Maths Makes Waves!

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1 Introduction

Waves are everywhere, from the smallest scales to the largest, from waves which oscillate in a pico second, to ones which change over may years, and from waves with length scales smaller than the size of the atom, to waves which stress across the galaxy. Waves lie at the heart of much of modern technology, physics, chemistry, biology, medical imaging, music, popular culture and are all around us. The study of waves also led to major advances in mathematics (both pure and applied), connecting such seemingly diverse subjects as trigonometry, geometry, differential equations (both partial and ordinary), functional analysis, harmonic analysis, fluid mechanics, quantum theory, electromagnetism, integrable systems and thermodynamics, to name just a few. Calculating problems involving waves led to the 19th Century development of the analogue computer, and the problems posed by studying waves remain some of the biggest challenges that we face.

We can celebrate the extraordinary range of scales that see wave-like phenomena by taking a whistle stop tour through wavy time and space. Most waves occur when a state is in a near balance between different forces and the wave is a periodic trade off between the effects of these forces. The notable feature of all waves are a repeatability (a coherence) in both time and space. When we look up into the sky and see extraordinary patterns in the clouds above us we often respond with awe and wonder and this amazing display by nature, and what we are seeing are gravity waves in the clouds. The smallest waves are those that lie at the heart of the atom, and are the waves that describe the elementary particles of matter, in the celebrated wave/particle duality. The waves satisfied by electrons have wavelengths of around 500 pm, and these waves are used to image tiny objects in an electron scanning microscope. Moving up the scales, a wave with which we are all very familiar is light, which is just one of the many forms of electro-magnetic radiation. Light typically has a wavelength of the order of 500 nm.

Electro-magnetic radiation is also manifest as X-rays, infra red, and radio waves. The shortest of the radio waves in common experience are the micro-waves used in mobile phones and in micro-wave cookers. The typical wavelength of the waves in a micro-wave cooker is about 10 cm with a frequency of about 3 GHz. (This is about the same dimensions as the food itself. Micro-wave

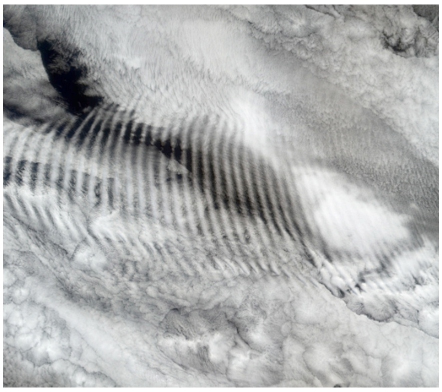


Figure 1: Gravity waves in clouds seen from space

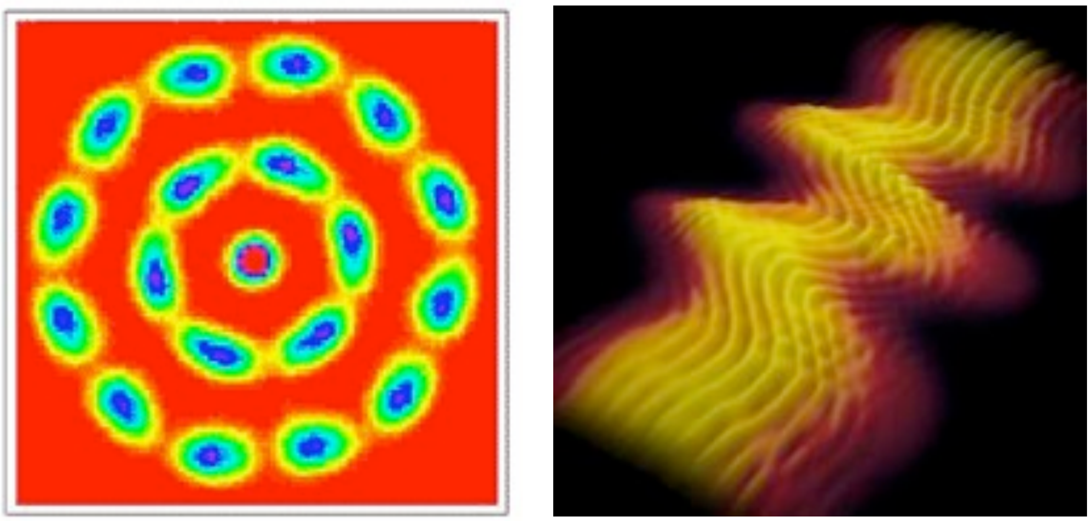


Figure 2: (a) Electron wave. (b) Light wave

cookers typically set up standing wave patterns in the food which can lead to very uneven cooking patterns). Another form of wave with which we are all very familiar are sound waves. These are caused by density variations in the air (or water) around us. Typically sound waves have a frequency in the kilo-Hertz and a wavelength of about 50 cm. Going up a scale again, we often see waves of the order of a metre. One of my favourite examples of waves occurring in nature are the beautiful ripple patterns that you see in sand after the tide has gone out. These are caused by the interaction between the sand particles (which are dragged along by the tide) and gravity. The wavelength of the ripples depends on the size and mass of the sand particles and the speed of the tide. On a larger length scale we can also see waves in the deserts caused by the interaction between the wind and the sand. This is the cause of sand dunes. Of course if we look beyond the beach to the sea itself we can see many different waves. Waves close to the sea-shore are primarily gravity waves, largely created by the effect of the wind on the sea. The wave breaking which we are all familiar with is caused by the water in the base of the wave slowing down as it comes into contact with the beach, with the water above going faster. The wavelength of the gravity waves The sea has also other types of waves, the most destructive being Tsunamis, which are an example of *shallow water waves*, so called because their wavelength is huge, and much larger than the depth H of the ocean. The speed c of a Tsunami is given by

$$c = \sqrt{gH}$$

where g is the acceleration due to gravity.

If we look above the ocean we see another form of gravity wave, this time in the atmosphere. If we have a mass of heated and thus bouyant air, then this can oscillate up and down in the atmosphere with an interplay between the buoyant and gravitational effects. Atmospheric gravitational waves can have wavelengths from tens of metres to kilometres. They are responsible for many of the beautiful cloud patterns that we see (especially in the morning and evening). The atmosphere also supports much larger waves, called Rossby waves, which are giant oscillations in high-altitude winds that are a major influence on our weather. Rossby waves occur on length scales comparable to the size of the Earth and are caused by the interplay between the Coriolis forces caused by the Earths rotationand the shear forces in large scale air masses. When the oscillations in the Rossby waves become very pronounced, they produce the cyclones and anticyclones and are responsible for typical weather patterns at mid-latitudes.

Figure 3: (a) Ripples in the sand. (b) A breaking water wave

' Now we move away from Earth and look for waves that occur on a Galactic length scale. These are the *gravitational waves* predicted by Einstein's General Theory of Relativity. These waves are enormous and typically have wavelengths of

Thus, waves truly do lie at the heart of modern applied mathematics (interpreted in the broadest possible way), they have a long and distinguished history and they show the way forward for the technologies of the future. It was therefore very fitting that the 2011 British Festival of Science gave the Mathematics Section the opportunity not only to highlight the importance and theory of waves, but also to celebrate two big anniversaries of perhaps the most important wave equation of all.

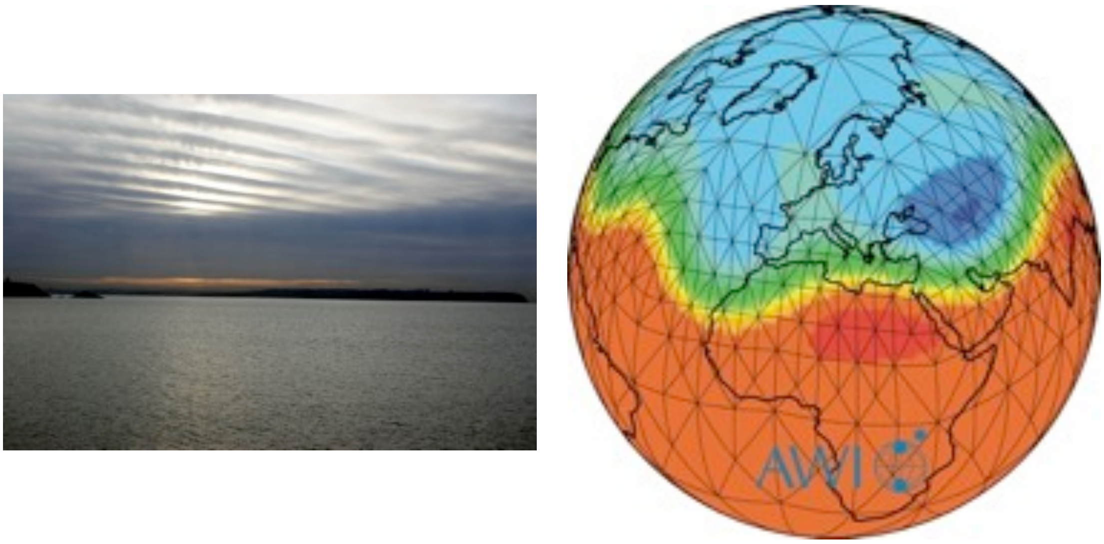


Figure 4: (a) A gravity wave causing cloud ripples in the atmosphere. (b) A planetary scale Rossby wave

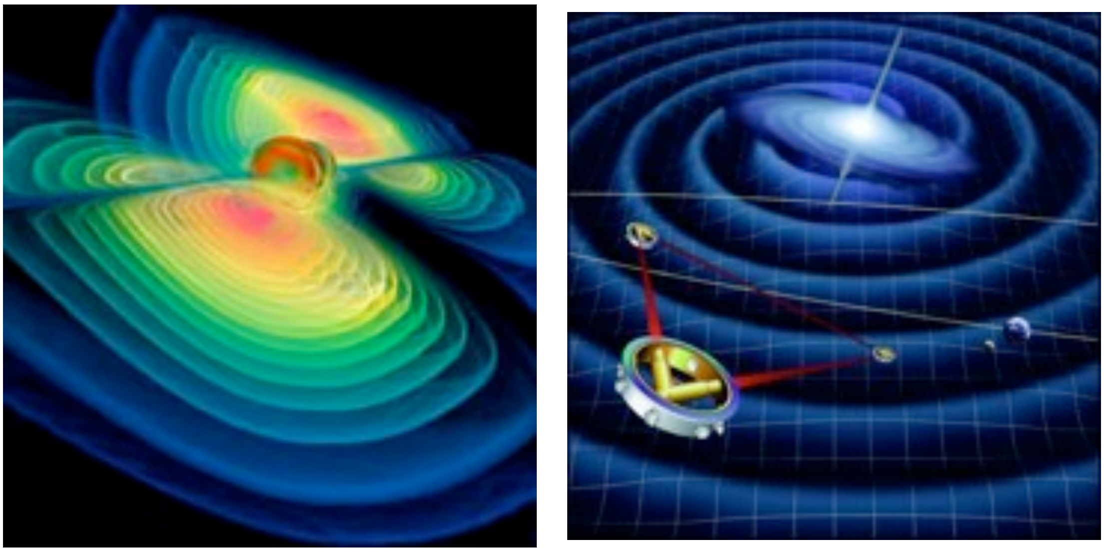


Figure 5: (a) Gravitational wave. (b) A breaking water wave

2 Schrödinger and his wave equation

Erwin Schrödinger was one of the greatest scientists that has ever lived. He was born in ?? Vienna in 1887 in and died in Dublin in 1961. The event at the British Science Festival celebrated the 50th anniversary of his death. Schrödinger completely defies the stereotype of a boring scientist that only wears a lab coat and never comes out of the laboratory. Instead Schrödinger had a most exotic lifestyle, a string of mistresses and, arguably, a number of wives at once. It was 85 years ago in 1926, whilst he was on holiday with one of his mistresses (and by all accounts having an active time of it) that he derived his celebrated wave equation. Schrödinger was grappling with the great challenge then facing physics, namely how to understand the microscopic nature of matter which seemed to obey very different rules from normal space and time. Schrödinger was trying to make sense of the observation both that it was very hard to localise the position of a particle and also that at a sub-atomic scale, particles seemed to have many of the properties of waves. In order to reconcile this he introduced a *wave function* $\psi(x, y, z, t)$ which described the probability distribution of states with different energies. He then postulated that the wave-function for a particle of mass m moving in a potential $V(x)$ obeyed a *linear partial differential equation* similar to that of a wave on a string which was given by

Figure 6: Erwin Schrödinger

$$i\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V(x,y,z,t)\psi. \tag{1}$$

(Note the fact that this equation is linear. This allows us to superimpose quantum states, and this concept is a back bone of modern quantum physics.) The wave equation (1) is pretty well one of the most important partial differential equations of mathematical physics. It is up there with Newton's law of Gravitation, the Euler Equations of fluid motion and the Dirac equation for the electron, in giving us a deep understanding of the universe ranging from (to quote Feynman) the Hydrogen atom to a frog. Indeed one of the first applications of (1) was in explaining the spectrum of the Hydrogen atom (though, sadly, not of a frog). To quote Wikipedia

> *The 1926 paper has been universally celebrated as one of the most important achievements of the twentieth century, and created a revolution in quantum mechanics, and indeed of all physics and chemistry*

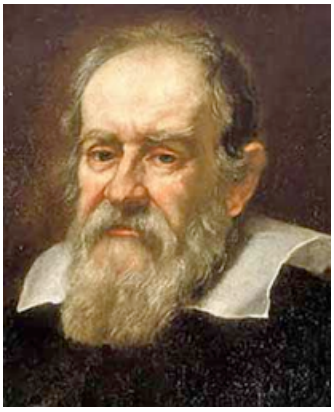
Thanks to the power of modern computers, Schrödinger's (wave) equation can now be solved directly for reasonably large systems of atoms and electrons, and this allows the chemistry of the systems to be determined in advance of synthesizing a material. Schrödingers equation is also important in the design of micro-electroninc systems where the small size of the components means that quantum effects become very important. In every way, Schrödingers equation dominates our modern lives at it is fitting that in 2011 we also celebrated its 85th anniversary.

3 A brief history of waves

Whilst waves as we have seen are very much up-to-date, our understanding and appreciation of waves has a very long history. Probably the first waves to be studied in a systematic fashion were sound waves produced by musical instruments. It was the Greeks that first noticed that the pitch of a sound produced by plucking a string (on say a Lyre) depended on the *length of the string.* Simliarly (although for somewhat different reasons) the pitch of the note obtained when a blacksmith struck an iron bar depened on the length of the bar. It was also noticed that certain notes when played together sounded better than others and this was the beginnings of the science and art of harmony. The first mathematicians to study this phenomenon was the great Pythagoras. Pythagoras is of course well known to school children for his discovery of Pythagoras' Theorem (which he did not in fact discover). He is less known for his work on music (which he did do). If it was more widely appreciated that a mathematician laid down both the theory and (via the scale) the practice of much of modern music, then maybe mathematicians would have a higher reputation in modern society? What Pythagoras observed was that if a string under tension had length l and gave a note of a certain pitch, then the lengths of the strings which gave notes with a harmonious pitch were *those which had lengths which were simple fractions of* l. As an example, a string of length $l/2$ gave a note an Octave higher than the original, and indeed sounded very similar to the original. A string of length $2l/3$ gave a note which was a *perfect fifth* above the original. On a piano, if the base note is a C then the perfect fifth above C gives a G. Similarly, the notes C and F form a *perfect 4th*, with the string lengths in ratio $3l/4$ and C and E form a *major third* with strings in the ratio $4/5$. It is immediate to see that the numbers $1/2, 2/3, 3/4, 4/5$ are all simple fractions with low order denominators. Pythagoras therefore found a profound link between simple fractional proportion and music. He extended this to give the *just scale* of eight notes C D E F G A B C so that the stings had the lengths

$$l, 9l/8, 5l/4, 4l/3, 3l/2, 5l/3, 15l/8, 2l$$

and this scale formed the backbone of Western music until begin replaced by the equally tempered scale in the 18th Century. We now fast forward about two thousand years to the life of the great Italian scientist Galileo 1564-??, Galileo is well known for his work on the dynamics o moving bodies and also for his work in astronomy. He also did profound work on the theory of waves. Galileo was trying to understand Pythagoras' resuls, and realised that musical notes in stringed instruments arise from waves on the strings. The frequency or pitch of the note was then inversely proportional to the length of the string (and directly proportional to the tension of the string). He alo established the very remarkable link between waves and triangles, showing that the simplest wave of amplitude A and of frequency f (and angular frequency ω) in time could be described by



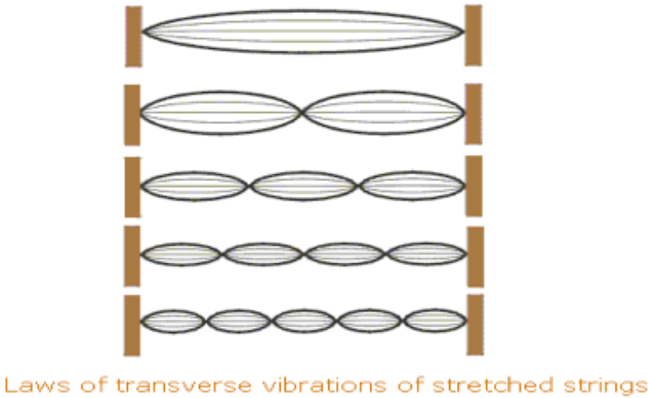


Figure 7: (a) Galileo. (b) Waves on a string

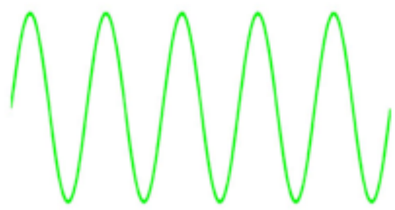


Figure 8: Simple sine wave

a *sine wave* so that

$$u(t) = A\sin(\omega t) = A\sin(2\pi f t). \tag{2}$$

Personally I continue to find this link between waves and triangles remarkable, and it demonstrates a deep underlying unity in science and mathematics. More generally, of course, a wave varies in time and space and can be described by the formula

$$u(x,t) = A\sin\left(2\pi\left(\frac{x}{\lambda} - ft\right)\right) = A\sin\left(\frac{2\pi}{\lambda}(x - ct)\right) \tag{3}$$

where λ is the *wavelength* and

$$c = f\lambda$$

is the *wave speed.* For waves in air at sea level, $c = 320ms^{-1}$. The note of middle C has $f = 261.6$ Hz, $\lambda = 1.2$ m and the note of G above has $f = 392$ Hz and $\lambda = 0.8$ m. When we plot one wave against another in a Lissajous figure we get a clear hint about the nature of harmony. The graph of the notes C:G plotted in this way is simple and elegant, whereas that of the discordant pairing E:F is much more complex These observations all of course beg the question of exactly why the simplest waves occurring in nature are sine waves. One simple explanation is to consider a wave as being the height of a point on a disc rotating at a uniform angular speed ω. However, a much more complete explanation comes from understanding the equations that lead to waves on strings. As a precursor to this, consider the motion of another object studied by Galileo, namely

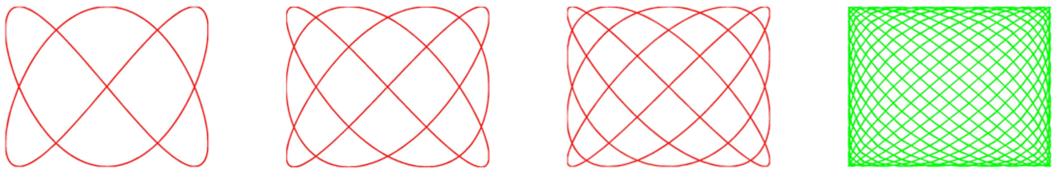


Figure 9: (a) Lissajous figure for C:G. Note the elegant form of this figure, (b) C:F, (c) C:E, (d) E:F, note the complicated form of this figure.

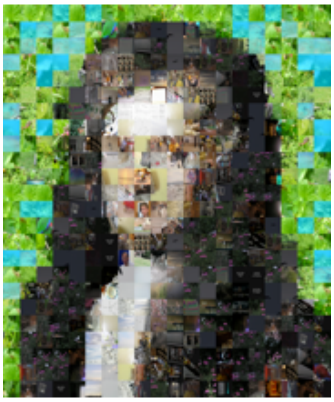
the simple pendulum swinging with an angle θ in a uniform gravitational field. Not very long after Galileo's death, Sir Isaac Newton formulated this motion (or at least the motion of swing of small amplitude) as a *second order differential equation* in θ of the form

$$a\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} + c\,\theta = 0. \tag{4}$$

It is not clear whether Newton had an analytic solution of this equation. More probably he had a geometrical solution, and quite possibly a numerical solution (using what is now called the Leap-Frog method). However, Euler certainly had one taking the form

$$\theta(t) = A\,e^{-\alpha t}\,\sin(\omega t + \varphi),$$

where A, ω, α and φ all depend on a,b and c through the solution of an appropriate quadratic equation. This is of course a *damped sine wave.* Here A is the amplitude, ω the angular frequency, φ the phase and α the level of damping (which drops to zero if $b$is zero). Here again we see a profound (and very fortunate!) link between trigonometric functions and the solution of a differential equation describing wave-like motion. The most interesting feature of this being that (4) describes not only the small swings of a pendulum, but the small amplitude motion of a vast number of physical phenomena which are periodic in time. It is this simple fact which forms a deep link between trigonometric functions, waves and periodic motion. Until recently, all radio (and TV) receivers contained a large number of *tuned circuits*, carefully designed to satisfy differential equations of the form (4) for varying electronic signals of different frequencies. By careful selection of the values of a,b and c such circuits could then be tuned to waves of a certain frequency, allowing the selection of a signal of that frequency. It is this process which lies at the heart of the frequency selectivity of (until recently) most radio receivers and transmitters.



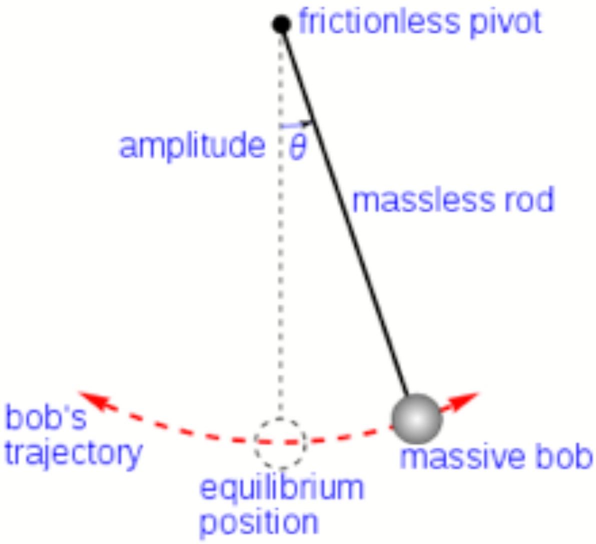


Figure 10: (a) Isaac Newton. (b) A simple pendulum

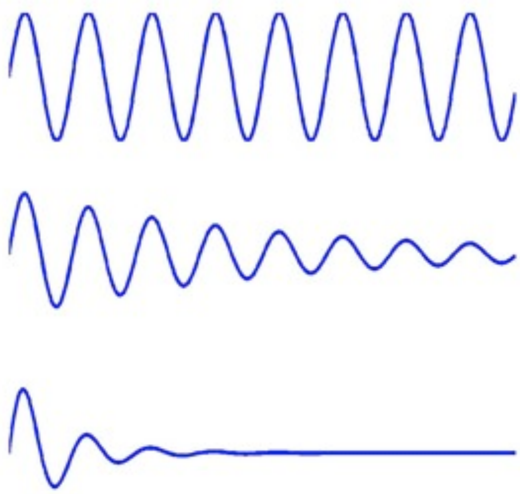


Figure 11: A damped sine wave with increasing levels of damping

4 Fourier's revolution

To misquote George Orwell, if one (sine) wave is good then many (sine) waves are better. More generally, what happens if you combine a very large number, or possibly even an infinite, number of waves of different frequency. This question was first asked by the French mathematician/magistrate Jean Baptiste Fourier. Fourier was interested in the way that heat was conducted along a metal bar. By a process of physical reasoning he derived what is now called the heat equation for the temperature $T(x,t)$ of the bar at time $t > 0$ an at a position x along the bar, in the form of the linear partial differential equation

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}, \tag{5}$$

where κ is the thermal conductivity. In general this equation is hard to solve. However, Fourier observed that a particular solution was given by the expression

$$T(x,t) = e^{-kappa\omega^2 t} \left(A\cos(\omega x) + B\sin(\omega x)\right), \tag{6}$$

where ω is general at this stage of the calculation. We can see immediately that this expression is a spatial wave (of angular frequency ω) which decays in time. Of course most functions dont take this special form. Fourier's genius was to recognise that a general solution of (5) could be made up of sums of solutions of the form (6). In the case of an *infinite bar* this sum takes the form of an integral so that

$$T(x,t) = \int_{-\infty}^{\infty} e^{-\kappa\omega^2} \left(A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)\right)\, d\omega. \tag{7}$$

In contrast, if the bar has a finite length L then ω takes the discrete values $2\pi n/L$ for $n = 0, 1, 2\ldots$ and

$$T(x,t) = \sum_{n=0}^{\infty} e^{-4\kappa\pi^2 n^2/L^2} \left(A_n\cos(2\pi nx/L) + B_n\sin(2\pi nx/L)\right). \tag{8}$$

These are respectively the Fourier Transform and the Fourier series decomposition of the function $T(x,t)$. This was an extraordinary insight of Fourier and not only gave a means of solving the heat equation, but brought in a whole new area of mathematics (Fourier analysis or Harmonic analysis), studying functions $f(x)$ (and the equations that they solve) by decomposing them as sums of waves so that

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n\cos(2\pi nx/L) + B_n\sin(2\pi nx/L). \tag{9}$$

where

$$A_n = \frac{1}{\pi}\int_0^L \cos(2\pi nx/L) f(x)\, dx \quad \text{and} \quad B_n = \frac{1}{\pi}\int \sin(2\pi nx/L) f(x)\, dx.$$

The values A_n, B_n, the *Fourier coefficients*, give a huge amount of information about the function $f(x)$. The partial sums S_K of this series are given by

$$S_K(x) = \frac{A_0}{2} + \sum_{n=1}^{K} A_n\cos(2\pi nx/L) + B_n\sin(2\pi nx/L). \tag{10}$$

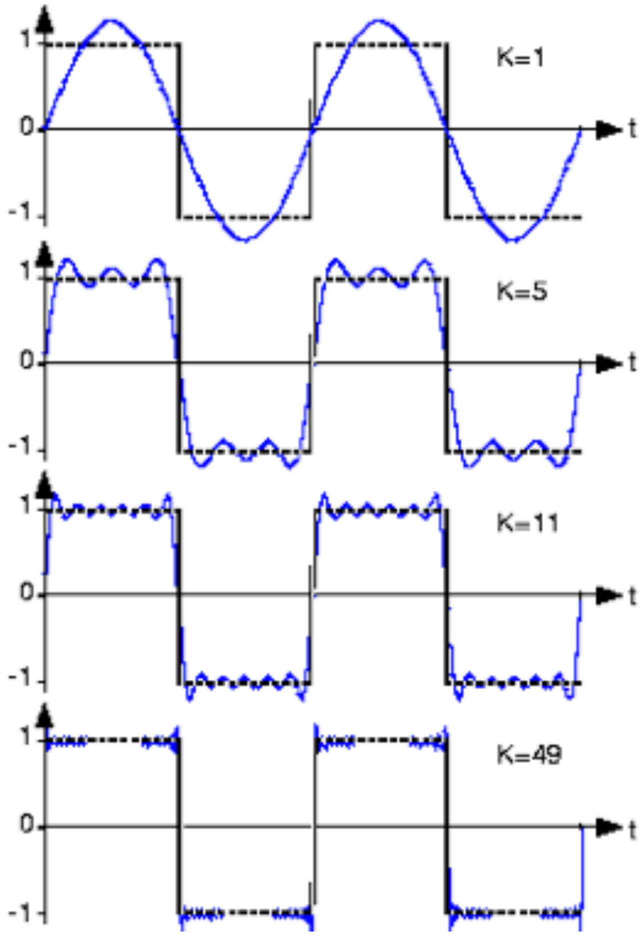


Figure 12: A sequence of partial sums of a Fourier series, converging slowly to a square wave. Observe the oscillations at the discontinuities of the square wave due to the Gibb's phenomenon. These can be reduced by using appropriate *window functions*

Usually (but not always) $S_K \to f(x)$ as $K \to \infty$, however this convergence can be slow if $f(x)$ is not a smooth function. For example, if $f(x)$ is a square wave of unit amplitude then it can be represented by the slowly converging Fourier series

$$f(x) = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\sin((2k+1)x)}{2k+1}.$$

The Fourier coefficients in this series converge slowly to zero, and the square wave is often described as being *rich in harmonics.* It is used to test the response of electronic circuits to signals of many different frequencies. However, the slow decay of the Fourier coefficients means that only do you need to sum many terms to get an accurate representation of the square wave, but also that the partial sums S_K display rapid oscillations, called the Gibb's phenomenon, close to the discontinuities of the square wave.

Figure 13: A synthesiser, user to create a rich variety of different sounds

[?] The process of decomposing a periodic function into a series of sine waves, and finding the values of the Fourier coefficients in (9), lies at the heart of many processes both in nature and in modern technology. The ear detects sound in the eustachion tube in such a manner that there are different sensors for the different pitched sounds, with the tube acting as a filter for the different pitches. Thus our own appreciation of sound follows from an analogue application of Fourier's formula. Similarly, a spider can detect different sounds by doing a Fourier decomposition of the vibrations that it senses through its legs. Similarly, complex waveforms such as a square wave, or the sound of a church organ or even human speech can be *synthesised* by combining sine waves of the appropriate amplitude and phase. Furthermore, a sound signal can be *compressed* by finding its Fourier series decomposition and deleting the terms with high frequency and/or small Fourier coefficients. The resulting waveform then sounds very similar to the original but can be stored with significantly less computer memory. Essentially this process of lossy data compression is used to compress music and/or videos to form the MPEG files used in the IPod and other similar devices.

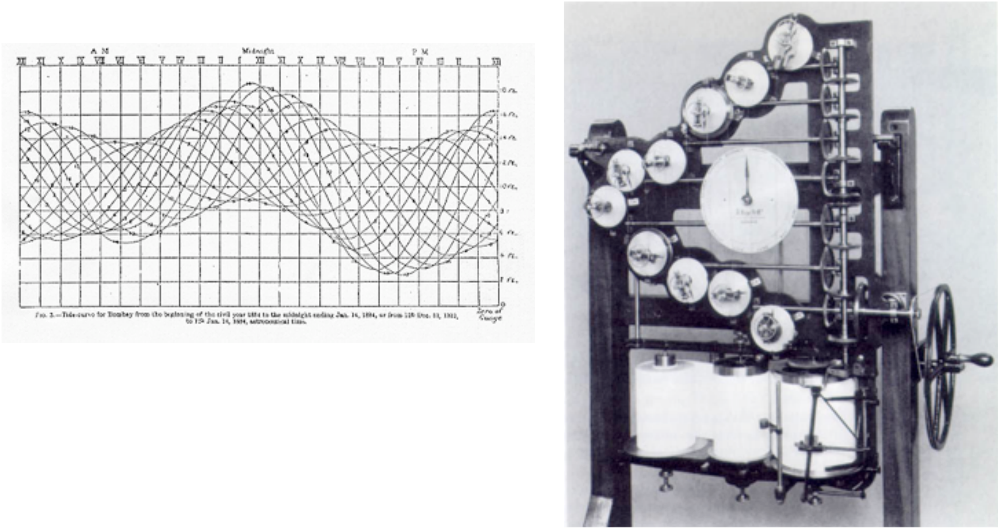


Figure 14: (a) The height of the Bombay tides over several days. (b) Kelvin's tidal predictor now in the Science Museum, London. An analogue computer used to calculate the tidal height by summing a 10 term Fourier series.

Modern *software defined radio* replaces the tuned circuits that we described in the last section, with *digital filters* that achieve the same (indeed better) task by converting the analogue signal received at the aerial into a digital signal, and then manipulating the Fourier coefficients of this signal to process it.

Closely linked to existence of the Fourier series (9) the need to evaluate it quickly and accurately. At first sight (and to the mathematicians at the time) this is a formidable task as it means evaluating the value of many sine waves at many different points and then adding them up. As we have seen, Fourier series can be quite slow to converge, and thus it may be necessary to do this calculation for a large sum. One early major success in computing this sum was achieved by Lord Kelvin in the 19th Century. Kelvin was one of the greatest scientists of the 19th century, famous for his work in thermodynamics, his development of many mathematical techniques and (as we shall see) his work on the telegraph. Kelvin was also very famous for his work with the Navy. Indeed one of his inventions was the Binnacle, which contained the ship's compass and allowed the compass to be used accurately in iron ships, correcting for the magnetic perturbations of the iron hull. In ???? Kelvin was given the challenge of predicting the tides in Bombay. The tides are effectively a global water wave and occur roughly every twelve hours and 25 minutes and are due to the tidal attraction of the gravitational fields of the Moon and of the Sun on the Earth and of its oceans. This tidal force of both the Moon and the Sun is proportional to the inverse cube of the distance of the attracting body from the Earth, and as a result is roughly the same for each. As the period of the Moon rotating around the Earth and of the Earth rotating around the sun are not in synchrony the tides vary in height from day to day, so that there are stronger (spring) tides at new and full Moon when the Moon and Sun pull together, and weaker (neap) tides at the half Moon when they are at 90°. The tides are also highest close to the Equinoxes. Whilst the pattern of the tides does repeat, at least approximately, it only does so every 19 years. This makes the prediction of the tidal height difficult, especially as it is also affected by the local geography. It has however been known since the time of the French mathematician Laplace, that the tides are composed of a number of periodic waves. These include a 12:25:14.164 cycle due to the gravity of the moon, 24:00 and 24:50:28.328 cycles caused by the differences in the two tidal bulges, a 27.2122 day cycle caused by change in lunar declination (Moons angle to the Earth), a 27.5546 day cycle caused by a regular change in the Earth-Moon distance, a 29.5306 day cycle caused by the phases of the moon, and an annual cycle. The amplitude and phase of each of these depends on the location and the tides at Bombay differ from those in London. To predict the height of the Bombay tide, Kelvin decomposed it into 8 (later 10 and then 20 and even more) different harmonic components and used past data to find the amplitude and phase of each. Doing this was hard enough, but then adding up the contributions seemed beyond the abilities of the human calculators of the day. Kelvin overcame this by inventing a *mechanical* calculator which used an arrangement of cog wheels and pulleys to add up the various components. The resulting machine was called the *Kelvin tidal predictor*, and it can still be seen in the Science Museum in London. Not long afterwards the Americans built a bigger, shinier, but essentially similar, version of their own. These mechanical tidal predictors worked very well indeed, and can arguably be called one of the first ever *analogue* computers. Indeed, they were so effective that they were used to predict the tidal heights in Normandy in advance of the D-Day landings, and they remained in use before being replaced by electronic computers in the 1960s.

5 The wave equation, waves in fluids and waves down wires

As we saw earlier in this article, the waves that many of us are very familiar with (from childhood) are waves on the sea. These turn out to be quite hard to describe mathematically. There are basically several types of water waves, often generated either by the wind or by a moving body. There are for example, small waves in deep water, large waves in shallow water and breaking waves.

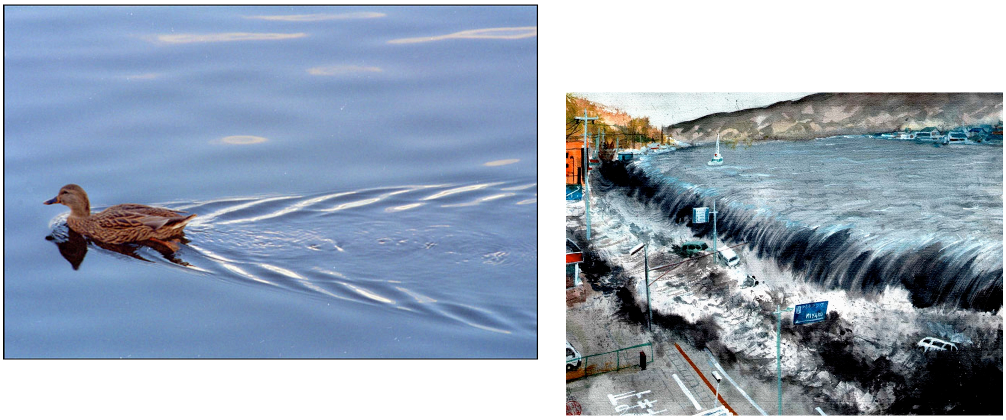


Figure 15: (a) The wake of a duck (b) A destructive Tsunami.

For water to feel 'deep to a wave, its depth must be greater than half of the wavelength of the wave. This is true for most ocean waves, however Tsunamis have wavelengths of many hundreds of kilometers and to them the mighty ocean is just a shallow pond! A water wave typically arises as an exchange of gravitational potential energy (as the height of the wave goes up and down) with the kinetic energy of the movement of the waver itself. Such waves are usually called *gravity waves*In shallow water, all of the water is involved in the wave motion and in deep water it is only the top surface. This is why these waves move rather differently. Associated with any water wave is its wavelength Λ and its speed c. If the water is deep and the waves are not too large in amplitude, then we have the relation

$$c = \sqrt{\frac{g\Lambda}{2\pi}} \approx 1.25\sqrt{\Lambda},$$

where $g = 9.81ms^{-2}$ is the acceleration due to gravity. These waves obey a form of the *wave equation*. From this we can make many predictions, such as the nature of the wake of a ship, or indeed the wake of a duck! (One of the nicest predictions is that of the angle of the duck's wake).We can also see that waves with the longest wavelength travel fastest. This means that after a storm, in which the wind generates many waves of different wavelengths, it is the ones with the longest wavelength that first arrive at the beach.

In contrast, if the water is shallow and has a depth of h then the speed of the wave is

$$c = \sqrt{gh} \approx 3.13\sqrt{h}.$$

Both of these relations can be obtained from Bernouilli's equation which relates the height of a moving fluid to its velocity. A more exact description of the Tsunami behaviour can be obtained from solving the *Shallow Water Equations*, which are a simplification of the Euler equations describing fluid motion. These equations play a very important role both in understanding water waves and also in may other phenomena such as the flight of an aircraft or the flow of a river. If you look at the equation for the speed of a shallow water wave you will see that the larger h is the faster the wave moves. As a wave approaches the shore this means that the top of the wave is moving faster than the bottom, which also slows down due to friction with the ground. As a result the wave leans forward into the shape so much admired by surfers, until it eventually breaks. At that point we really don't have the maths to work out what happens next!

Many other types of waves appear in fluids, for example *roll waves* which you can see on a rainy day as water flows down a road, *tidal bores* of which the most famous is the Severn Bore and the deep water waves that play an important role in our weather and climate.

As I hope you agree, there is simply no limit to the role that waves play in our lives, and the mathematical study of them continues to be a fascinating one.

Figure 16: (a) A roll wave in a narrow river (b) The Severn Borei.