoothing of the phase transition, offers significantly speed up in the computational time with only a small effect on the overall accuracy, provided that the parameter τ is

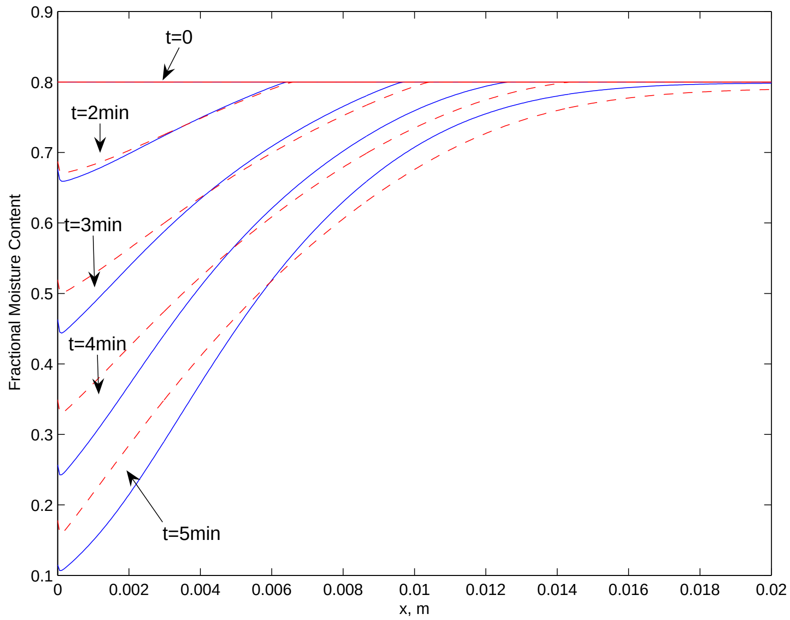

Figure 12: Moisture content profiles of a sample of mashed potato heated from an initial state of $M = 0.8$, comparing constant and varying dielectric properties. Constant dielectric properties in dash, varying dielectric properties in solid.

chosen carefully.

In summary, the results of this paper indicate that for sufficiently large lengths of the foodstuff, and reasonable heating times the Lambert's law approximation with constant dielectric properties and the smoothed Enthalpy equation leads to a good approximation of the field and temperature inside a foodstuff heated from chilled with a phase change at T_b. In a subsequent paper [3] (see also [6]) we apply these approximations to calculate the temperature and moisture content of a foodstuff with a more realistic geometry, and compare the results with experiment.

6 Acknowledgements

It is a pleasure to thank the Knowledge Transfer Network in Industrial Mathematics, EPSRC and the Campden and Chorleywood Food Research Association Group for partly funding this work through a CASE award. We would also like to thank Greg Hooper, Kevin Parrott, Philip Richardson and Heather Tewkesbury for their very helpful advice and expertise.

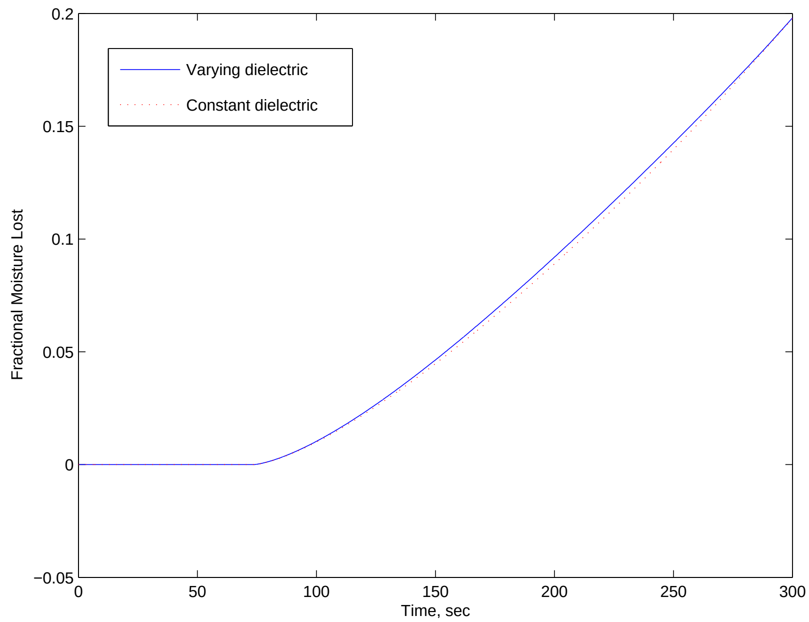

Figure 13: The total moisture lost from a sample of mashed potato heated in a microwave oven, compating constant and varying dielectric properties.

References

[1] Ayappa, K.G., Davis, H.T., Crapiste, G., Davis, E.A., Gordon, J. Microwave Heating: An Evaluation of Power Formulations. *Chemical Engineering Science*, 46:4:1005–1016, 1991.

[2] Basak, T. Analysis of resonances during microwave thawing of slabs. *International journal of heat and mass transfer*, 46:4279–4301, 2003.

[3] Budd,C.J., Hill, A.D.C. and Hooper. G. Experimental and analytical studies of the microwave heating of slab-sided moist foodstuffs. *In preparation*, 2008.

[4] Dincov, D.D., Parrott, K.A., Pericleous, K.A. A new computational approach to microwave heating of two-phase porous materials. *International journal of numerical methods for heat and fluid flow*, 14:6:783–802, 2004.

[5] G. Hooper. ????? *?????*, ??:??–??, 200?

[6] Hill, A.D.C. Mathematical Models for the Microwave Heating of Chilled Food. *PhD Thesis, University of Bath*, 2006.

[7] Liu, C.M., Wang, Q.Z., Sakai, N. Power and temperature distribution during micrwave thawing, simulated by using Maxwell's equations and Lam-

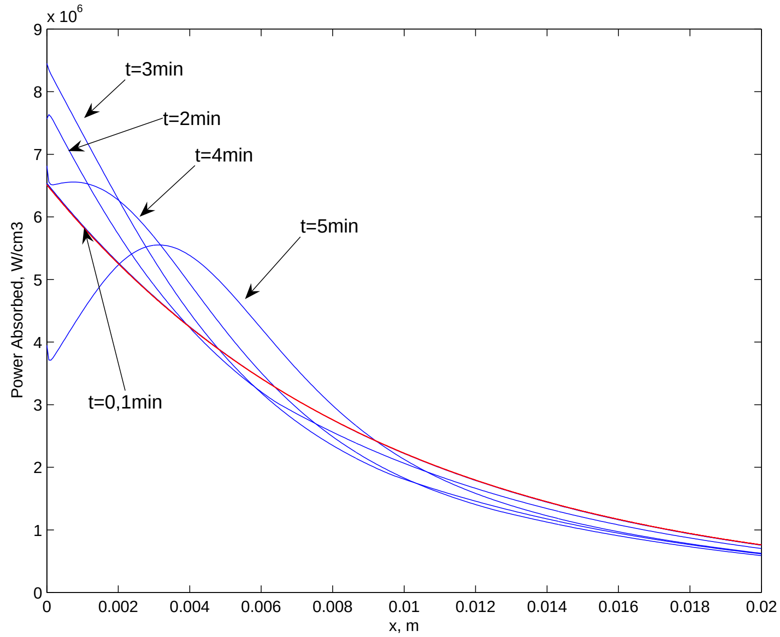

Figure 14: The power density profiles in the sample, comparing constant and varying dielectric properties.

bert's law. *International Journal of Food Science and Technology*, 40:9–21, 2005.

[8] Metaxas, A.C. *Foundations of Electroheat*. John Wiley and Sons, Inc., 605 Third Avenue, New York, New York 10158-0012, 1986.

[9] Ni, H., Datta, A.K., Torrance, K.E. . Moisture transport in intensive microwave heating of biomaterials: a multiphase porous media model. *International journal of heat and mass transfer*, 42:1501–1512, 1999.

[10] Rao, M.A., Rizvi, S.S.H. *Engineering Properties of Foods*. Marcel Dekker, Inc., 270 Madison Avenue, New York, New York 10016, 1986.

[11] Regier, M., Housova, J., Hoke, K. Dielectric Properties of Mashed Potato. *International Journal of Food Properties*, 4:3:431–439, 2001.

[12] Tayler, A.B. *Mathematical models in applied mechanics*. Oxford university press, Great Claredon Street, Oxford OX2 6DP., 1986.

[13] The Mathworks Inc. *MATLAB*. The Mathworks Inc., Natick, MA, USA, 2006.

[14] Tong, C.H., Lund, D.B. . Microwave heating of baked dough products with simultaneous heat and moisture transfer. *Journal of food engineering*, 19:319–339, 1993.

[15] Voller, V., Cross, M. Accurate solutions of moving boundary problems using the enthalpy method. *International journal of heat and mass transfer*, 24:545–556, 1980.