ON SELF-SIMILAR BLOW-UP IN EVOLUTION EQUATIONS OF MONGE–AMPÈRE TYPE: A VIEW FROM REACTION-DIFFUSION THEORY

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ABSTRACT. We use techniques from reaction-diffusion theory to study the blow-up and existence of solutions of the parabolic Monge–Ampère equation with power source, with the following basic 2D model

0.1 (0.1)
$$u_t = -|D^2u| + |u|^{p-1}u \quad \text{in} \quad \mathbb{R}^2 \times \mathbb{R}_+,$$

where in two-dimensions $|D^2u| = u_{xx}u_{yy} - (u_{xy})^2$ and $p > 1$ is a fixed exponent. For a class of "dominated concave" and compactly supported radial initial data $u_0(x) \geq 0$, the Cauchy problem is shown to be locally well-posed and to exhibit finite time blow-up that is described by similarity solutions. For $p \in (1, 2]$, similarity solutions, containing domains of concavity and convexity, are shown to be compactly supported and correspond to surfaces with flat sides that persist until the blow-up time. The case $p > 2$ leads to single-point blow-up. Numerical computations of blow-up solutions without radial symmetry are also presented.

The parabolic analogy of (0.1) in 3D for which $|D^2u|$ is a cubic operator is

$$u_t = |D^2u| + |u|^{p-1}u \quad \text{in} \quad \mathbb{R}^3 \times \mathbb{R}_+,$$

and is shown to admit a wider set of (oscillatory) self-similar blow-up patterns. Regional self-similar blow-up in a cubic radial model related to the fourth-order M-A equation of the type

$$u_t = -|D^4u| + u^3 \quad \text{in} \quad \mathbb{R}^2 \times \mathbb{R}_+,$$

where the cubic operator $|D^4u|$ is the catalecticant 3×3 determinant, is also briefly discussed.

1. Introduction: our basic parabolic Monge–Ampère equations with blow-up

1.1. **Outline.**

Fully nonlinear parabolic partial differential equations with spatial Monge–Ampère (M-A) operators arise in many problems related to optimal transport and geometric flows [9], image registration [31], adaptive mesh generation [14, 5], the evolution of vorticity in meteorological systems [9], the semi-geostrophic equations of meteorology, as well as being extensively studied in the analysis literature; see Taylor [50, Ch. 14,15], Gilbarg–Trudinger [28, Ch. 17], Gutiérrez [29], and Trudinger–Wang [52], as a most recent reference. To describe such equations, we consider a given function $u \in C^2(\mathbb{R}^N)$, for which D^2u denotes the corresponding $N \times N$ *Hessian* matrix $D^2u = \|u_{x_ix_j}\|$, so that in two-dimensions, $d = 2$,

$$|D^2u| \equiv \det D^2u = u_{xx}u_{yy} - (u_{xy})^2. \tag{1.1}$$

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Similarly, in three dimensions

M133 (1.2)
$$|D^2u| = \begin{vmatrix} u_{xx} & u_{xy} & u_{xz} \\ u_{xy} & u_{yy} & u_{yz} \\ u_{xz} & u_{yz} & u_{zz} \end{vmatrix}.$$

A general parabolic *Monge–Ampère* (M-A) *equation* with a nonlinear source term, then takes the form

MA.1 (1.3)
$$u_t = g\big(\det D^2u\big) + h(x, u, D_x u) \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+$$

with proper initial data $u(0, x) = u_0(x)$. Such PDEs with various nonlinear operators $g(\cdot)$ and $h(\cdot)$ have a number of important applications.

The origin of such fully nonlinear M-A equations dates back to Monge's paper [45] in 1781, in which Monge proposed a civil-engineering problem of moving a mass of earth from one configuration to another in the most economical way. This problem has been further studied by Appel [1] and L.V. Kantorovich [36, 37]; see references and a survey in [19]. Other key problems and M-A applications include: logarithmic Gauss and Hessian curvature flows, the Minkowski problem (1897) and the Weyl problem (with Calabi's related conjecture in complex geometry), etc.

For increasing functions $g(s)$, the equation (1.3) is parabolic if $D^2u(\cdot, t)$ remains positive definite for $t > 0$, assuming that $D^2u_0 > 0$ and local-in-time solutions exist [41, p. 320]. Provided that $g(s)$ does not grow too rapidly, for example if

$$g(s) = \ln s, \;\; g(s) = -\tfrac{1}{s}, \;\text{ and }\; g(s) = s^{\frac{1}{N}} \quad \text{for } s > 0,$$

it is known [35, 29, 7] that the solutions of M-A exist for all time.
In general, however, the questions of local solvability and regularity for M-A equations even in 2D such as the system

Kh1 (1.4)
$$(u_{xx} + a)(u_{yy} + c) - (u_{xy} + b)^2 = f$$

in the hyperbolic ($f < 0$) and mixed type (f of changing sign) are difficult, and there are some counterexamples concerning these basic theoretical problems and concepts; see [38]. Note that classification problem for the M-A equations such as (1.4) on finding their simplest form was already posed and partially solved by Sophus Lie in 1872-74 [42]; see details and recent results in [39].

For other functions $g(s)$ in (1.3) with a faster growth rate as $s \to \infty$, and for certain nonlinear source terms h, solutions which are locally regular may evolve to blow-up in a finite time T. This gives special singular asymptotic patterns, which can also be of interest in some geometric applications; on singular patterns for M-A flows, see [10, 11] The analysis literature currently suffers from a lack of understanding about the formation of such singularities in the fully nonlinear M-A equation despite their relevance to such problems as front formation in meteorology [9]. This paper aims to make a start at studying such blow-up singular behaviour by using techniques derived from studying reaction-diffusion equations to look at some special parabolic

M-A problems, which lead to the finite time formation of singularities. In particular, we consider (1.5), (1.2), and some other higher-order PDEs as formal basic equations demonstrating that M-A models can exhibit several common features of blow-up, which have been previously observed in PDEs with classic reaction-diffusion, porous medium, and the p-Laplacian operators. Indeed, we will approach the study of M-A type operators by developing the related theory of the p-Laplacian operator. Our interest in this paper will be an understanding of the various forms of singularity that can arise in some parabolic Monge–Ampère models with a *polynomial source term* as well as extending the general existence theory for such problems.

1.2. **Model equations and results. Model 1** Our first model fully nonlinear PDE is given by

M1

$$u_t = (-1)^{d-1}|D^2u| + |u|^{p-1}u \quad \text{in} \quad \mathbb{R}^d \times \mathbb{R}_+, \tag{1.5}$$

where $p > 1$ is a given constant. Such models are natural counterparts of the porous medium equation with reaction/absorption, and of thin film (or Cahn–Hilliard-type, $n = 0$) models,

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$$u_t = \Delta u^m \pm u^p \quad \text{and} \quad u_t = -\nabla \cdot (|u|^n \nabla \Delta u) \pm \Delta |u|^{p-1}u. \tag{1.6}$$

Our principle interest lies in the study of those solutions which have large isolated spatial maxima tending towards singularities forming in the finite time T. Such solutions are locally concave close to the peak. The choice of sign of the principal operator in (1.5) ensures local well-posedness (local parabolicity) of the partial differential equation in such neighborhoods. Significantly, the existence theory for such locally concave solutions is rather different from the usual theory of the M-A operator, which is restricted to globally convex solutions, and we will look at it detail in Section 4. The initial data $u_0(x) \geq 0$ is assumed to be bell-shaped (this preserves "dominated concavity") and sometimes compactly supported. We firstly study radially symmetric solutions in two and three spatial dimensions, and will show analytically, by extending the theory of p-Laplacian operators, and demonstrate numerically, that whilst the Cauchy problem is locally well-posed, and admits a unique radially symmetric weak solution, certain of these solutions become singular with finite-time blow-up. We will also find a set of self-similar blow-up patterns, corresponding to single-point blow-up if $p > d$, regional blow-up for $p = d$, and to global blow-up for $p \in (1, d)$. The stability of these will be investigated numerically, and we will find that monotone self-similar blow-up profiles appear to be globally stable.

We will also present some analytic and numerical results for the time evolution of non radially symmetric solutions in two dimensions. These computations will give some evidence to conclude that stable non-radially symmetric blow-up solution profiles also exist, though this leads to a number of difficult open mathematical problems.

Model 2 As a second model equation, we will look at higher-order fully nonlinear M-A spatial operators associated with the equations of the form

1.cjb

$$u_t = (-1)^{d-1}|D^4u| + |u|^d u \quad \text{in} \quad \mathbb{R}^d \times \mathbb{R}_+, \tag{1.7}$$

where $|D^4u|$ is the determinant of the 4th derivative matrix of u. We will find similar results on the blow-up profiles to those for the second-order operator. A principal feature of compactly supported solutions to (1.7) is that these are infinitely oscillatory and changing sign at the interfaces, and this property persists until blow-up time.

The layout of the remainder of this paper is as follows. In Section 2 we look at single-point blow-up, regional blow-up, and the global one of the radially symmetric solutions of the polynomial M-A equation in two and three spatial dimensions. We will combine both an analytical and a numerical study to determine the form and stability of the self-similar blow-up solutions. In Section 3, we will extend this analysis to look at solutions, which do not have radial symmetry, and will give numerical evidence for the existence of stable blow-up profiles in non radial geometries. In Section 4, we will look at more general properties associated with M-A type flows, in particular the existence of various conservation laws. Finally, in Section 5 we will study the forms of the blow-up behaviour for the equations with higher-order operators as in (1.7).

2. Parabolic M-A equations in $\mathbb{R}^2$: blow-up in radial geometry

S2

2.1. **Radially symmetric solutions: first results on blow-up.** The Hessian operator $|D^2u|$ restricted to radially symmetric solutions in $\mathbb{R}^d$ takes the form of a non-autonomous version of the p-Laplacian operator. Namely, for solutions $u = u(r,t)$, with the single spatial variable $r = |(x, y, ...)| > 0$, equation (1.5) takes the form

2.1
$$u_t = (-1)^{d-1} \tfrac{1}{r^{d-1}} \, (u_r)^{d-1} u_{rr} + |u|^{p-1} u \quad \text{in} \quad \mathbb{R}_+ \times \mathbb{R}_+, \tag{2.1}$$

and then for $r = 0$ we have the symmetry condition

$$u_r = 0.$$

In this section, we shall consider the nature of the blow-up solutions for this problem and will identify different classes (single-point, regional and global) of self-similar radial solutions, giving some numerical evidence for their stability. However, we note at this stage (and will establish in the next section) that (possibly stable) non-radially symmetric blow-up solutions of the underlying PDE also exist. One can see that (2.1) implies that the equation is (at least, degenerate) parabolic if

par1
$$(-1)^{d-1}(u_r)^{d-1} \ge 0 \quad \Longrightarrow \quad \begin{cases} u_r \le 0 & \text{for even } d, \\ u_r & \text{is arbitrary for odd } d. \end{cases} \tag{2.2}$$

For the local well-posedness of the above M-A flow, (2.2) is always assumed. In all the cases, the differential operator $(-1)^{d-1} \frac{1}{r^{d-1}} \, (u_r)^{d-1} u_{rr}$ is regular in the class of monotone decreasing, sufficiently smooth, and strictly concave at the origin functions, so that the local well-posedness of (2.1) is guaranteed for the initial data satisfies the regularity and monotonicity constraints

2.2
$$u(r,0) = u_0(r) \ge 0, \quad u_0 \in C^1([0,\infty)), \quad u_0'(0) = 0, \quad u_0'(r) \le 0 \ \text{ for } \ r > 0. \tag{2.3}$$

The corresponding p-Laplacian counterpart of (2.1) is then

2.3
$$u_t = \tfrac{1}{r^{d-1}} \, |u_r|^{d-1} u_{rr} + |u|^{p-1} u \quad \text{in} \quad \mathbb{R}_+ \times \mathbb{R}_+, \tag{2.4}$$

which is locally well-posed by parabolic regularity theory; see e.g., [15]. By the Maximum Principle (MP), the assumptions in (2.3) guarantee that the solution $u = u(r,t)$ satisfies the monotonicity condition

2.4
$$u_r(r,t) \le 0. \tag{2.5}$$

Therefore, equations (2.1) and (2.4) coincide in this class of monotone solutions. Note that for local well-posedness, we do not need any *concavity-type* assumptions that are usual for standard M-A flows.

The phenomenon of blow-up for the solutions of the model (2.4) can be studied by using techniques derived from the study of reaction diffusion equations (see [49, Ch. 4]). By a comparison of the solution with sub-and super-solutions of the same equation of self-similar form, we can show that, for nonnegative solutions u, there exists a critical Fujita exponent

Fuj1
$$p_0 = d + 2 \quad \text{such that:} \tag{2.6}$$

(i) for $p \in (1, p_0]$, any $u(x,t) \not\equiv 0$ blows up in finite time, and

(ii) in the supercritical range $p > p_0 = d + 2$, solutions blow-up for large enough data, while for small ones, the solutions are global in time.

The proof of blow-up in the critical case $p = p_0$ is most delicate and demands a monotonicity/asymptotic rescaled construction; see e.g., [21, 24].

For the remainder of this paper we shall always assume that the initial data are such that blow-up always occurs.

2.2. **Blow-up similarity solutions.** The M-A equation with the $|u|^{p-1}u$ source term is invariant under the scaling group

$$t \to \lambda t, \quad r \to \lambda^{\frac{p-d}{2d(p-1)}} r, \quad u \to \lambda^{-\frac{1}{p-1}} u.$$

Accordingly, a self-similar blow-up profile (with blow-up at the origin $r = 0$, which is assumed to belong to the blow-up set) is described by the following solutions:

M5
$$u_S(r,t) = (T-t)^{-\frac{1}{p-1}} f(z), \quad z = \tfrac{r}{(T-t)^\beta}, \quad \beta = \tfrac{p-d}{2d(p-1)}. \tag{2.7}$$

Here $f \ge 0$ is a solution of the following ordinary differential equation,

M6a
$$\tfrac{1}{z^{d-1}}\,(-1)^{d-1}(f')^{d-1}f'' - \beta f' z - \tfrac{1}{p-1}\, f + |f|^{p-1} f = 0, \;\; f'(z) \le 0, \;\; f'(0) = f(+\infty) = 0. \tag{2.8}$$

The condition on $f(+\infty)$ (and the consequent requirement that the solutions of (2.8) should decay to zero as $z \to \infty$) is necessary to ensure that the self-similar solutions correspond to solutions of the original Cauchy problem of the PDE. In the case of monotone decreasing solutions with $f'(z) \le 0$ the equation (2.8) becomes

M6
$$\tfrac{1}{z^{d-1}}\,|f'|^{d-1} f'' - \beta f' z - \tfrac{1}{p-1}\, f + |f|^{p-1} f = 0, \;\; f'(z) \le 0 \text{ in } \mathbb{R}_+; \quad f'(0) = f(+\infty) = 0. \tag{2.9}$$

In particular, in the case of $d = 2$ to be studied in greater detail, we have to require the monotonicity assumption to allow the construction of smooth solutions. In the case of $d = 3$, we can relax this assumption, leading to a richer class of (possibly oscillatory) solutions. When considered as an initial value problem, the solutions of (2.8) lose regularity at the degeneracy

$f' = 0$ (except the origin $r = 0$) and are not twice differentiable at such points. However the existence of weak solutions with reduced regularity is guaranteed by the standard theory of the p-Laplacian operator, and these questions are standard in parabolic theory; see [15] and [27, Ch. 2]. Note that the ODE (2.9) has two constant equilibria given by

eq11 (2.10)
$$\pm f_0 = \pm(p-1)^{-\frac{1}{p-1}}$$

and that solutions close to these equilibria can be oscillatory, which is a crucial property to be properly treated and used.

If $p > 1$, then each solution of the form (2.7) blows up in finite time, however the nature of the scaling is very different in the three cases of $1 < p < d$, $p = d$ and $p > d$, corresponding to global (blow-up over the whole of $\mathbb{R}^d$), regional (blow-up over a sub-set of $\mathbb{R}^d$ with non-zero measure), and single-point blow-up respectively (zero measure blow-up set in general). Indeed, we will show that for $p \in (1, d]$ the solutions $f(z)$ of the ODE are compactly supported, while for $p > d$ they are strictly positive.

2.3. **Regional blow-up when $p = d$.** We begin with the case $p = d$, where, according to (2.7), the ODE becomes autonomous and $z = r$, so that

M7 (2.11)
$$\tfrac{1}{r^{d-1}}\, |f'|^{d-1} f'' - f + |f|^{p-1} f = 0, \;\; f'(r) \le 0 \;\text{ in }\; \mathbb{R}_+; \quad f'(0) = f(+\infty) = 0.$$

This problem falls into the scope of the well-known blow-up analysis for quasilinear reaction-diffusion equations, [49, Ch. 4].

Pr.1 **Proposition 2.1.** *The problem* (2.11) *has a non-trivial, monotone, compactly supported solution $F_0(z) \ge 0$ such that $F_0(0) > 1$. The support of $F_0(z)$ is given by $[0, L_S]$ with the asymptotic behaviour near the interface: as $z \to L_S$,*

as1N (2.12)
$$F_0(z) = A(L_S - z)_+^{\frac{d+1}{d-1}}(1 + o(1)), \quad \textit{where} \quad A = \left[\tfrac{(d-1)^{d+1}}{2(d+1)^d}\right]^{\frac{1}{d-1}}.$$

Proof. The result follows the lines of the ODE analysis in [49, pp. 183-189] and uses a shooting approach in which (2.8) is considered as an IVP with shooting parameter $f(0)$. If $f(0)$ is too large then the solutions of the IVP diverge to $-\infty$ and if it is not large enough then the solution has an oscillation about f_0 in a manner to be described in more detail below. The self-similar solution occurs at the point of transition between these two forms of behaviour. $\square$

The form of the proof leads to a *numerical method based on shooting* for constructing an approximation to the function F_0. To do this we specify the value of $F_0(0)$, take $F_0'(0) = 0$ and solve (2.11) as an initial value problem in r using an accurate numerical method (typically a variable step BDF method). In this numerical calculation the term $|f'|$ is replaced by $\sqrt{\varepsilon^2 + (f')^2}$ with $\varepsilon = 10^{-5}$. This allows the numerical method to cope with the loss of regularity when $f' = 0$. The value of $f(0)$ is then steadily increased from f_0 and the transition between oscillatory and divergent behaviour determined by bisection. At this particular value $f(0) \equiv F_0(0)$ the solution approaches zero as $z \to L_S$ and for $z > L_S$ we have $F_S(0) \equiv 0$. Unlike studies of reaction-diffusion equations, the proof of the uniqueness of the solution of (2.11) is not straightforward. However, the numerical calculations strongly indicate the conjecture that such a compactly

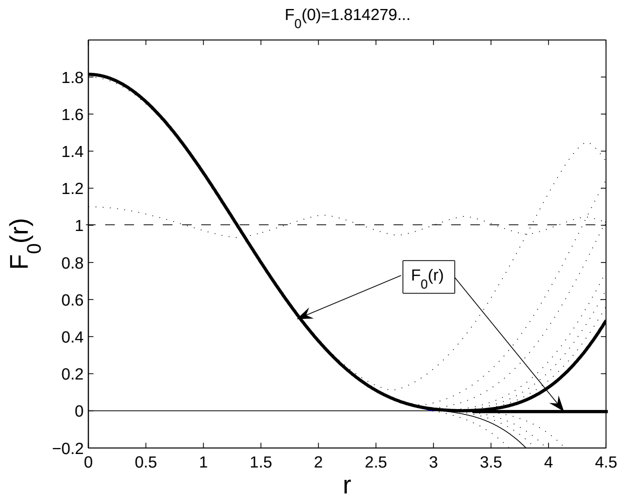


FIGURE 1. $p = d = 2$: the similarity profile F_0 obtained by shooting in the ODE (2.11) from the origin $r = 0$ with the parameter of shooting given by $f(0) > 0$. F1

supported monotone decreasing profile $F_0(z)$ is indeed unique. In the case of $d = 2$, its support is

M8 (2.13) $$L_S = 3.26... , \quad \text{with} \quad F_0(0) = 1.814279... .$$

Similarly, in the case of $d = 3$,

M8cjb (2.14) $$L_S = 2.303... , \quad \text{with} \quad F_0(0) = 1.366... .$$

In Figure 1, we show the resulting profile (bold) in the case of $p = d = 2$, obtained by shooting as described above, together with an oscillatory solution of the IVP when $f(0) \approx f_0$ and some nearby divergent solutions.
In the case of $d = 3$, we can relax the monotonicity requirement on the function $f'(z)$. In this case there exists a countable set $\{F_k^P\}$ of compactly supported profiles that change sign precisely k times for any $k = 0, 1, 2, ...$. A numerical shooting calculation of both $F_0(z)$ (dotted) and of $F_1^P(z)$ (bold), together with some nearby oscillatory solutions, is shown in Figure 2. This figure both explains how these further profiles can be obtained numerically and indicates how their existence can be justified rigorously along the lines of the proofs in [2, 23]. Here, we have

$$L_S^{(1)} = 3.95... , \quad \text{with} \quad F_1^P(0) = 1.6513... .$$

It follows immediately that the variable separable solution given by

M9 (2.15) $$u_S(r, t) = \tfrac{1}{T-t}\, F_0(r)$$

describes *regional blow-up* which is localized in the disc/ball $\{r \leq L_S\}$.

We now make a further numerical calculation to investigate the stability of such a blow-up profile (in the restricted class of radially symmetric solutions). To do this we use a semi-discrete numerical method in which we discretise the Monge–Ampère spatial operator on a fine spatial

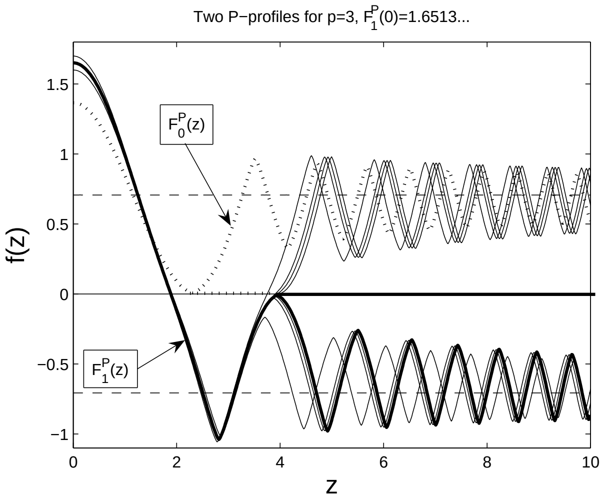


FIGURE 2. $p = d = 3$: the similarity profiles F_0 (dotted) and F_1^P (bold) and some nearby oscillatory solutions, obtained by shooting in the ODE (2.11) from the origin with the parameter of shooting $f(0) > 0$. FSh

mesh. This leads to a set of ordinary differential equations for the discrete approximation to the solution $u(r, t)$. These (stiff ordinary differential) equations are then solved using an accurate variable order BDF method. In this calculation we substitute a spatial domain $r \in [0, L]$ for the infinite interval and impose a Neumann condition at the boundaries $r = 0$ and $r = L$. For a calculation with $p = d = 2$ we take $L = 8$ and use a spatial discretisation step size of $L/1000$. We present in the following figures some calculation showing the evolution of $u(r, t)$ and the scaled function $u(r, t)/u(0, t)$ taking as initial data

$$u(r, 0) = 10\, \mathrm{e}^{-\alpha x^2/2}.$$

We consider two values of α to give profiles which lie above and below the self-similar solution. With $\alpha = 0.1$ the solutions initially lie above the self-similar solution and in this case we see clear evidence in Figures 3 and 4 for evolution towards self-similar regional blow-up with a blow-up time of $T \approx 0.1099$.

We also plot in Figure 5 the value of $m(t) = 1/u(0, t)$. For small values of m this figure is very close to linear, and a linear fit gives

$$m(t) \approx -0.5553t + 0.0610, \quad \text{so that} \quad u(0, t) \approx \tfrac{1.805}{0.1099 - t},$$

which is in good agreement with the earlier calculation of the self-similar profile.

For comparison we now take initial data $\alpha = 10$. In this case the value of $u(0, t)$ initially decreases, and then increases with a blow-up time of $T \approx 1.4371$. Asymptotically the blow-up is almost identical to that observed earlier, however we can see clearly that this time the function $u(r, t)/u(0, t)$ approaches the regional self-similar profile from below.

Very similar figures arise in the case of $d = 3$ indicating that the regional self-similar solution is also stable in this case.

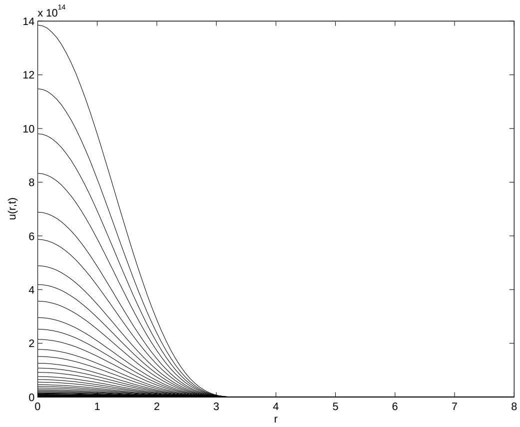


FIGURE 3. $p = d = 2$: Regional blow-up of the function $u(r, t)$ with $\alpha = 0.1$. CJB

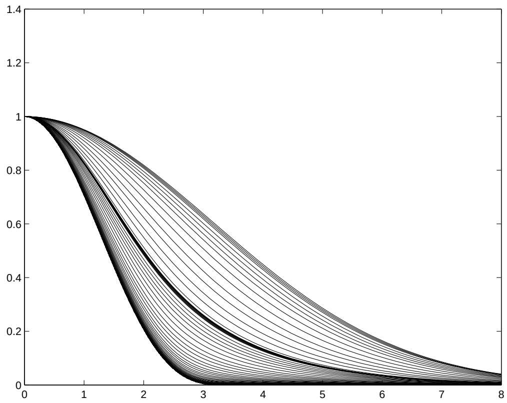


FIGURE 4. $p = d = 2$: Regional blow-up of the scaled function $u(r, t)/u(0, t)$ with $\alpha = 0.1$ showing convergence to the self-similar profile with support $[0, 3.26]$ from above. CJB

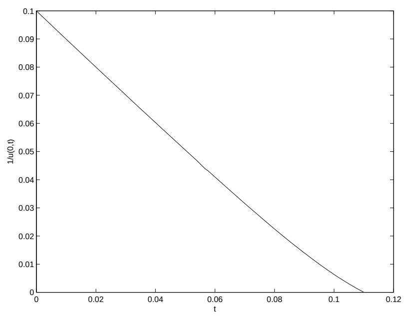


FIGURE 5. $p = d = 2$: Evolution of $1/u(0, t)$ with $\alpha = 0.1$ showing that $u(0, t)$ increases towards infinity. Note the linear behaviour $\sim T - t$ close to the blow-up time. CJB

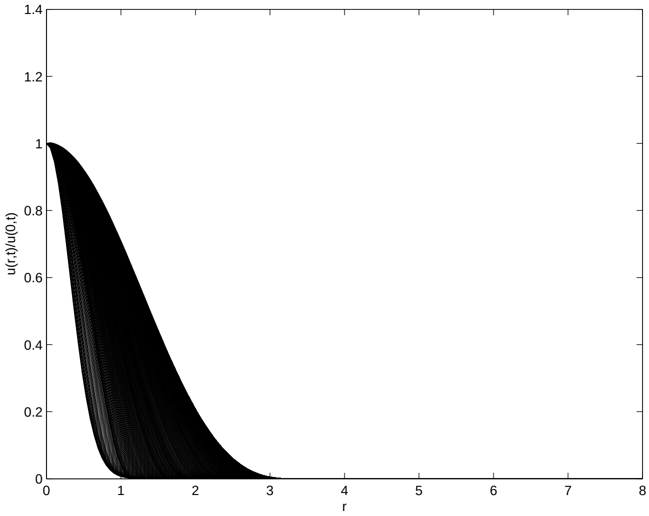


FIGURE 6. $p = d = 2$: Regional blow-up of the scaled function $u(r,t)/u(0,t)$ with $\alpha = 10$ showing convergence to the self-similar profile with support $[0, 3.26]$ from below. CJB

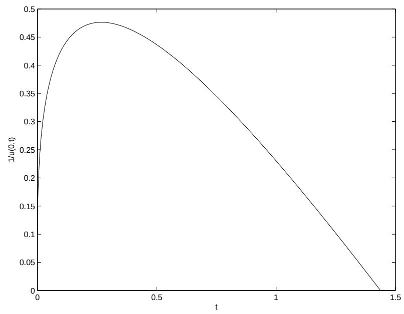


FIGURE 7. $p = d = 2$: Evolution of $1/u(0,t)$ with $\alpha = 10$ showing that $u(0,t)$ initially decreases before tending towards infinity. Note the linear behaviour $\sim T - t$ close to the blow-up time. CJB

2.4. **Single point blow-up for $p > d$: P and Q profiles.** If $p > d$ then $\beta > 0$ and single point blow-up occurs at the origin. The self-similar blow-up profiles can then take various forms. Initially we consider the monotone profiles for general d.

PrLS **Proposition 2.2.** *For any $p > d$, the ordinary differential equation problem* (2.9) *admits two strictly positive solutions $F_0^P(z) > 0$ and $F_0^Q(z) > 0$, which each satisfy the asymptotic expansion*

C01 (2.16)
$$F_0(z) = C_0 z^{-\frac{2d}{p-d}}(1 + o(1)) \quad \text{as } z \to +\infty \quad (C_0 > 0),$$

and for which (i) *$F_0^P(z)$ is strictly monotone decreasing, $F_0'(z) < 0$ for $z > 0$, with*

s2 (2.17)
$$F_0(0) > f_0, \quad \text{and}$$

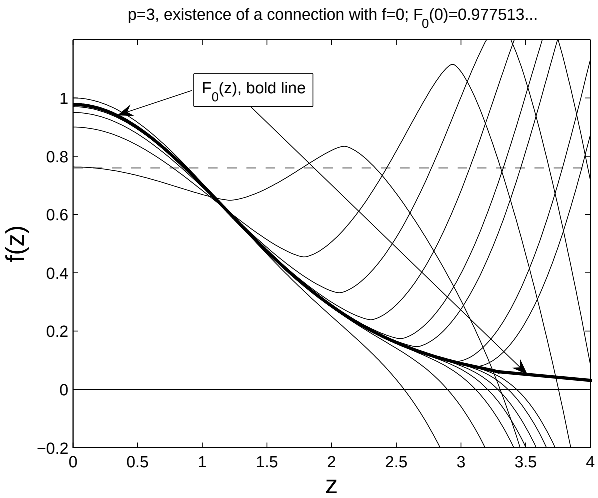


FIGURE 8. Similarity P-profile F_0 for $d = 2$ and $p = 3$ in (2.9) obtained by shooting from the origin $y = 0$ with the parameter of shooting $F_0(0) > 0$. F2

(ii) $F_0^Q(z) \equiv f_0$ *on some interval* $z \in [0, a_0]$ (*i.e., it has flat sides in this ball*), *has the behaviour close to the interface given by*

ff11N (2.18)
$$F_0(z) = f_0 - \tfrac{1}{2}\,\beta a_0^2 (z - a_0)_+^2 (1 + o(1)) \quad \textit{as} \quad z \to a_0,$$

and is strictly monotone decreasing and smooth for $z > a_0$.

The main ingredients of the proof of both P-type profiles of the form (i) and Q-type profiles of type (ii) are explained in [23, 2]. Both solutions are again constructed by shooting, with $f(0)$ being the shooting parameter for the P-type profiles and a_0 for the Q-type profiles. In Figure 8, we show the similarity profile of the P-type solution $F_0(z)$ (bold) for the case of $d = 2$ and $p = 3$ together with some nearby divergent solutions. In this case we find that $F_0(0) = 0.9751...$. Numerically there is strong evidence for uniqueness of this solution, but a proof of this uniqueness result is open.

Similarly, in Figure 9, we show the results of shooting to find the Q-type profile when for $d = 2$ and $p = 3$ and we obtain numerically that $a_0 = 2.292...$.

In the case of $d = 3$, we may extend these results to construct countable families of non-monotone P and Q-type solutions described as follows:

PrLS3 **Proposition 2.3.** *If* $d = 3$ *then for any* $p > 3$, *problem* (2.8) *admits the following two countable families of solutions:*

(i) *A P-type family* $\{F_k^P(z) > 0,\, k \geq 0\}$ *such that*

s23 (2.19)
$$F_k(0) \neq f_0 \quad \textit{and} \quad F_k(z) = C_0 z^{-\frac{2d}{p-3}} (1 + o(1)) \;\; \textit{as} \;\; z \to +\infty,$$

where $C_k > 0$ *is a constant, and each* $F_k(z)$ *has precisely* $k + 1$ *intersections with the constant solution* f_0; *and*

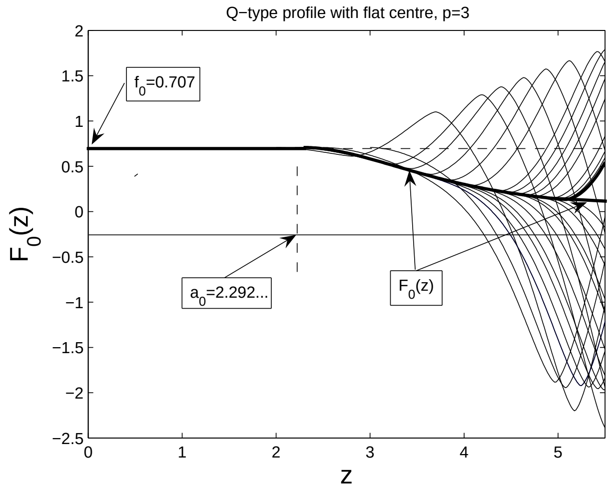


FIGURE 9. Similarity Q-profile F_0 for $d=2$, $p=3$ in (2.9) obtained by shooting from $z=a_0>0$, which is the parameter of shooting. F2Q

(ii) *A Q-type family* $\{F_k^Q(z),\, k \ge 0\}$, *where each* $F_k^Q(z) = f_0$ *on some interval* $z \in [0, a_k]$, $\{a_k > 0\}$ *is strictly monotone decreasing, with the following behaviour at the interface:*

ff11N3 (2.20)
$$F_k(z) = f_0 + (-1)^k \sqrt{\tfrac{8}{9}\, \beta a_k^3}\, (z-a_k)_+^{\frac{3}{2}}(1+o(1)) \quad as \quad z \to a_k,$$

and has precisely k *intersections with* f_0 *for* $z > a_k$ *and has the asymptotic behaviour* (2.19).

For the main concepts of the proof, see [23, 2].

In Figure 10 (a,b), we show the similarity profiles $F_0^P(z)$ $F_1^P(z)$ for $d=3, p=4$. Construction of further P-type profiles is similar and the following holds:

$$F_k^P(0) \to f_0 \quad \text{as} \quad k \to \infty,$$

and moreover the convergence is from above for even $k=0,2,4,...$, and from below for odd $k=1,3,5,\dots$. Uniqueness of each F_k with $k+1$ intersections with equilibrium f_0 is a difficult open problem. We claim that as $k\to\infty$, both families $\{F_k^P(z)\}$ and $\{F_k^Q(z)\}$ satisfying

$$F_k^P(0) \to f_0 \quad \text{and} \quad a_k^Q \to 0$$

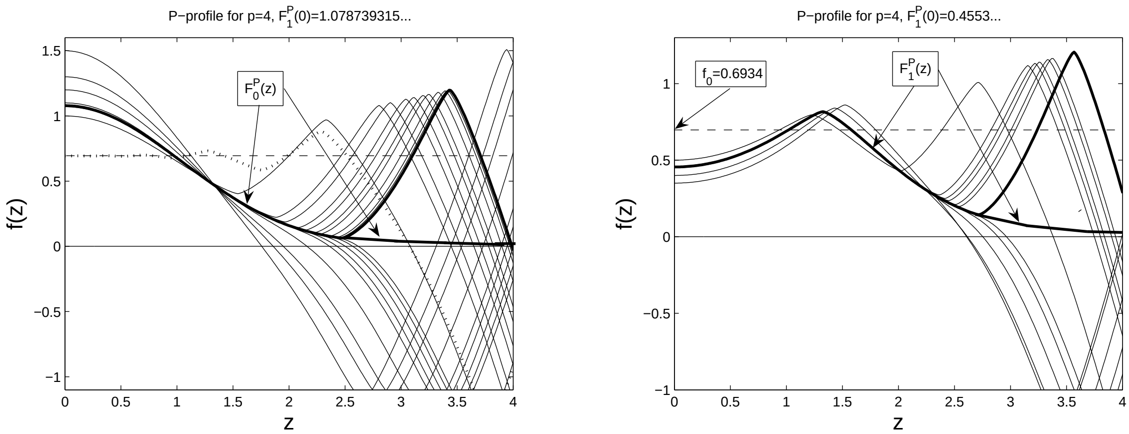
converge to a unique S-type profile $F_\infty^S(z)$ such that

fs1 (2.21) $\quad F_\infty^S(0) = f_0$ and has infinitely many intersections with f_0 for small $z>0$.

The first two similarity profiles of Q-type with $d=3$ and $p=4$, with a flat centre part, are shown in Figure 11.

Passing to the limit $t \to T^-$ in (2.7), where $z \to +\infty$, for any fixed $r>0$, we find from (2.17) the following final-time profile for both P and Q patterns:

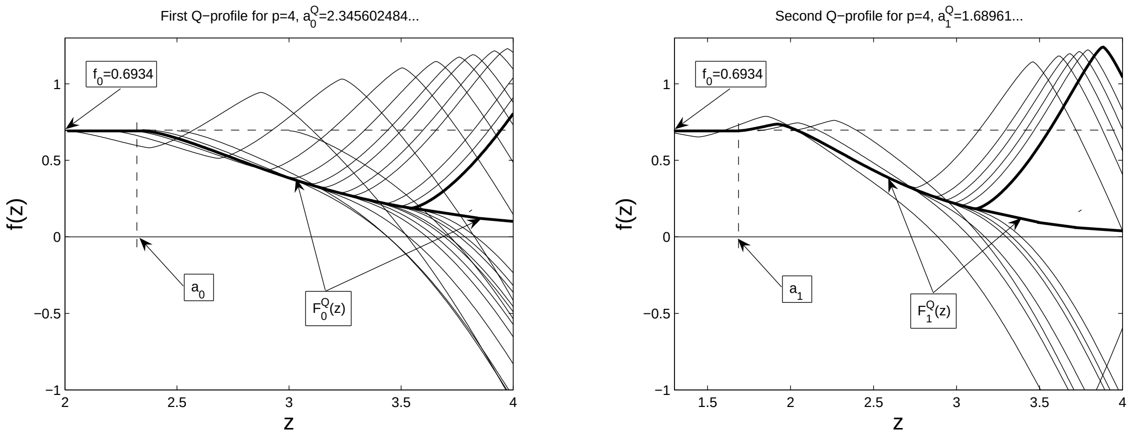
ft1 (2.22)
$$u_S(r, T^-) = C_0 r^{-\frac{2}{p-1}} < \infty \quad \text{for any} \quad r > 0,$$



(a) $k = 0$ (b) $k = 1$

F23

FIGURE 10. P-type profiles of the ODE (2.9) for $d = 3$, $p = 4$.



(a) $k = 0$ (b) $k = 1$

FQQ1

FIGURE 11. Q-type profiles of the ODE (2.9) for $p = 4$.

with $C_0 > 0$ fixed in (2.16). This implies single point blow-up at the origin $r = 0$ only in both cases.

We now make a *numerical study* of the stability of these solutions with single-point blow-up. Using a similar method to that described in the previous section (including taking Neumann boundary conditions with $L = 8$) we can study the nature of the time dependent blow-up solutions in this case. Usually when studying single-point blow-up the narrowing of the solution peak as the reaction time T is approached, requires the use of a spatially adaptive mesh [4]. However in the M-A systems, as the blow-us the width of the solution peak scales relatively slowly (as $(T-t)^{1/8}$ when $d = 2$) it is not necessary to use an adaptive method to study the nature of the blow up solutions in this case provided that the spatial grid is fine enough. Taking $d = 2$ and $p = 3$ and initial data $u(r,t) = 10\mathrm{e}^{-r^2}$ we find that the solution blows up in a time $T = 0.028476$. Indeed, plotting $1/u^2(0,t)$ as a function of t we find a close to linear solution,

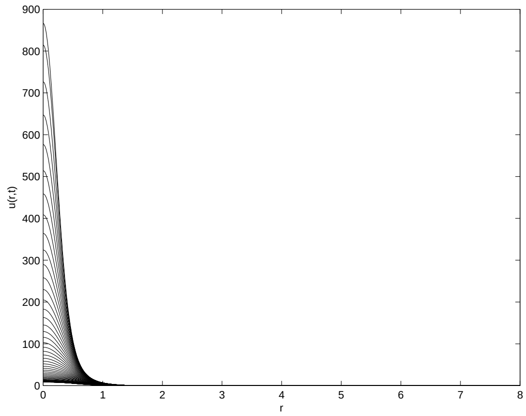


FIGURE 12. The solution $u(r,t)$ when $d = 2$ and $p = 3$ as it evolves toward a P-type singular solution at time $t = 0.028476$ F2c

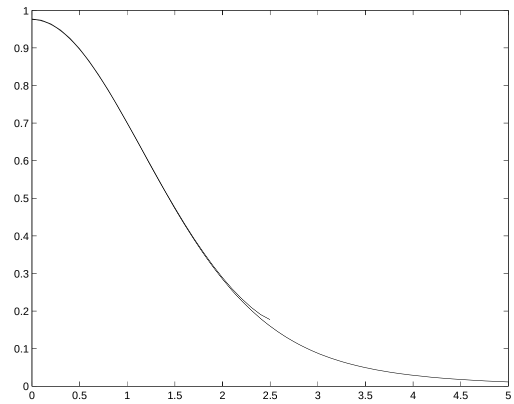


FIGURE 13. The rescaled solution $(T-t)^{1/2}u(r,t)$ plotted as a function of the similarity variable $y = r/(T-t)^{1/8}$ compared with the similarity profile of the P-type solution F_0 obtained by shooting. F2c

which has a best fit with the equation $u(0,t) = 0.975/(T-t)^{1/2}$ with T as above. Plotting the rescaled solution $(T-t)^{1/2}u(r,t)$ as a function of the similarity variable $z = r/(T-t)^{1/8}$ we find close agreement to the similarity solution constructed above. This strongly implies that the monotone P-type similarity solution is stable in the rescaled variables. A similar result is observed for calculations when $d = 3$. However, the Q-type and non-monotone P-type blow-up profiles all appear from these calculations to be unstable.

2.5. **S-type periodic solutions.** The main part in the existence analysis for (2.9), following the methods described in [23, 2], relies on constructing two solutions to the initial value problem with *different oscillatory structure*, and then deducing the existence of an intermediate solution with the correct properties at infinity by applying continuity arguments. To construct such solutions we must determine the oscillatory properties of the solutions $f(z)$ about the positive equilibrium f_0. To study these we consider solutions of the form

l1 (2.23) $$f(z) = f_0 + Y(z), \quad \text{where } Y \text{ is small and solves the reduced equation}$$

l2 (2.24) $$\tfrac{1}{z^{d-1}}\,|Y'|^{d-1}Y'' - \beta Y'z + Y = 0,$$

which remains nonlinear. In order to study oscillations of $Y(z)$ about 0, we can exploit the scaling invariance of (2.24) and introduce the *oscillatory component* φ as follows:

l3 (2.25) $$Y(z) = z^{\alpha}\varphi(s), \quad s = \ln z, \quad \text{where} \quad \alpha = \tfrac{2d}{d-1}.$$

Substituting (2.25) into (2.24) in (for example) the case of $d = 2$ yields the following *autonomous ODE*:

l4 (2.26) $$|\varphi' + 4\varphi|(\varphi'' + 7\varphi' + 12\varphi) - \beta\varphi' + \tfrac{1}{p-1}\,\varphi = 0.$$

Setting $\varphi' = \psi(\varphi)$ yields the first-order ODE system

l5 (2.27) $$\psi \frac{\mathrm{d}\psi}{\mathrm{d}\varphi} = \frac{\beta\psi - \frac{1}{p-1}\,\varphi}{|\psi + 4\varphi|} - 7\psi - 12\varphi,$$

which we can study by using phase-plane analysis, and particular identify a limit cycle.

Pr.lc **Proposition 2.4.** *The ODE* (2.27) *admits a stable limit cycle on the* $\{\varphi, \psi\}$*-plane, which generates a periodic solution* $\varphi_*(s)$ *of* (2.26)*.*

Proof. Equations (2.27) from PME and p-Laplacian theory with limit cycles have been studied since 1980's; see [23] and extra references and related results in [2]. The limit cycle exists for all $p > d$. $\square$

As an immediate consequence, the gradient-dependent ODE (2.9) also admits an S-type solution $F^S(z)$ with infinitely many oscillations about the equilibrium f_0 near the origin. Since the behaviour (2.23), (2.25) violates the monotonicity, this S blow-up profile is not associated with the original M-A equation when $d = 2$ but does correspond to a possible solution when $d = 3$. The existence of F^S then follows by shooting from the origin using the 1D asymptotic bundle constructed using (2.23) and the representation (2.25) with the periodic solution $\varphi_*(s)$, i.e.,

ff11 (2.28) $$f(z) = f_0 + z^{\frac{2d}{d-1}}\varphi_*(s_0 + \ln z) + \ldots\,.$$

Here the shift s_0 is the only shooting parameter. Using this parameter, it follows from the oscillatory structure of the expansion (3.14) that there exists an s_0 such that $f = F^S$ has the power decay at infinity given by (2.17) with a positive constant C_0. We do not have clear evidence for the uniqueness of such $F^S(z) > 0$ and it appears from the numerical calculations that the associated self-similar blow-up profiles are unstable.

2.6. **Global blow-up for $p \in (1, d)$.** Many of the mathematical features of the analysis of the similarity blow-up structures in this case are the same as for $p > d$, though the evolution properties are quite different. It follows from (2.7) that $p \in (1, d)$ corresponds to the *global blow-up*, where

l6 (2.29) $$u_S(x,t) \to \infty \quad \text{as} \quad t \to T^- \quad \text{uniformly on bounded intervals in } \mathbb{R}_+.$$

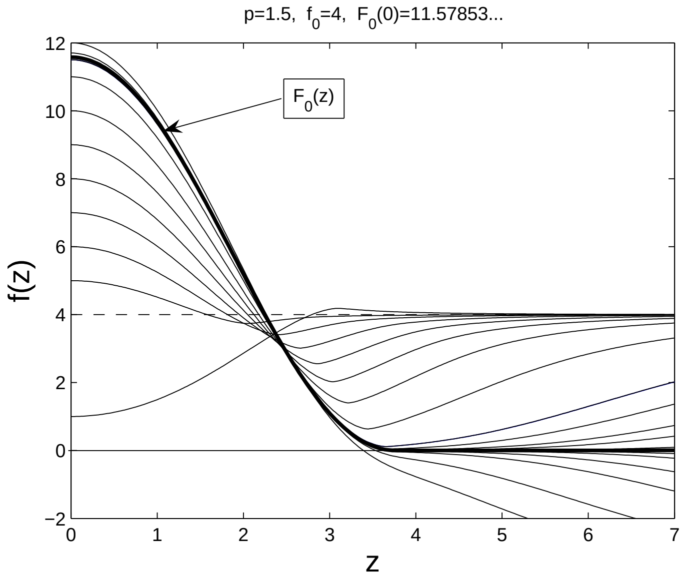


FIGURE 14. Shooting compactly supported similarity profile $F_0(z)$ (the bold line) for $p = 1.5$ in (2.9); $F_0(0) = 11.5785\ldots$. F4

As in [49, pp. 183-189], we obtain existence of a similarity profile. Since the PDE (2.1) is non-autonomous in space, uniqueness remains an open problem, since the geometric Sturmian approach to uniqueness (see [6] for main results and references) does not apply.

Pr.HS **Proposition 2.5.** *For any $p \in (1, 2)$, problem* (2.9) *admits a compactly supported solution $F_0(y)$.*

The numerical shooting construction of F_0 is shown in Figure 14 by the bold line for $d = 2$ and $p = \frac{3}{2}$.

SectNR 3. EXAMPLES OF BLOW-UP IN A NON-RADIAL GEOMETRY IN $\mathbb{R}^2$

It is immediate that the M-A equation (1.5) admits non-radially symmetric blow-up solutions which do not become more symmetric as $t \to T^-$. These solutions can be obtained directly from a non-symmetric transformation of the radially symmetric blow-up solutions described in the previous section and we describe below. However, it is unclear from the analysis whether these solutions are stable or not. In this section we consider these solutions and make some numerical computations to infer their stability.

3.1. **Regional blow-up for $p = d = 2$: the existence of non-radially symmetric blow-up solutions.** For $p = d = 2$, (1.5) is

M1NN (3.1)
$$u_t = -|D^2 u| + u^2 \quad \text{in} \quad \mathbb{R}^2 \times \mathbb{R}_+.$$

This partial differential equation admits self-similar blow-up solutions (2.7), i.e.,

R0 (3.2)
$$u_S(x, y, t) = \tfrac{1}{T-t}\, f(x, y).$$

Here the function $f(x,y)$ now satisfies the following "elliptic" M-A equation, with decay to zero at infinity:

R1
$$\mathbf{A}(f) \equiv -|D^2 f| + f^2 - f = 0 \quad \text{in} \quad \mathbb{R}^2. \tag{3.3}$$

The radial compactly supported solution $F_0(r)$ described in Proposition 2.1 also solves (3.3). The existence of non-radial smooth solutions of (3.3) now follows from an invariant group of scalings. Indeed, if $f(x,y)$ is a solution of (3.3) then so is the function

R66
$$f_a(x,y) = f\big(\tfrac{x}{a}, ay\big) \quad \text{for any} \;\; a > 0. \tag{3.4}$$

Indeed any rotation of this function is also a solution. Therefore, by taking the radial profile $F_0(y)$ supported in $[0, L_S]$ as a solution of (2.11), we can obtain the non-radially symmetric blow-up solution

R67
$$u_{\rm S}(x,y,t) = \tfrac{1}{T-t}\, F_0\big(\sqrt{(\tfrac{x}{a})^2 + (ay)^2}\,\big), \tag{3.5}$$

This blows up in the ellipsoidal *localization domain* given by

R68
$$E_a = \big\{(x,y): \quad \big(\tfrac{x}{a}\big)^2 + (ay)^2 < L_S^2 \quad (a \ne 1)\big\}. \tag{3.6}$$

Note that its area does not depend on a:

$$\operatorname{meas} E_a = \pi L_S^2 \quad \text{for any} \;\; a > 0.$$

Thus, M-A equations such as (3.1) do not support the (unconditional) symmetrization phenomena, found for many semilinear and quasilinear parabolic equations; see [27, p. 50] for references and basic results. In classic parabolic theory, results on symmetrization are well known and are connected with the moving plane method and Aleksandrov's Reflection Principle; see key references in [28, Ch. 9] and [27, p. 51]. However, all these approaches are based on the Maximum Principle that fails for M-A parabolic flows like (3.1). We conjecture that (3.3) does not admit other non-symmetric solutions but have no proof of this result.

3.2. **On the linearized operator.** Checking the stability properties of the non-radial solutions of (3.3), one can easily derive the linearized operator about the radial state $F_0(r)$

L1
$$\begin{aligned} \mathbf{A}'(F_0)Y = {} & \big[F_0'' \tfrac{y^2}{r^2} + F_0'\big(\tfrac{1}{r} - \tfrac{y^2}{r^3}\big)\big] Y_{xx} + \big[F_0'' \tfrac{x^2}{r^2} + F_0'\big(\tfrac{1}{r} - \tfrac{x^2}{r^3}\big)\big] Y_{yy} \\ & -2\big(F_0'' - \tfrac{1}{r}\, F_0'\big) \tfrac{xy}{r^2}\, Y_{xy} - Y. \end{aligned} \tag{3.7}$$

Moreover, since $\mathbf{A}$ is potential in L^2, this Frechet derivative is symmetric, so there is a hope to get a "proper" self-adjoint extension of the linear operator (3.7). Unfortunately, we should recall that (3.7) cannot be treated as elliptic in the domain of convexity of F_0. In addition, since $F_0(r)$ is compactly supported, the operator (3.7) have singular coefficients at $r = L_S$ and will be inevitably defined in a complicated domain with possibly singular weights, which makes rather obscure using such operators in studying the angular stability or unstability of the radial profile $F_0(r)$. In any case, it is convenient to note that, due to the symmetry (3.4), the stability is *neutral*, i.e.,

L2
$$\exists\, \hat\lambda = 0, \quad \text{with the non-radial eigenfunction} \quad \hat\psi_0 = \tfrac{\mathrm{d}}{\mathrm{d}a}\, f_a\big|_{a=1} = -\tfrac{x^2 - y^2}{r}\, F_0'(r) \tag{3.8}$$

(we naturally assume that the eigenfunction belongs to the domain of the self-adjoint extension). In other words, the stability/unstability will depend on an appropriate and delicate centre manifold behaviour. More precisely, if we perform the linearization $f(\tau) = F_0 + Y(\tau)$ of the corresponding to (3.3) non-stationary flow

L3
$$f_\tau = \mathbf{A}(f) \quad \Longrightarrow \quad Y_\tau = \mathbf{A}'(F_0)Y - |D^2 Y| + Y^2. \tag{3.9}$$

Then the corresponding formal centre subspace behaviour[1] by setting

$$Y(\tau) = a(\tau)\hat{\psi}_0 + w^\perp, \quad w^\perp \perp \hat{\psi}_0 \quad \big(\|w^\perp(\tau)\| \ll |a(\tau)| \;\text{ for }\; \tau \gg 1\big)$$

leads, on projection (as in classic theory, we have to assume at the moment a certain completeneee/closure of the orthonormal eigenfunction subset, which are very much questionable problems), to the equation

L5
$$\dot{a} = \gamma_0 a^2 \quad \text{for } \tau \gg 1, \text{ where } \quad \gamma_0 = \langle -|D^2 F_0| + F_0^2, \hat{\psi}_0 \rangle. \tag{3.10}$$

Therefore, for $\gamma_0 > 0$, stability/instrability of such flows depend on how the sign of γ_0 is associated with the sign of the expansion coefficient $a(\tau)$.

More careful checking by using equation (3.3) and the eigenfunction in (3.8) of changing sign shows that

L6
$$\gamma_0 = \langle F_0, \psi_0 \rangle = 0, \tag{3.11}$$

so that this centre subspace angular evolution according to (3.10) is formally absent at all. Of course, this is not surprising, since our functional setting assumes fixing the domain $\{r < L_S\}$ of definition of the functions involved, and this clearly prevents any angular evolution on the centre subspace that demands changing this domain.

In view of such a non-justifying formal linearized/invariant manifold analysis, we will next rely on also rather delicate numerical techniques to check angular stability of blow-up similarity profiles and solutions.

SectNR2

3.3. **Single point blow-up in non-radial geometry: similarity solutions for $p > d = 2$.** We can use a similar method to study non-radially symmetric single-point blow-up profiles. The self-similar solution (2.9),

uu0
$$u_S(x,y,t) = (T-t)^{-\frac{1}{p-1}} f(\xi,\eta), \quad \xi = x/(T-t)^\beta, \;\; \eta = y/(T-t)^\beta, \quad \beta = \tfrac{p-2}{4(p-1)}, \tag{3.12}$$

now leads to a more complicated elliptic M-A equation

uu1
$$-|D^2 f| - \beta \nabla f \cdot \zeta - \tfrac{1}{p-1} f + |f|^{p-1} f = 0 \quad \text{in} \quad \mathbb{R}^2 \quad \big(\zeta = (\xi,\eta)^T\big). \tag{3.13}$$

As before, the group of scalings (3.4) leaves equation (3.13) invariant. Therefore, taking the radial solution $F_0(z)$ from Proposition 2.2, we obtain the family

ff1
$$F_a(\xi,\eta) = F_0\Big(\sqrt{(\tfrac{\xi}{a})^2 + (a\eta)^2}\,\Big) \tag{3.14}$$

[1]Existence of a centre manifold by standard invariant manifold theory [44] is a very difficult open problem, which seems hopeless.

of non-radial solutions of (3.13) (together with all rotations of these). However, this set of solutions may not exhaust all the non-symmetric patterns. To see this, we consider the linearization (2.23) about the constant equilibrium f_0 which leads to a nonlinear M-A elliptic problem:

uu2
$$-|D^2 f| - \beta \nabla f \cdot y + f \equiv -\big[f_{\xi\xi} f_{\eta\eta} - (f_{\xi\eta})^2\big] - \beta(f_\xi \xi + f_\eta \eta) + f = 0 \quad \text{in} \quad \mathbb{R}^2. \tag{3.15}$$

This fully nonlinear PDE does not admit separation of variables. We can see from (3.13) that if $f(z) \to 0$ as $z \to \infty$ sufficiently fast, the far-field behaviour is governed by the linear terms,

uu3
$$-\beta(f_\xi \xi + f_\eta \eta) - \tfrac{1}{p-1}\, f + ... = 0. \tag{3.16}$$

Solving this gives the following typical asymptotics (cf. (2.17)):

uu4
$$f(z) = C\big(\tfrac{z}{|z|}\big)|z|^{-\frac{4}{p-2}} + ... \quad \big(z = (\xi, \eta)^T\big), \tag{3.17}$$

where $C(\mu) > 0$ is an arbitrary smooth function on the unit circle $\{|\mu| = 1\}$ in $\mathbb{R}^2$. The constant function $C_0(\mu) \equiv C_0 > 0$ gives the radially symmetric similarity profile as in Proposition 2.2. Furthermore the π-periodic function $C(\mu)$ given in the polar angle φ by

$$C_1(\varphi) = \tfrac{1}{2}\big(\tfrac{1}{a^2} + a^2\big) + \tfrac{1}{2}\big(\tfrac{1}{a^2} - a^2\big)\cos 2\varphi$$

generates the ellipsoidal solutions (3.14). We conjecture that other solutions are possible with $C_l(\varphi)$ having smaller periods $\frac{2\pi}{3}$, $\frac{\pi}{2}$, ... However, at present, the existence of such is unknown.

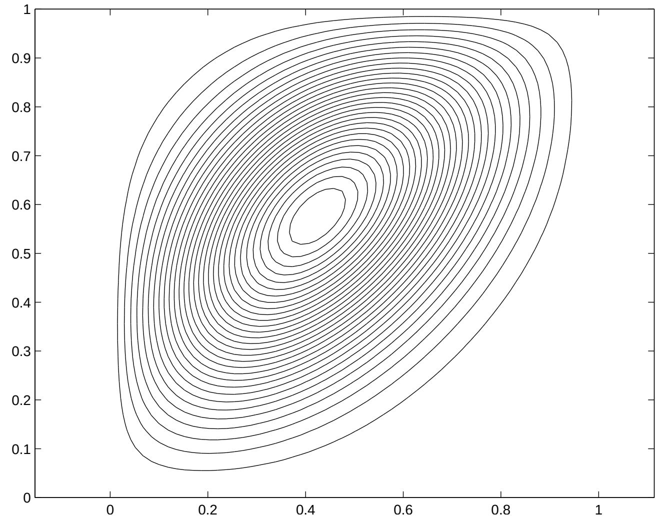
3.4. **Numerical computations of the non-radially symmetric time-dependent solutions.** We now consider a numerical computation of the blow-up profiles when $\Omega = [0,1] \times [0,1]$ is the unit square (so that $d = 2$), and we took $p = 3$. It is convenient in this calculation to impose Dirichlet boundary conditions. The PDE is solved by using a semi-discrete method for which the square is divided into a uniform grid (typically a 100×100 mesh) and the spatial Monge–Ampère operator discretised in space by using a second-order 9 point stencil. The resulting time dependent ODE system is very nonlinear and an implicit solver is very inefficient. Accordingly it was solved using an explicit, adaptive Runge–Kutta method with a small tolerance. The discretisation in space leads to certain chequer-board type instabilities[2] and these are filtered out at each stage by using a suitable averaging spatial filter applied to the ODE system. For initial data satisfying the Dirichlet condition we took

$$u(0,t) = 10^4 \mathrm{e}^{-4r^2} \sin(\pi x)\sin(\pi y), \quad \text{where} \quad r^2 = a^2(\hat{x} - 0.4)^2 + \tfrac{1}{a^2}\,(\hat{y} - 0.6)^2,$$

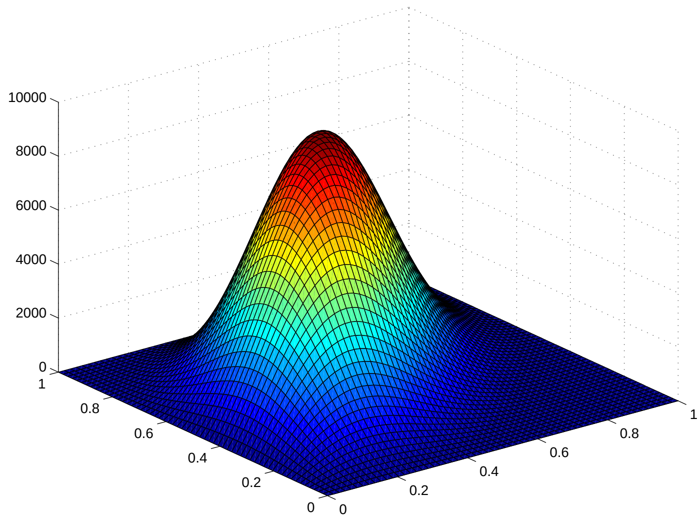
for which $a = 2$ and $\hat{x}$, $\hat{y}$ were a set of coordinates rotated at an angle of $\frac{\pi}{4}$. This data were chosen to have an elliptical set of contours close to its peak.

This system led to blow-up of the discrete system in a computed finite time $T \approx 1.17 \times 10^{-4}$. (Note that the computed blow-up time decreases when the spatial mesh is refined). In Figure 15 (a,b) we present the initial solution and its contours and in Figure 16 (a,b) a solution much closer to the blow-up time (so that it is approximately 10 times larger than the initial profile). Note that the elliptical form of the contours has been preserved during the evolution giving some evidence for the stability of the elliptical blow-up patterns. We also give in Figure 17 a plot of the solution a slightly later time. Note in this case evidence for an instability close

[2]We recall that the M-A flow under consideration is supposed to have some natural instabilities in the areas, where the concavity of solutions is violated; these questions will be discussed.



(a) Contours



(b) Profile

Fc1

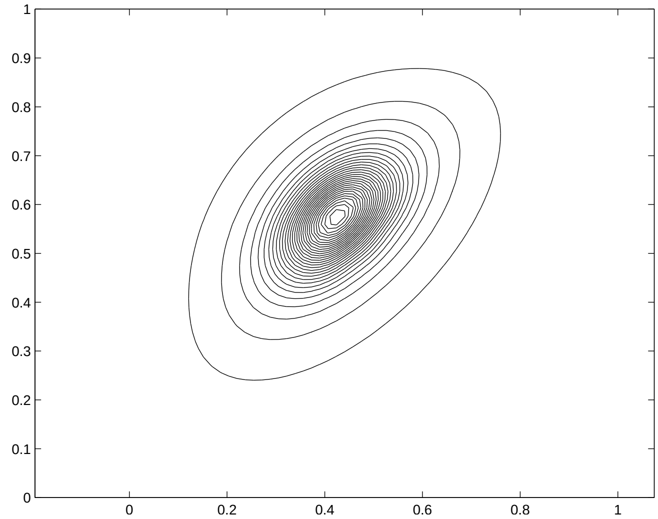
FIGURE 15. Initial solution profile and contours for $d = 2$, $p = 4$.

to a point where the solution profile loses convexity. It is not clear at present whether this is a numerical or a true instability. Certainly all of the numerical methods used had extreme difficulty in computing a significant way into the blow-up evolution.

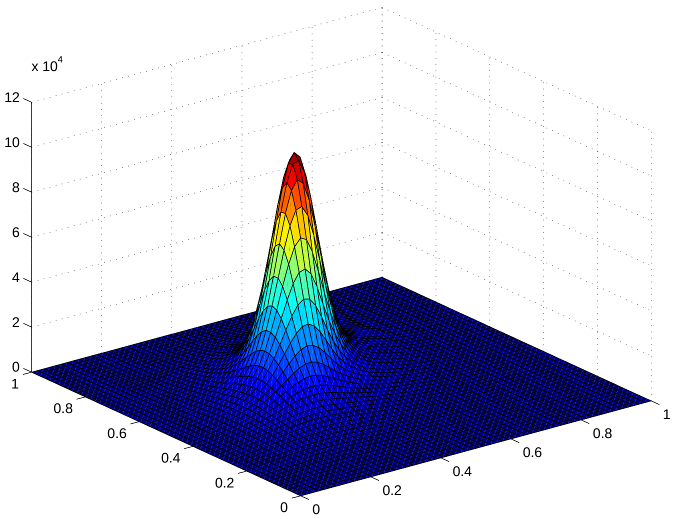
4. EXISTENCE, UNIQUENESS, AND REGULARITY OF THE SOLUTIONS: DISCUSSION AND OPEN PROBLEMS

SectEx

The previous sections have studied the existence and blow-up of the radially symmetric solutions, and the last calculation gives some indication of an instability in the elliptical blow-up profile. The latter observation leads us naturally into the more general question of the local



(a) Contours



(b) Profile

Fc2 FIGURE 16. Profile and contours of the solution for $d = 2, p = 4$ closer to the blow-up time.

existence and regularity of the solutions of the forced M-A equation. In this section we will briefly address these question by (mainly) considering a regularised form of the M-A equation. We also review some existing results on some of these questions for equations such as (3.1) and related PDEs. As we are now interested mainly in local existence and stability properties of the M-A equation, it is appropriate to ignore the quadratic reaction term u^2, and concentrate on the properties of the fully nonlinear M-A operator. Accordingly we will consider the Cauchy problem

R4 (4.1) $$u_t = -|D^2u| \equiv -u_{xx}u_{yy} + (u_{xy})^2 \quad \text{in} \quad \mathbb{R}^2 \times \mathbb{R}_+, \quad u(x,0) = u_0(x) \geq 0 \quad \text{in} \quad \mathbb{R}^2,$$

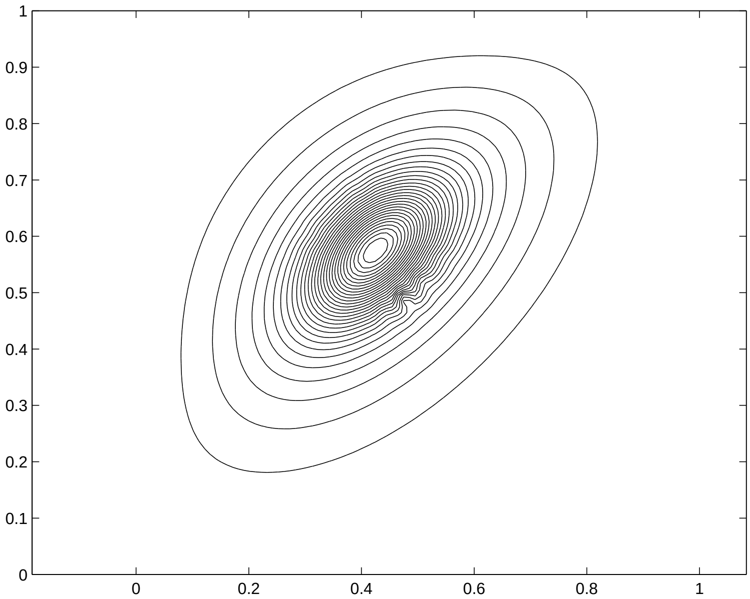


FIGURE 17. The solution at a slightly later time showing a possible instability at a point where the profile loses concavity. Fc3

with sufficiently smooth compactly supported initial data satisfying some extra necessary conditions (for example having "dominated concavity"). The questions of local solubility and regularity in M-A theory have still not been developed in detail. [3]. We note that the classical theory of this operator for convex (concave) solutions; see [28, 35, 30], etc. cannot be applied to the more general solution profiles considered in the last section. Furthermore, even for convex solutions,the local regularity theory for the basic M-A equation

$$\det D^2 u = f(x) > 0 \quad \text{in a convex bounded domain} \quad \Omega \subset \mathbb{R}^N$$

is rather involved with a number of open questions; see [33] as a guide. On the other hand, even for 2D stationary M-A equations of changing convexity-concavity (see (1.4)), there are counterexamples concerning local solvability and regularity, [38]. The study of finite regularity solutions of the simplest degenerate (at 0) M-A equation in $\mathbb{R}^2$ with radial homogeneous $f(x)$,

DDDD1
$$\det D^2 u = |x|^\alpha \quad \text{in} \quad B_1 \tag{4.2}$$

has some surprises [12]; e.g. for $\alpha > 0$, there exist a radial and a non-radial $C^{2,\delta}$ solutions, whilst for $\alpha \in (-2, 0)$ only the radial solution exists (this is about a delicate study of a single point blow-up singularity for (4.2)); see also [48] for regularity of radial solutions for (4.2) with the right-hand side $f(\frac{1}{2}\,|x|^2, u, \frac{1}{2}\,|\nabla u|^2)$. Equation (4.2) has the origin in Weyl's classic problem (1916).

[3] "Yet, it is remarkable that the basic question of whether there exist any examples of local nonsolvability, has remained open for this well-studied class of equations", [38, p. 665].

We do not plan to discuss such delicate questions seriously here (especially for our problems which have such a strong degeneracy and even a changing of type), and we will restrict our discussion to the first auxiliary aspects of such singularity phenomena for (4.1).

4.1. **Source-type similarity solutions.** The easiest solutions to construct are the similarity solutions, for the radial equation

81 (4.3)
$$u_t = -\tfrac{1}{r}\, u_r u_{rr}, \quad \text{so that}$$

82 (4.4)
$$u_*(r,t) = t^{-\frac{1}{3}} F(y), \quad y = \tfrac{r}{t^{1/6}}, \quad \text{where} \quad F(y) = \tfrac{1}{48}\left(d^2 - y^2\right)_+^2, \;\; d > 0.$$

4.2. **Scaling group: non-symmetric solutions.** As we have seen before, (4.1) admits a variety of non-radial solutions. Indeed, the equation is invariant under the following group of scaling transformations:

sc1 (4.5)
$$u(x,y,t) \mapsto \tfrac{a^2 b^2}{c}\, u\big(\tfrac{x}{a}, \tfrac{y}{b}, \tfrac{t}{c}\big), \quad a,\, b,\, c \ne 0.$$

Therefore, (4.1) does not support the symmetrization phenomena that are typical for many nonlinear parabolic PDEs.

Using also the time-translation, we obtain from (4.4) the following 4-parametric family of exact solutions:

zz1 (4.6)
$$u_*(x,y,t) = \tfrac{a^2 b^2}{c^{2/3}}\, (T+t)^{-\frac{1}{3}} \big[d^2 - c^{\frac{1}{3}} (T+t)^{-\frac{1}{3}} \big(\tfrac{x^2}{a^2} + \tfrac{y^2}{b^2}\big)\big]_+^2.$$

4.3. **No order-preserving semigroup in non-radial geometry.** We recall that in the radial geometry, the semigroup for the parabolic equation (4.3) is obviously order-preserving. It turns out that in the non-radial setting, this is not the case.

Pr.No **Proposition 4.1.** *In general, sufficiently smooth solutions of* (4.1) *do not obey a comparison principle.*

Proof. We take two exact solutions from (4.6): $u(x,y,t)$ with $a=b=c=T=1$ and the general solution $u_*(x,y,t)$, and show that the usual comparison is violated in this family of non-radial solutions. Comparing positions of the interfaces at the x- and y-axes and the maximum values at the origin yields for initial data at $t=0$ that

y1 (4.7)
$$u_*(x,y,0) \le u(x,y,1) \quad \text{if} \quad \tfrac{ad}{c^{1/6}} T^{\frac{1}{6}} < 1, \;\; \tfrac{bd}{c^{1/6}} T^{\frac{1}{6}} < 1, \;\; \tfrac{a^2 b^2 d^4}{c^{2/3}} T^{-\frac{1}{3}} < 1.$$

On the other hand, the comparison is violated for large $t \gg 1$ if

y2 (4.8)
$$\tfrac{ad}{c^{1/6}}\, (T+t)^{\frac{1}{6}} > t^{\frac{1}{6}}, \quad \text{i.e.,} \quad \tfrac{ad}{c^{1/6}} > 1.$$

It is easy to see that the system of four algebraic inequalities in (4.7) and (4.8) has, e.g., the following solution:

$$a = 1, \quad b = \tfrac{1}{8}, \quad c = 2, \quad d = 2, \quad T \in \big(\tfrac{1}{2^8}, \tfrac{1}{2^5}\big). \quad \square$$

4.4. **Towards well-posedness.** We present now some very formal speculations, which nevertheless hint that the non-fully concave M-A flow (4.1) may be better well-posed than might be expected. Actually, exactly this behaviour was observed in a number of numerical experiments discussed above. Assume that, due to an essentially deformed spatial shape of the solution (say, by means of choosing special "ellipsoidal" initial data), we consider the *unstable area* that is characterized as follows: $|u_{xy}|^2 \ll |u_{xx}u_{yy}|$ and, e.g., $u_{yy} \ge c_0 > 0$, i.e., the flow

fl1 (4.9)
$$u_t = -u_{yy}u_{xx} + ... \,, \quad u(x,y,t) > 0 \quad \big(\text{cf.} \quad u_t = -c_0 u_{xx} + ...\big)$$

is backward parabolic with respect to the spatial variable x. Then, let us assume that the positive solution $u(x,y,t)$ is going to produce a blow-up singularity in finite time as $t \to T^-$, and, say, let it be a Dirac delta function of a positive measure. . Of course, we do not mean precisely that in this fully nonlinear equation, but can expect that a certain such tendency as t moves towards T can be observed, as the linear PDE in the braces in (4.9) suggests. Hence, if such a tendency of approaching a $\sim \delta(x-x_0)$ in x is observed, then, obviously, at this unstable subset

fl2 (4.10)
$$u_{xx} \le -c_0 < 0 \quad \Longrightarrow \quad u_t = (-u_{xx})u_{yy} + ... \ \text{becomes well-posed parabolic in } y$$

(here we again assume that u_{xy} does not play a role at this stage). In other words, such a simple localized pointwise singularity $\sim \delta(\mathbf{x} - \mathbf{x}_0)$ in both variables x and y is unlikely. This means that the PDE (4.1) can exhibit a certain "self-regularization" even in the case of not fully concave data. We are not aware of any rigorous mathematical justification of such a phenomenon, and will continue to discuss this subject below using other arguments.

4.5. **ε-regularization: on formal extended semigroup.** We now propose to construct a unique *proper* solution of (4.1) as a limit, as $\varepsilon \to 0$, of a family of smooth regularized solutions $\{u_\varepsilon\}$ of the regularized fourth-order uniformly parabolic equation

R5 (4.11)
$$u_\varepsilon : \quad u_t = -\varepsilon \Delta^2 u - |D^2 u| \quad \text{in} \quad \mathbb{R}^2 \times \mathbb{R}_+,$$

with the same initial data u_0 as the original problem. Global and even local solvability of the Cauchy Problem for (4.11) is a difficult open problem. Here, $\mathbf{A}(u) = |D^2u|$ is a potential operator in $L^2(\mathbb{R}^2)$ with the inner product denoted by $\langle \cdot, \cdot \rangle$. The potential is given by

$$\Phi(u) = \int_0^1 \langle u, \mathbf{A}(\rho u) \rangle \, \mathrm{d}\rho = \tfrac{1}{3} \int u |D^2 u|.$$

Hence, equation (4.11) admits two integral identities obtained by multiplication by u and u_t,

R6 (4.12)
$$\begin{gathered} \tfrac{1}{2} \tfrac{\mathrm{d}}{\mathrm{d}t} \int u^2 = -\varepsilon \int (\Delta u)^2 - \int u |D^2 u|, \\ \int_0^T \int (u_t)^2 = E(u(T)) - E(u_0), \quad E(u) = -\tfrac{\varepsilon}{2} \int (\Delta u)^2 - \tfrac{1}{3} \int u |D^2 u|. \end{gathered}$$

In particular, writing the last identity as

R61 (4.13)
$$\int_0^T \int (u_t)^2 + \tfrac{\varepsilon}{2} \int (\Delta u)^2 + \tfrac{1}{3} \int u |D^2 u| = -E(u_0)$$

we see that for a uniform bound on $u_\varepsilon \ge 0$ it is necessary to have the following "dominated concavity property": for all $t \ge 0$,

DC1 (4.14) $$|D^2 u_\varepsilon| > 0 \quad \text{in domains, where } u_\varepsilon \ge 0 \text{ is not small.}$$

To support this, as a formal illustration, we prove an "opposite nonexistence" result:

Pr.Non1 **Proposition 4.2.** *Assume that, for smooth enough* u_0,

DC2 (4.15) $$\textstyle\int u_0 |D^2 u_0| < 0.$$

Then $u_\varepsilon(\cdot, t)$ *for* $\varepsilon \ll 1$ *is not bounded in* L^2 *for large* $t > 0$.

Recall that, for the problem (4.11), the conservation of the mass (i.e., an L^1 uniform estimate) is available; see Section **??**. Note that (4.15) is not true in the radial case for decreasing $u_0(r) \not\equiv 0$, since by (4.3),

$$\textstyle\int u_0 |D^2 u_0| = \int r u_0 \, \frac{1}{r} \, u_0' u_0'' = -\frac{1}{2} \int (u_0')^3 > 0.$$

Proof. Estimating from the second identity in (4.12)

$$\textstyle\int u |D^2 u| \le -\frac{3\varepsilon}{2} \int (\Delta u)^2 + 3 \big[\frac{\varepsilon}{2} \int (\Delta u_0)^2 + \frac{1}{3} \int u_0 |D^2 u_0| \big]$$

and substituting into the first one yields

DC4 (4.16) $$\textstyle\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int u^2 \ge \frac{\varepsilon}{2} \int (\Delta u)^2 - \frac{3\varepsilon}{2} \int (\Delta u_0)^2 - \int u_0 |D^2 u_0| \ge -\frac{1}{2} \int u_0 |D^2 u_0| > 0$$

for sufficiently small $\varepsilon > 0$. Hence, under the hypothesis (4.15),

DC5 (4.17) $$\textstyle\int u^2(t) \ge \frac{1}{2} \big| \int u_0 |D^2 u_0| \big| \, t \to +\infty \quad \text{as} \quad t \to \infty. \quad \square$$

It seems that the divergence (4.17) of $\{u_\varepsilon\}$ actually means that the approximated solution $u(x,t)$ is not global and must blow-up in finite time. This would imply global nonexistence of solution of (4.1) if (4.15) violates the dominant concavity hypothesis (4.14) at $t = 0$. However, we do not know whether reasonable data satisfying (4.15) actually exist. For instance, the standard profiles in (4.6) do not obey (4.15) (since they correspond to uniformly bounded L^2-solutions of (4.1)).

In general, identities (4.12) cannot provide us with estimates that are sufficient for passing to the limit as $\varepsilon \to 0$, so extra difficult analysis is necessary. The main difficulty is that the Hessian potential $\Phi(u)$ is not definite in the present functional setting and the operator $\mathbf{A}(u)$ is not coercive in the class of not fully concave functions. To avoid such a difficulty, another uniformly parabolic ε-regularization may be considered useful such as, e.g.,

R10 (4.18) $$u_\varepsilon: \quad u_t = -\varepsilon \Delta[(1+u^2)\Delta u] - |D^2 u|.$$

Unfortunately, the first operator is not potential in L^2 so deriving integral estimates become more tricky. On the other hand, using degenerate higher-order p-Laplacian operators such as

R11 (4.19) $$u_\varepsilon: \quad u_t = -\varepsilon \Delta(|\Delta u|^p \Delta u) - |D^2 u|$$

can be more efficient for $p > 1$. Here both operators are potential in $L^2(\mathbb{R}^2)$, and moreover, the p-Laplacian one is monotone, which can simplify derivation of necessary estimates; see Lions' classic book [43, Ch. 2]. Nevertheless, using various ε-regularizations as in (4.11), (4.18), or (4.19) does not neglect the necessity of the difficult study of boundary layers as $\varepsilon \to 0$.

Thus, according to extended semigroup theory (see [22, Ch. 7]), the *proper* solution of the Cauchy problem (4.1) is given by the limit

R7
$$u(x,t) = \lim_{\varepsilon \to 0} u_\varepsilon(x,t). \tag{4.20}$$

In general, as we have mentioned, existence (and hence uniqueness) of such limits assumes delicate studied of ε-boundary layers which can occur in the singular limit $\varepsilon \to 0$. In particular, the uniqueness of such a proper solution would be guaranteed by the fact that the regularized sequence $\{u_\varepsilon\}$ does not exhibit $O(1)$ oscillations as $\varepsilon \to 0$, so (4.20) does not have different particular limits along different subsequences $\{\varepsilon_k\} \to 0$. Another important aspect is to show that the proper solution does not depend on the character of the ε-regularizations applied. Such a strong uniqueness result is known for the second-order parabolic problems [22, Ch. 6,7] and is based on the MP. For higher-order PDEs, all such uniqueness problems are entirely open excluding, possibly, some very special kind of equations. We hope that the potential properties of the Hessian operator $|D^2u|$ and hence identities like (4.12) can help for passing to the limit in (4.11) or other regularized PDEs and to avoid studying in full generality difficult singular boundary layers. These questions remain open.

4.6. **Riemann's problems: a unique solution via a formal asymptotic series.** We now continue to study the passage to the limit $\varepsilon \to 0$ of the solutions of the regularized problem (4.11). We assume that the origin 0 belongs to the boundary of $\operatorname{supp} u_0$ and

u1
$$u_0(x) = O(\|x\|^4) \quad \text{as} \quad x \to 0 \quad \big(X = (x,y)^T\big). \tag{4.21}$$

This class of data specifies a form of Riemann problem (with a given type of singular transition to 0). We next perform the following scaling in (4.11):

u2
$$u(x,t) = \varepsilon v_\varepsilon(\zeta,t), \quad \zeta = x/\varepsilon^{\frac{1}{4}}, \tag{4.22}$$

so that $v_\varepsilon(\zeta,t)$ solves an ε-independent equation with ε-dependent data,

u3
$$v_\varepsilon : \quad v_t = \mathbf{A}_0(v) \equiv -\Delta^2 w - |D^2 w|, \quad v_{0\varepsilon}(\zeta) = \tfrac{1}{\varepsilon}\, u_0(\zeta \varepsilon^{\frac{1}{4}}). \tag{4.23}$$

According to (4.21), we assume that there exists a finite limit on any compact subset

u4
$$v_{0\varepsilon}(\zeta) \to v_0(\zeta) \quad \text{as} \quad \varepsilon \to 0, \tag{4.24}$$

where, without loss of generality, by $v_0(\zeta)$ we mean a fourth-degree polynomial.

As usual in asymptotic expansion theory (see e.g. Il'in [34]), the crucial is the first nonlinear step, where we find the first approximation $V_0(\zeta,t)$ satisfying the uniformly parabolic PDE

u5
$$V_0 : \quad V_t = \mathbf{A}_0(V), \quad V(\zeta,0) = v_0(\zeta). \tag{4.25}$$

It can be shown by classic parabolic theory [16, 20] that the fourth-degree growth of $v_0(\zeta)$ as $\zeta \to \infty$ guarantees at least local existence and uniqueness of V_0.

As the next step, we define the second term of approximation, $v = V_0 + w_0$, where w_0 solves the linearized problem

V13
$$w_0 : \quad w_t = \mathbf{A}_0'(V_0)w, \quad w(0,y) = v_{0\varepsilon}(y) - v_0(y), \quad \text{etc.}, \tag{4.26}$$

again checking that this linear parabolic problem is well-posed by the classical theory of [16, 20].

Eventually, this means that we formally express the solution via the asymptotic series

V14
$$v_\varepsilon(y,t) = V_0(y,t) + \textstyle\sum_{j\ge 0} w_j(y,t;\varepsilon), \tag{4.27}$$

where each term w_k for $k \ge 1$, is obtained by linearization on the previous member, by setting

$$V_k = V_{k-1} + w_k \equiv V_0 + \textstyle\sum_{j\le k} w_j,$$

which gives for w_k a non-homogeneous linear parabolic problem

V15
$$w_k: \quad w_t = \mathbf{A}_0'(V_{k-1})w - \big[(V_{k-1})_t - \mathbf{A}_0(V_{k-1})\big], \quad w(0,y)=0, \tag{4.28}$$

with similar assumptions on the well-posedness.

Thus, (4.27) gives a unique formal representation of the solution. In asymptotic expansion theory, the convergence of such series and passing to the limit $\varepsilon \to 0$ are often extremely difficult even for lower-order PDEs, where the rate of convergence or asymptotics are also hardly understandable. For instance, the asymptotic expansion for the classic Burgers' equation

Bur1
$$u_t + uu_x = \varepsilon u_{xx} \tag{4.29}$$

contains $\ln \varepsilon$ terms (close to shock waves), and a technically hard proof uses multiple reductions of (4.29) to the linear heat equation, which is illusive for our M-A PDEs; see [34]. For practical reasons, it is important that, as an intrinsic feature of asymptotic series, each term in (4.27) (including the first and the simplest one w_0) reflects the actual rate of convergence of u_ε given by (4.20) as $\varepsilon \to 0$ to the proper solution (4.20).

Sect4ord

5. Examples of blow-up for a fourth-order M-A equation with $-|D^4u|$

We now conclude this paper by extending the earlier results and methods to radially symmetric problems with a higher order operator, looking at blow-up solutions in these cases.

5.1. **On derivation of the higher-order radial M-A model.**

To do this we introduce radial models related to the fourth-order M-A equation

ma4
$$u_t = -|D^4u| + u|u|^{p-1}, \quad p>1, \quad \text{in} \quad \mathbb{R}^2 \times \mathbb{R}, \tag{5.1}$$

with the catalecticant determinant $|D^4u|$ given by

F41
$$\det D^4u \equiv \det \begin{bmatrix} u_{xxxx} & u_{xxxy} & u_{xxyy} \\ u_{xxxy} & u_{xxyy} & u_{xyyy} \\ u_{xxyy} & u_{xyyy} & u_{yyyy} \end{bmatrix}, \tag{5.2}$$

This operator plays an important role in the theory of quartic forms. For instance, each such form in two variables can be expressed via a sum of three fourth powers of linear forms and via two powers, provided that $\det D^4u = 0$; see [32]. It then follows from (5.2) that

D4
$$\begin{aligned} |D^4u| = {} & u_{xxxx}u_{yyyy}u_{xxyy} + 2u_{xxxy}u_{xyyy}u_{xxyy} \\ & - (u_{xxyy})^2 - (u_{xyyy})^2u_{xxxx} - (u_{xxxy})^2u_{yyyy}. \end{aligned} \tag{5.3}$$

In particular, for radial functions $u = u(r)$, we have

DD1
$$
\begin{aligned}
u_{xxxx} &= \tfrac{x^4}{r^4}\, u^{(4)} + \tfrac{6x^2y^2}{r^5}\, u''' + \tfrac{3(y^4-4x^2y^2)}{r^6}\, u'' + \tfrac{3(-y^4+4x^2y^2)}{r^7}\, u', \\
u_{yyyy} &= \tfrac{y^4}{r^4}\, u^{(4)} + \tfrac{6x^2y^2}{r^5}\, u''' + \tfrac{3(x^4-4x^2y^2)}{r^6}\, u'' + \tfrac{3(-x^4+4x^2y^2)}{r^7}\, u', \\
u_{xxyy} &= \tfrac{x^2y^2}{r^4}\, u^{(4)} + \tfrac{x^4+y^4-4x^2y^2}{r^5}\, u''' + \tfrac{-2(x^4+y^4)+11x^2y^2}{r^6}\, u'' + \tfrac{2(x^4+y^4)-11x^2y^2}{r^7}\, u', \\
u_{xxxy} &= \tfrac{x^3y}{r^4}\, u^{(4)} + \tfrac{3(xy^3-x^3y)}{r^5}\, u''' + \tfrac{3(2x^3y-3xy^3)}{r^6}\, u'' + \tfrac{3(-2x^3y+3xy^3)}{r^7}\, u', \\
u_{xyyy} &= \tfrac{xy^3}{r^4}\, u^{(4)} + \tfrac{3(x^3y-xy^3)}{r^5}\, u''' + \tfrac{3(2xy^3-3x^3y)}{r^6}\, u'' + \tfrac{3(-2xy^3+3x^3y)}{r^7}\, u'.
\end{aligned}
\tag{5.4}
$$

Balancing and mutual cancellation of the most singular terms of (5.4) as $r \to 0$ leads to the following model radial parabolic equation associated with the M-A flow (5.1):

ma50
$$u_t = -\tfrac{1}{r^2}\,(u_{rrr})^2 u_{rrrr} + u|u|^{p-1} \quad \text{in} \quad \mathbb{R}_+ \times \mathbb{R}_+. \tag{5.5}$$

Looking at it as a parabolic equation, we pose at the origin the symmetry (regularity) conditions

sr10
$$u_r(0,t) = u_{rrr}(0,t) = 0. \tag{5.6}$$

Obviously, (5.5) is uniformly parabolic with smooth (analytic) solutions in any domain of non-degeneracy $\{u_{rrr} \ne 0\}$. In particular, checking the regularity of the operator in (5.5) and passing to the limit $r \to 0$ yields

rr10
$$-\tfrac{1}{r^2}\,(u_{rrr})^2 u_{rrrr} \to -(u_{rrrr})^3, \tag{5.7}$$

so that, at the origin, the differential operator is non-degenerate and regular if $u_{rrrr} \ne 0$. Thus, regardless the degeneracy of the equation (5.5), this radial version of fourth-order M-A flows is well-posed, at least, locally in time.

As before, it is clear, on considering the relative magnitude of the terms, that the value of $p = 3$ is critical and separates regional blow-up when $p = 3$ from single point blow-up when $p > 3$. In the case of $p = 3$ we can, as before, seek blow-up similarity solutions of (5.5) of the form

rr30
$$u_{\rm S}(r,t) = \tfrac{1}{\sqrt{T-t}}\, f(r). \tag{5.8}$$

These functions $f(r)$ then satisfy the ODE

rr40
$$\mathbf{A}(f) \equiv -\tfrac{1}{r^2}\,(f''')^2 f^{(4)} + f^3 - \tfrac{1}{2}\, f = 0 \quad \text{for} \quad r > 0, \tag{5.9}$$

with the symmetry conditions generated by (5.6) at the origin,

symm1L
$$f'(0) = f'''(0) = 0. \tag{5.10}$$

Indeed, (5.9) is a difficult fourth-order ODE to analyze, with non-monotone, non-autonomous, and non-potential operators. Any (at least) 3D phase-plane analysis or shooting arguments are rather difficult. However, it is amenable to a numerical study using the same methods as described in Section 2. In Figure 18, we present the results of such a calculations showing the existence of a compactly supported solution of (5.9) associated with a regional blow-up profile. We did not detect any other P-type profiles that have more than one oscillation about the constant equilibrium

$$f_0 = \tfrac{1}{\sqrt{2}}.$$

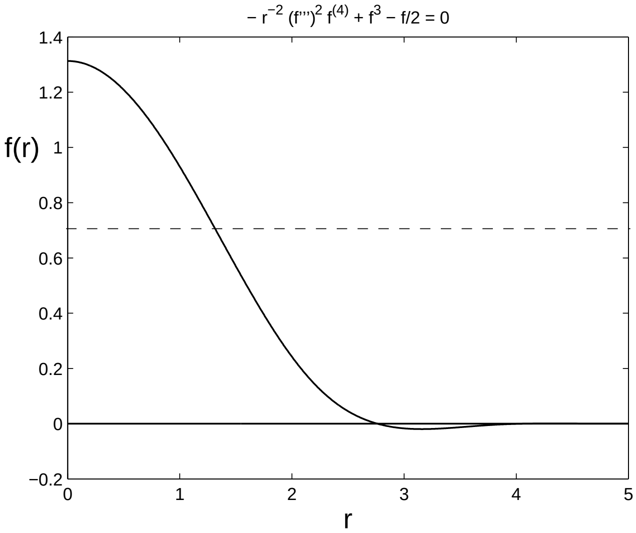


FIGURE 18. A unique compactly supported nonnegative solution of the ODE (5.9). F4S

As before, the stability of the similarity blow-up (5.8) can be studied in terms of the rescaled solution

w1 (5.11)
$$w(x,\tau) = \sqrt{T-t}\; u(r,t), \quad \tau = -\ln(T-t) \to +\infty,$$

where w solves the rescaled parabolic equation with the same elliptic operator

w2 (5.12)
$$w_\tau = \mathbf{A}(w) \equiv -\tfrac{1}{r^2}\,(w_{rrr})^2 w_{rrrr} + w^3 - \tfrac{1}{2}\,w \quad \text{in} \quad \mathbb{R}_+ \times \mathbb{R}_+.$$

Since $\mathbf{A}$ is not potential in any suitable metric, so (5.12) is not a gradient system, passage to the limit as $\tau \to +\infty$ to show stabilization to the stationary profile $f(r)$ represents a difficult open problem. Note that the main operator $-\frac{1}{r^2}\,(w_{rrr})^2 w_{rrrr}$ is gradient and admits multiplication by $r^2\, w_{rr\tau}$ in $L^2(\mathbb{R})$,

$$-\textstyle\int (w''')^2 w'''' w''_\tau = \frac{1}{12}\,\frac{\mathrm{d}}{\mathrm{d}\tau}\,\int (w''')^4 \quad ('= D_r),$$

but w^3 (and also w) does not.

5.2. $p=3$**: on oscillatory structure close to interfaces.** In addition, Figure 18 shows that, locally, close to the finite interface point $r = r_0$, sufficiently smooth solutions of (5.9) are oscillatory. This kind of non-standard behaviour of solutions of the Cauchy problem deserves a more detailed analysis. To describe this, we introduce an extra scaling by setting

o10 (5.13)
$$f(r) = (r_0 - r)^\gamma \varphi(s), \quad s = \ln(r_0 - r), \quad \text{where} \quad \gamma = 3.$$

Substituting (5.13) into (5.9) and neglecting the higher-degree term f^3 for $r \approx r_0^-$, we obtain the following equation for the *oscillatory component* $\varphi(s)$:

o20 (5.14)
$$(f''')^2 f^{(4)} = -\lambda_0 f \quad \Longrightarrow \quad (P_3(\varphi))^2 P_4(\varphi) = -\lambda_0 \varphi, \quad \lambda_0 = \tfrac{1}{2}\, r_0^2,$$

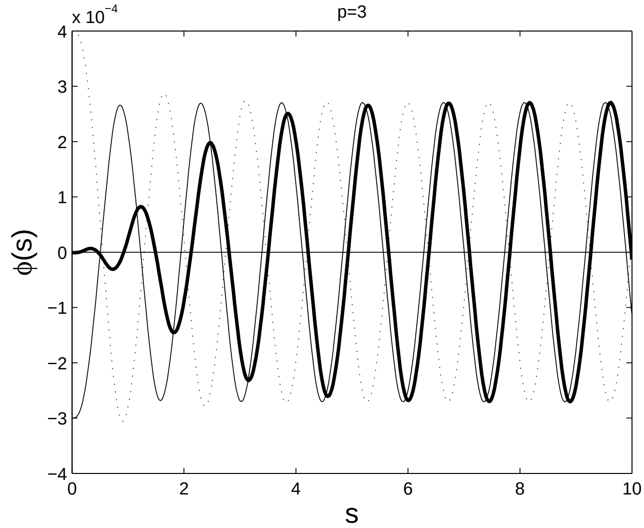


FIGURE 19. Stabilization to a stable periodic orbit $\varphi_*(s)$ for the ODE (5.14), $\lambda_0 = 1$. Fos

where P_k are linear differential polynomials obtained by the recursion procedure (see [25, p. 140])

$$
\begin{gathered}
P_1(\varphi) = \varphi' + \gamma\varphi, \quad P_2(\varphi) = \varphi'' + (2\gamma - 1)\varphi' + \gamma(\gamma - 1)\varphi, \\
P_3(\varphi) = \varphi''' + 3(\gamma - 1)\varphi'' + (3\gamma^2 - 6\gamma + 2)\varphi' + \gamma(\gamma - 1)(\gamma - 2)\varphi, \\
P_4(\varphi) = \varphi^{(4)} + 2(2\gamma - 3)\varphi''' + (6\gamma^2 - 18\gamma + 11)\varphi'' \\
+ 2(2\gamma^3 - 9\gamma^2 + 11\gamma - 3)\varphi' + \gamma(\gamma - 1)(\gamma - 2)(\gamma - 3)\varphi.
\end{gathered}
$$

In Figure 19, we show the typical behaviour of solutions of the second ODE in (5.14) demonstrating a fast stabilization to the unique stable periodic solution. According to (5.13), this periodic orbit $\varphi_*(s)$ describes the generic character of oscillations of solutions of the ODE (5.9). Periodic solutions for *semilinear* ODEs of the type (5.14) are already known for the third- [17, § 7] and fifth-order operators (with $P_5(\varphi)$) [18, § 6, 12]. The quasilinear equation (5.14) is more difficult, and existence and uniqueness of $\varphi_*(s)$ remain open.

As a key application of the expansion (5.13), we note that this shows that the ODE (5.9) generates a 2D asymptotic bundle of solutions close to interfaces

2d1 (5.15)
$$
f(r) = (r_0 - r)^3 \varphi(s + s_0) + \dots \quad \text{as} \quad r \to r_0^-,
$$

with two parameters, $r_0 > 0$ and $s_0 \in \mathbb{R}$ as the phase shift in the periodic orbit. The 2D bundle perfectly suits shooting also *two* boundary conditions (5.10), though a proper topology of shooting needs and deserves extra analysis.

5.3. **Structures of single point blow-up for $p > 3$.** As before, single point blow-up occurs for $p > 3$ The blow-up similarity solutions are now

rr30L (5.16)
$$
u_{\rm S}(r,t) = (T-t)^{-\frac{1}{p-1}} f(z), \quad z = r/(T-t)^{\beta}, \quad \text{where} \quad \beta = \tfrac{p-3}{12(p-1)},
$$

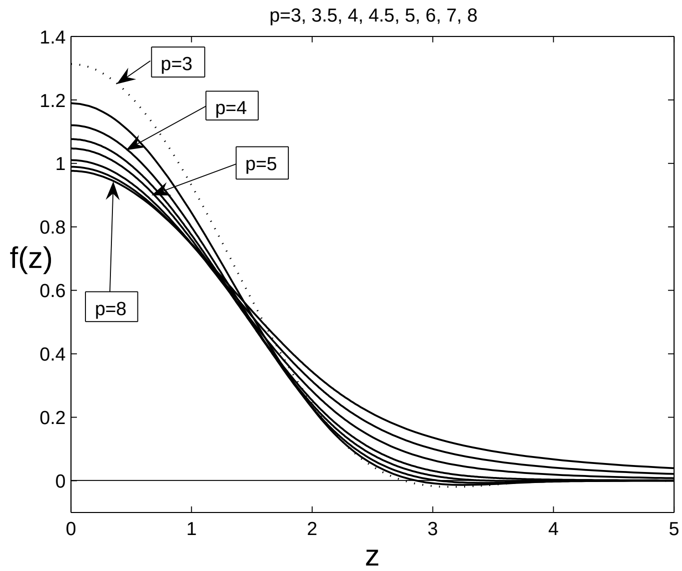


FIGURE 20. Single point blow-up profile satisfying the ODE (5.17) for $p \in (3, 8]$. F4L

and f solves the ODE

rr40L (5.17) $$\mathbf{A}(f) \equiv -\tfrac{1}{z^2}\,(f''')^2 f^{(4)} - \beta f' z - \tfrac{1}{p-1}\, f + |f|^{p-1} f = 0 \quad \text{for} \quad z > 0,$$

with the same symmetry conditions at $z = 0$, (5.10). For $p = 3$, (5.17) yields the simpler autonomous equation (5.9) for regional blow-up. For $p > 3$, this ODE is more difficult, and, following the results in Section 2, we expect to have blow-up profiles of P, Q and, possibly, S-type. Figure 20 demonstrates the first profiles of P-type for various $p \in [3, 8]$. The dotted line corresponds to the regional blow-up profiles, $p = 3$, from Figure 18. In particular, this clearly shows the continuity of the ("homotopic") deformation of solutions of (5.17) as $p \to 3^+$.

It is key to observe that the profiles for $p > 3$ have infinite interface and, moreover,

mm1 (5.18) $$f(z) > 0 \ \text{ in } \ \mathbb{R}_+ \ \text{ for } p \text{ larger than, about, } 5.$$

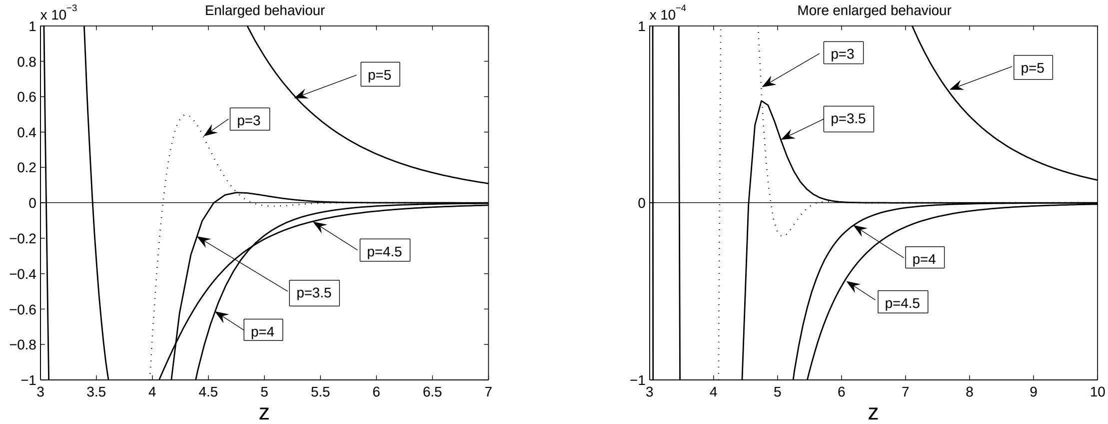
For smaller $p > 3$, the profiles continue to change sign as for $p = 3$, as the continuity in p suggests. In Figure 21, we show the enlarged behaviour of the profiles from Figure 20 in the domains, where these are sufficiently small. In both Figures (a) and (b), the profile for $p = 3.5$ has two zeros only.

It follows from (5.17) that the non-oscillatory profiles $f(z)$ have the asymptotic behaviour governed by two linear terms (other nonlinear ones are negligible on such asymptotics),

as1 (5.19) $$-\beta f' z - \tfrac{1}{p-1}\, f = 0 \quad \Longrightarrow \quad f(z) = C z^{\nu} + \dots, \quad \nu = -\tfrac{12}{p-3} < 0,$$

where $C = C(p) \neq 0$ for a.e. $p > 3$. The full 2D bundle of such non-oscillatory asymptotics includes an essentially "non-analytic" term of a typical centre manifold nature [46]:

nn1 (5.20) $$f(z) = C z^{\nu} + \dots + C_1 \mathrm{e}^{-b_0 z^{\mu}} + \dots, \quad \text{where} \quad \mu = \tfrac{11p-9}{p-3} > 0,$$



(a) enlarged, 10^{-3} (b) more enlarged, 10^{-4}

FFss1 FIGURE 21. Non-oscillatory behaviour of profiles from Figure 20 for $p > 3$.

$C_1 \in \mathbb{R}$ is the second parameter, and $b_0 = b_0(p) > 0$ is a constant that can be easily computed. The expansion (5.20) can be justified by rather technical application of standard fixed point theorems in a weighted spaces of continuous functions defined for large $z \gg 1$.

The full 2D bundle (5.20) poses a well-balanced matching problem to satisfy *two* symmetry conditions at the origin (5.10) for any $p > 3$, provided that $C \neq 0$.

On the other hand, Figure 21(b) clearly shows that $C(p)$ changes sign at some

pp1
$$\hat{p}_1 \in (4.5, 5), \quad \text{and} \quad C(\hat{p}_1) = 0. \tag{5.21}$$

Then, at $p = p_1$, using the oscillatory analysis presented below, we expect that the corresponding $f(z)$ is compactly supported.

Moreover, in view of the oscillatory behaviour near interfaces (cf. (5.13) for $p = 3$), we expect that there exists a monotone decreasing sequence of such critical exponents

ss11
$$\{\hat{p}_k\}_{k \geq 1} \to 3^+ \ \text{as} \ k \to \infty, \quad \text{such that} \quad C(\hat{p}_k) = 0, \tag{5.22}$$

so that $\hat{p}_1$ in (5.21) is just the first, maximal one. Possibly, such a mixture of compactly supported and non-compactly supported profiles $f(z)$ for $p > 3$ gets more complicated (see further comments below on oscillatory character of finite-interface solutions).

In the cases (5.22), the single point blow-up profile can be compactly supported, so we need to describe its local behaviour near the interface, which turns out to be different from that for $p = 3$ studied above.

5.4. **Finite interfaces: on oscillatory structures for $p > 3$.** The oscillatory structure of solutions is given by two first terms in (5.17), and hence have a different form, than in (5.13),

o10L
$$f(z) = (z_0 - z)^{\gamma} \varphi(s), \quad s = \ln(z_0 - z), \quad \text{where} \quad \gamma = \tfrac{9}{2}. \tag{5.23}$$

Substituting into (5.17) and neglecting other higher-degree terms, yields for $\varphi(s)$ the ODE

o20L
$$(P_3(\varphi))^2 P_4(\varphi) = -P_1(\varphi) \equiv -(\varphi' + \gamma\varphi), \tag{5.24}$$

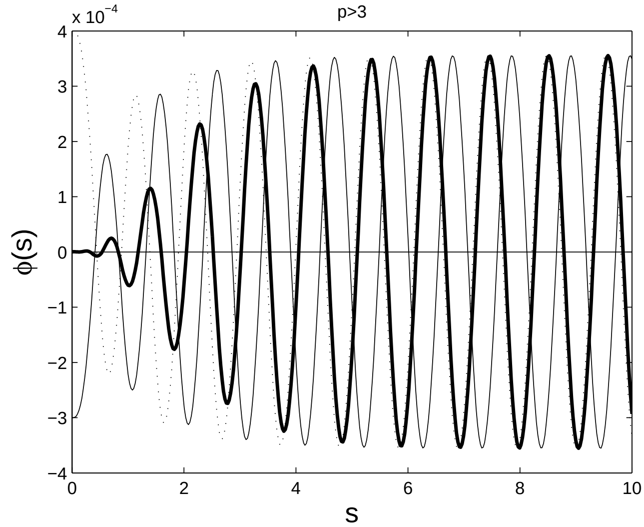


FIGURE 22. Stabilization to a stable periodic orbit $\varphi_*(s)$ for the ODE (5.24). Fos

where we have scaled out the constant multiplier $\beta z_0^3 > 0$ on the right-hand side. This ODE is even more difficult than (5.14), so we again rely on careful numerics.

In Figure 22, we show the typical behaviour of solutions of (5.24) demonstrating a fast stabilization to a unique and stable periodic solution. According to (5.23), this periodic orbit $\varphi_*(s)$ describes the generic character of oscillations of solutions of the ODE (5.17) in the critical case (5.21).

Since here the periodic orbit is stable as $s \to +\infty$, it is unstable as $s \to -\infty$ (in the direction towards the interface at $s = -\infty$ according to (5.23)), and moreover the stable manifold of $\varphi_*(s)$ as $s \to -\infty$ consists of the solution itself up to shifting. Therefore, the full equation (5.17) admits precisely

2dL (5.25) $2D$ bundle of small solutions, with parameters $z_0 > 0$ and $s_0 \in \mathbb{R}$,

where s_0 is again the translation in the oscillatory periodic component $\varphi(s + s_0)$. Therefore, shooting via 2D bundle precisely *two* symmetry conditions at the origin (5.10) represents a well-posed problem, which can admit solutions, and possibly a countable set of these. Nevertheless, Figures 21(a) and (b) justify that the actual behaviour is governed by the non-compactly supported 2D bundle (5.20) for a.e. $p > 3$, and we do not know any mathematical reason why the finite-interface bundle (also 2D) fails to be applied for $p > 3$ a.e. and not only for $p = p_k$.

5.5. **On multiplicity of solutions by branching.** Figure 23 shows two P-type profiles, $F_1(z)$ and $F_2(z)$. This poses a difficult open problem on multiplicity of solutions for $p > 3$ (and also for $p = 3$). Recall that operators in both equations (5.17) and (5.9) are not potential, so we cannot rely on well developed theory of multiplicity for variational problems; see e.g., [40, § 57].

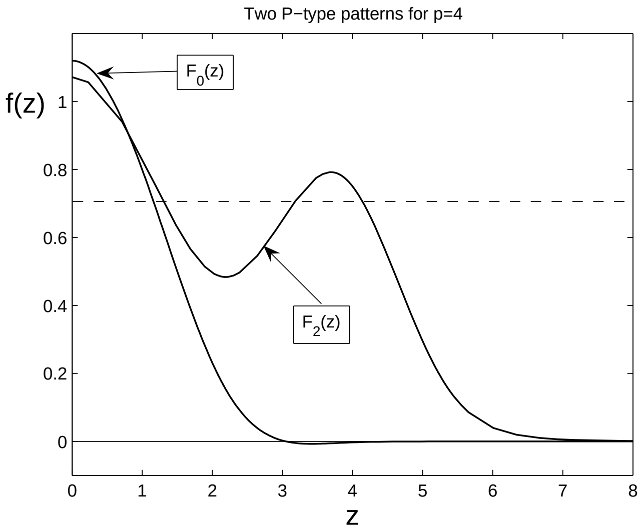


FIGURE 23. Two P-type profiles of the ODE (5.17) for $p = 4$. F4L

Nevertheless, we will rely on variational theory by introducing a family of approximating operators

aa1 (5.26)
$$\mathbf{A}_\mu(f) = \big[-\tfrac{\mu}{z^2}\,(f''')^2 + \mu - 1\big] f^{(4)} - \mu\beta f' z - \tfrac{1}{p-1}\, f + |f|^{p-1} f, \quad \mu \in [0,1].$$

Then $\mathbf{A}_1 = \mathbf{A}$, while for $\mu = 0$,

aa2 (5.27)
$$\mathbf{A}_0(f) = -f^{(4)} - \tfrac{1}{p-1}\, f + |f|^{p-1} f,$$

we obtain a standard non-coercive variational operator. The corresponding functional for $f \in H^2(-L, L)$ on a fixed interval with $L \gg 1$,

aa3 (5.28)
$$\Phi_0(f) = -\tfrac{1}{2} \int (f'')^2 - \tfrac{1}{2(p-1)} \int f^2 + \tfrac{1}{p+1} \int |f|^{p+1}$$

has at least a countable set of different critical points $\{f_k^{(0)},\ k = 0, 1, 2, ...\}$; see [47].

Thus we arrive at the branching problem from profiles $f_k^{(0)}$ at the branching point $\mu = 0$, which leads to classic branching theory; see [40, Ch. 6] and [53]. In general in the present ODE setting, where the linearized operator for $\mu = 0$,

aa4 (5.29)
$$\mathbf{A}_0'(f) Y = -Y^{(4)} + p|f|^{p-1} Y,$$

has a 1D kernel, branching theory [13, p. 401] suggests that each member $f_k^{(0)}$ generates a continuous branch $\{f_k^{(\mu)}\}$ at $\mu = 0$. The global continuation of those branches up to $\mu = 1$ remains a difficult open problem, that in the present ODE case admits an effective numerical treatment.

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