Error estimates for SISL methods and the Moving Mesh SISL method

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Framework

- Most problems in fluid mechanics and meteorology are advection-dominated.
- Explicit time integration is computationally efficient but the time step is constrained by wave speed and mesh resolution (CFL condition).
- Semi-implicit Semi-Lagrangian (SISL) methods are effective in dealing with large CFL numbers.
- The main source of error in SISL schemes is given by interpolation. This can be reduced with adaptive mesh methods (both r- and h-adaptive).

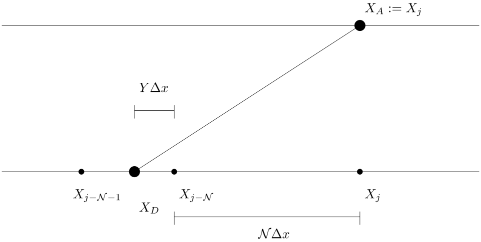
Combine moving mesh strategy with SISL to obtain a scheme with low error and no CFL restriction !

Content of the talk

- Describe SISL method.
- Find analytic expression for diffusive and dispersive errors when applying SISL to Burger's equation in 1D.
- Describe Moving Mesh SISL method.
- Show results of MMSISL applied to 1D Burger's equation and Shallow Water equations.
- Future work.

The Semi-Implicit Semi-Lagrangian (SISL) Method

Physical PDE: $\frac{Du}{Dt} = f(x, u_x, u_{xx})$, $\frac{D}{Dt} = \frac{\partial}{\partial t} + u\partial_x$



Kinematic Equation: $X_A - X_D = \int_{t^n}^{t^{n+1}} u(X(t), t)\, dt$

$$= (\mathcal{N} + Y)\Delta x, \quad \mathcal{N} \in \mathbb{N},\ 0 \leq Y < 1$$

Overview: The SISL scheme

Using a semi-implicit 2TL discretization leads to the system:

$$(X_A - X_D)/\Delta t = \frac{1}{2}(U_A^{n+1} + U_D^n) \tag{1}$$

$$(U_A^{n+1} + U_D^n)/(\Delta t) = \theta f_A^{n+1} + (1-\theta) f_D^n, \quad 0 \leq \theta < 1 \tag{2}$$

Solve for the unknown field U_A^{n+1} using the interpolated quantities U_D, f_D.

Source of fast waves in f_A are treated implicitly, while the remaining terms are treated explicitly.

Implementation

1. At time $t^n = n\Delta t$, set $X_D = X_A$ and discretize $f(x, u, u_x, u_{xx})$ at mesh points X_A.
2. Find an estimate of U_A^{n+1} by solving (2).
3. Find X_D by solving (1).
4. Interpolate f^n, U^n onto X_D using an interpolation procedure.
5. Apply iteratively step (1-4) to find U_A^{n+1}.

L2 error between numerical and exact solution is given by:

- Spatial and Temporal discretization of the physical PDE
- Location of the departure points
- Interpolation of f^n, U^n onto X_D
- Interpolation of the discrete solution U_A^{n+1}

Application of SISL to Burger's Equation

$$\frac{Du}{Dt} = \epsilon u_{xx} \quad 0 < \epsilon < 1,$$
$$u(-\infty, t) = c + \alpha,$$
$$u(+\infty, t) = c - \alpha.$$

The exact solution is a travelling wave at constant speed c:

$$u(x,t) = c - \alpha \tanh\left(\frac{\alpha(x - ct)}{2\epsilon}\right)$$

We expect ϵ to be very small in real applications, so that $\Delta x/\epsilon >> 1$.

The SISL method gives a solution with broader front than the exact travelling wave

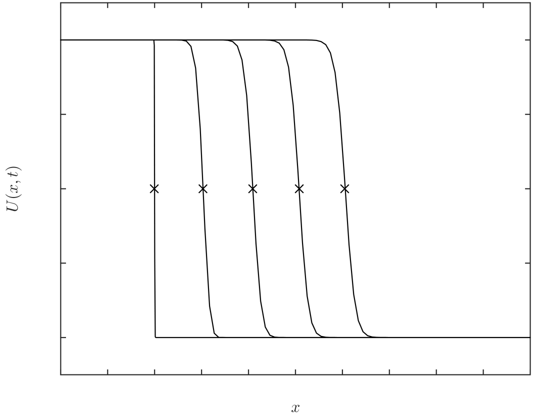
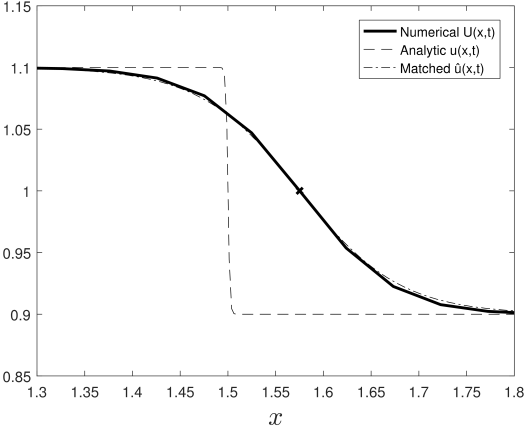


Figure: Numerical solution with parameters $\alpha = 0.1$, $\epsilon = 1e-4$, $c = 1$

SISL solution with *piecewise linear interpolation* is a rescaled tanh function:

$$\hat{u}(x,t) = c - \alpha \tanh\left(\frac{\alpha(x - \hat{c}t)}{2\hat{\epsilon}}\right)$$



Types of error

Dissipative error:

$$\hat{\epsilon} = O(\Delta x) >> \epsilon$$

Dispersive error:

$$\hat{c} = c + O(\alpha^2 \Delta x / \hat{\epsilon})$$

The errors are analysed using the modified equation analysis:

Find the modified nonlinear PDE satisfied exactly by the SISL solution

We obtain that the dominant source of error is given by the *interpolation step* in the SISL calculation.

Let $U_j^n = \hat{u}(j\Delta x, n\Delta t) = V(j\Delta x - \hat{c}n\Delta t) = V(z)$

Substitute it into the SISL scheme and expand the solution to find $V(z)$ as heteroclinic solution of the ODE:

$$\Delta t\left(\left(V-\widehat{c}\right)V'-\varepsilon V''\right)=\frac{\Delta t^2}{2}\left(\widehat{c}+V\right)\left(V'^2+\left(V-\widehat{c}\right)V''-\varepsilon V'''\right)$$
$$+\frac{1}{2}\left[\left(\left(2\mathcal{N}+1\right)\Delta x-V\Delta t\right)V\Delta t-\left(\mathcal{N}^2+\mathcal{N}\right)\Delta x^2\right]V''$$
$$+\mathcal{O}\left(\left(\Delta x+\Delta t\right)^3\right).$$

$$V(-\infty)=c+\alpha,\quad V(\infty)=c-\alpha\,.$$

Solve for $V(z)$ and c by expanding in powers of α.

Types of error

The linear interpolant depends on the CFL number

$$\nu_{CFL} = c\Delta t/\Delta x = \mathcal{N} + Y, \mathcal{N} \in \mathbb{N}, \quad 0 \le Y < 1$$

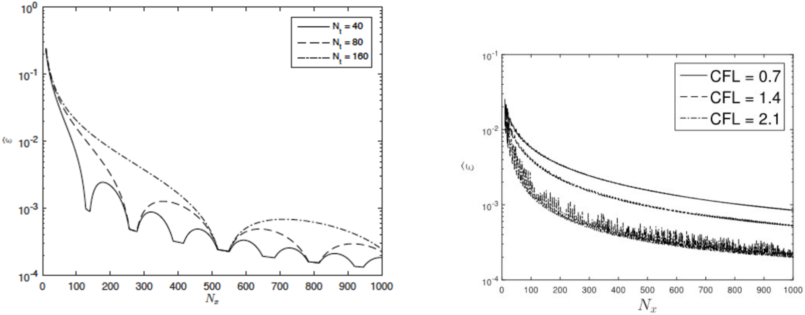
$$\hat{\epsilon} = \epsilon + \frac{cY(1-Y)\Delta x}{2\nu_{CFL}} + O(\alpha \Delta x^2)$$

$$\hat{c} = c - \frac{\alpha^2(1-2Y)\nu_{CFL}}{3cY(1-Y) + 6\epsilon\nu_{CFL}/\Delta x} + O(\alpha^2 \Delta x^2)$$

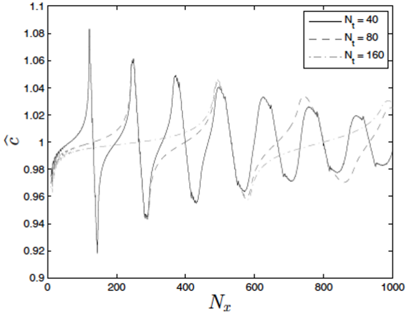
The dissipative error is smallest when Y $= 0$
(Right solution at the wrong time)
The dispersive error is smallest when Y $= 1/2$
(Wrong solution at the right time)

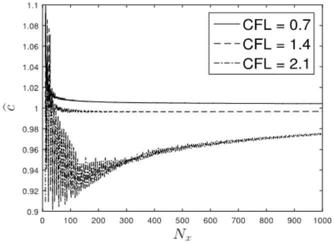
Dissipative error: Burger's equation

The error increases as the time step decreases !

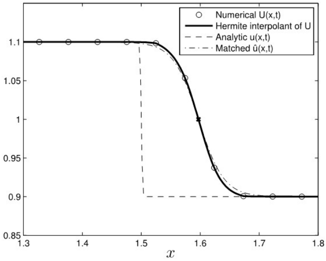


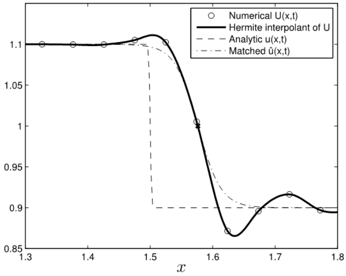
Dispersive Error: Burger's equation





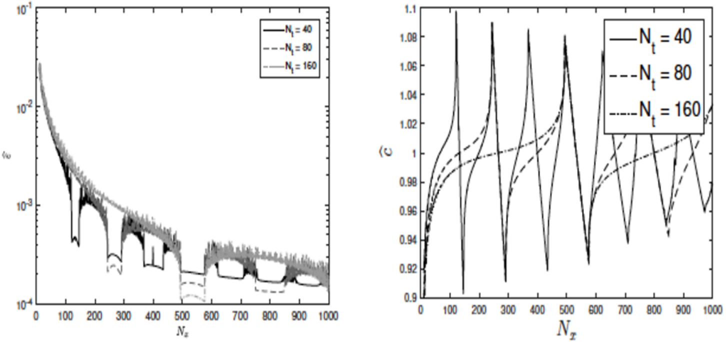
Hermite interpolation





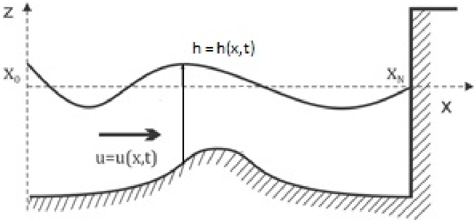
- Flux limited Hermite interpolant reduces diffusive error but has significant dispersive error.
- Non-monotonic Hermite interpolant has dispersive error and oscillatory approximation error.

Dispersion/Diffusion Error for Lagrange interpolation



1D Shallow-Water equation

$$
\begin{aligned}
\textit{Momentum equation}: \frac{Du}{Dt} + g\partial_x h &= 0,\\
\textit{Continuity equation}: \frac{Dh}{Dt} + \partial_x(hu) &= 0,\\
\textit{Boundary conditions}: \partial_x h(0,t) = \partial_x h(L,t) &= 0,\\
u(0,t) = u(L,t) &= 0.
\end{aligned}
$$



We perform numerical tests with the SISL scheme applied to the Dam-Break problem

$$h(x,0) = \begin{cases} h_l & \text{if } 0 \le x \le x_0 \\ h_r & \text{if } x_0 < x \le L \end{cases} \qquad u(x,0) = 0 \ m/s, \tag{16}$$

with $h_r = 3$, $h_l = 1$, $x_0 = 5$ and domain $[0, 10]$.

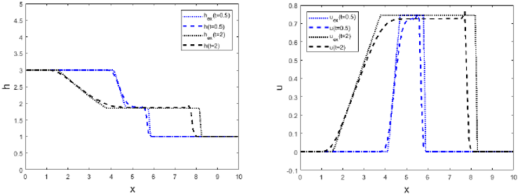
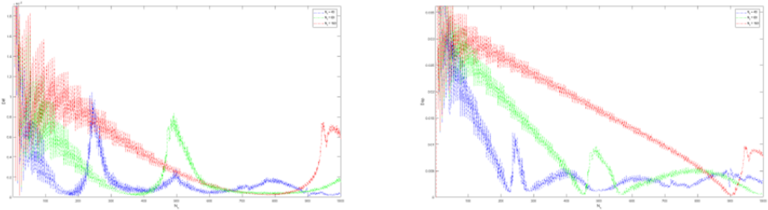


Figure: Numerical and exact solution of h, u at time $t = 0.5$ (blue) and $t = 2$ (black)

The solution develops a left-going rarefaction wave and a right-propagating shock wave.



Observations:

- Error is mainly Dispersive
- Highly oscillatory error for small N_x
- Peaks are observed for increasing N_x when Δt is reduced. As the interpolation is the main source of error in the SISL scheme, we see that these peaks are linked to the propagating shock wave, associated with integer-valued $\nu_{CFL} = v_s \Delta t / \Delta x$, where v_s is the analytical shock speed. The minimum error is obtained for half-integer ν_{CFL}.

The Moving Mesh SISL method

- Calculate the new mesh point locations X_j^n via equidistribution:

$$\int_{X_j^n}^{X_{j+1}^n} \rho(u(x,t^n))dx = \theta(t^n)$$

- Set arrival points: $X_{j,A}^{n+1} = X_j^n$
- Apply SISL iterative scheme on the non-uniform mesh

Advantages:

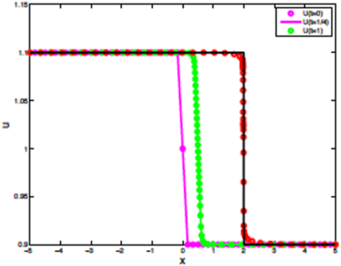
- Interpolation onto the new mesh is handled within the SISL method
- Spatial accuracy increased and no CFL restriction.
- Curvature monitor function $\rho = \sqrt{1+\beta u_{xx}^2}$ gives *optimal error estimates* for linear interpolation. In practice also arc-length monitor function is used: $\rho = \sqrt{1+\beta u_x^2}$

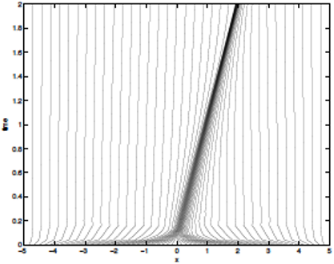
The MMSISL procedure

- At time level $t^n = n\Delta t$, we have a computed solution U_j^n at the current points X_j^n.
- Evaluate $\rho(U_j^n, X_j^n)$ to compute the new arrival points $X_{j,A}^{n+1}$.
- Use the kinematic equation (1) to compute the departure point $X_{j,D}^n$ from $X_{j,A}^{n+1}$.
- Apply the SISL iterative scheme to find the new computed solution $U_{j,A}^{n+1}$.

No interpolation onto the new mesh is needed !

The MMSISL scheme on Burger's equation





Dissipative/Dispersion error MMSISL

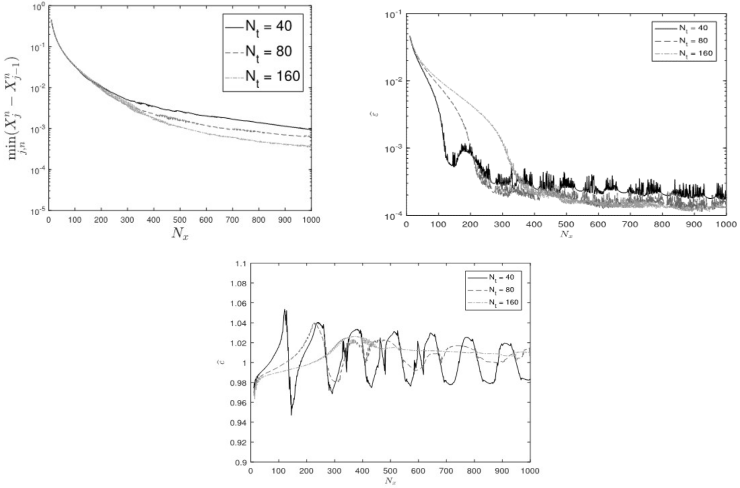


Figure: Arc-length monitor function for Burger's equation (MAX CFL $= 37.5$)

L2 error

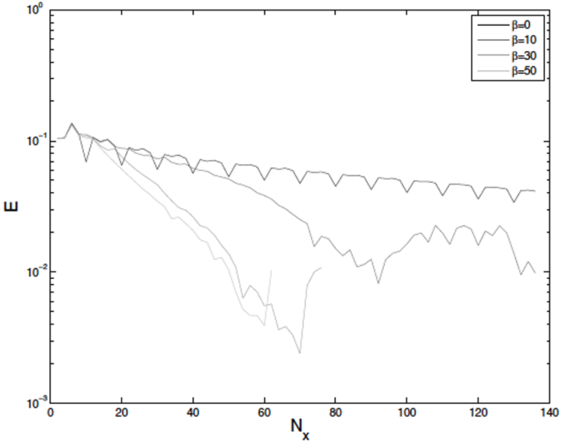


Figure: Error for arc-length monitor function $\rho = \sqrt{1 + \beta u_x^2}$

Dissipative/Dispersion error MMSISL

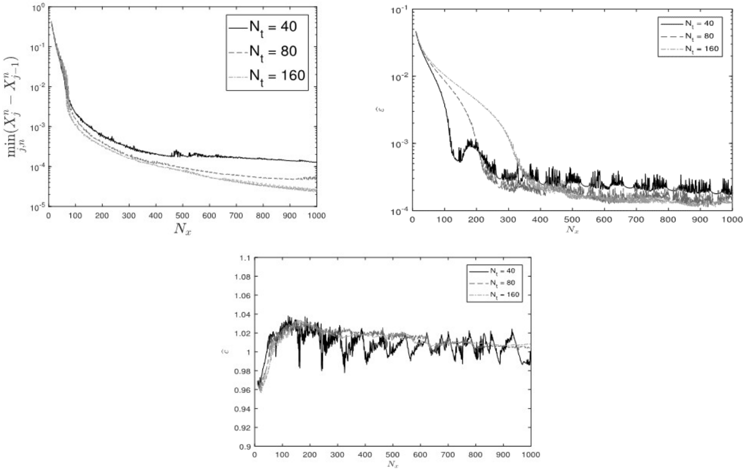
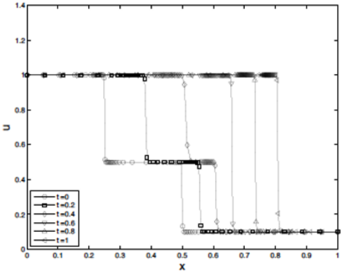
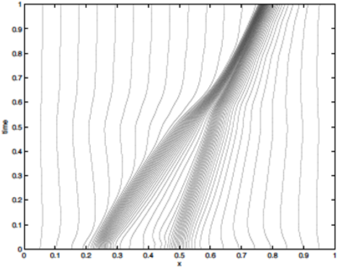


Figure: Curvature monitor function for Burger's equation (MAX CFL = 37.5)

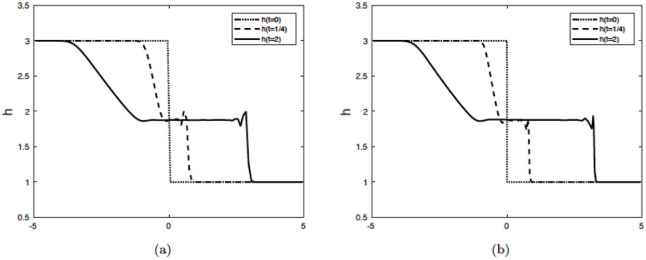
Two evolving fronts solution of Burger's equation

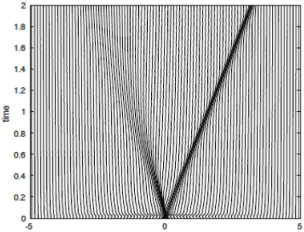
$$u(x,t) = \frac{0.1e^{\frac{-x+0.5-4.95t}{20\epsilon}} + 0.5e^{\frac{-x+0.5-0.75t}{4\epsilon}} + e^{\frac{-x+0.375}{2\epsilon}}}{e^{\frac{-x+0.5-4.95t}{20\epsilon}} + e^{\frac{-x+0.5-0.75t}{4\epsilon}} + e^{\frac{-x+0.375}{2\epsilon}}}.$$



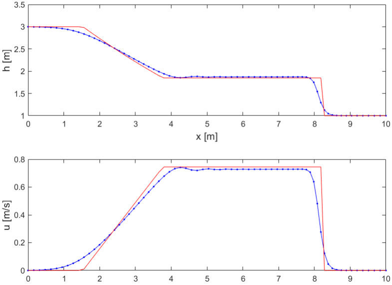


The MMSISL scheme on Dam break problem





Moving Mesh SWEs with Euler scheme



Solution affected by numerical dissipation and dispersion.

Number of mesh points N_x must be limited (< 100) to avoid instability.

Forward Euler problem unstable for $\nu_{CFL} < 1$ and $\beta > 1$.

Conclusions

- The SISL method allow for big time steps with no CFL constraint. Interpolation error poses a problem for spatial accuracy.
- The Moving Mesh SISL method reduces significantly the spatial error by adapting the mesh to the solution according to the equidistribution principle.
- Interpolation to the arrival points is handled automatically by the SISL method.
- Accurate and stable results have been showed for the Burger's equation and the Dam-Break problem.

Future Research

- Extend the MMSISL method to more complex problems in higher dimensions.
- 2D mesh adaptation by solving an optimal-transport Monge-Ampere equation with the Mixed Finite Element method.
- Use a posteriori-errori DG estimate as monitor function ρ.
- Implement mass conservative interpolation scheme by local Galerkin projection.

THANK YOU FOR YOUR ATTENTION !