Combinatorial geometries and torus strata on homogeneous compact manifolds

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Combinatorial geometries and torus strata on homogeneous compact manifolds

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CONTENTS

Introduction

In this article we consider the decomposition of a compact homogeneous space of a complex semisimple group G into orbits via the action of a Cartan subgroup. The geometry of these orbits is very interesting. Even more interesting is the decomposition of a manifold into orbit fibres with the same geometry.

The first part of the article (§§1, 2) is devoted to the case of the Grassmann manifold $G_k(\mathbf{C}^n)$ of k-dimensional subspaces of n-dimensional space. Most of the results in the first part were simultaneously and independently obtained by Goresky, MacPherson, and the authors [17]. The strata on the Grassmannian can be defined in three different ways: a) the union of all the orbits of the Cartan subgroup whose images under the moment mapping give the same polytope (see [4]); b) the collection of projective configurations that give the same combinatorial geometry; c) the intersection of the Schubert cells for all Borel subgroups that contain the given Cartan subgroup.

We note particularly that strata and their application to the Grassmannian are important for the general theory of hypergeometric functions (see [15], [11], [18], [19]). With each non-degenerate stratum on the Grassmannian there is associated a hypergeometric function having singularities on the strata adjacent to the given one.

Strata on the Grassmannian are studied in another context by Vershik and Mnëv.

The first part can be read without any specialist knowledge. It seems to us that an elementary exposition has the advantage of explaining the geometrical essence of the subject in more detail.

The decomposition of the Grassmannian that we construct turns out to be closely connected with the theory of matroids or of combinatorial geometries. In particular, the problem of describing all strata on the Grassmannian reduces to that of the representability of matroids (see [6]-[8]). In 2.1 and 2.3 we present the necessary standard facts of matroid theory (see, for example, [6]-[8]). We particularly recommend the book of Crapo and Rota [8] for further reading on this fascinating theory.

In §2 we study the important connection between matroids and convex polytopes of a special type, which apparently does not occur in the literature on matroid theory.

In the second part of the paper (§§3-9) the results of the first part are extended to the case of the compact homogeneous space connected with an arbitrary complex semisimple Lie group. It also contains a survey of results connected with toric varieties, the moment mapping, and hypersimplexes.

The Grassmannian $G_k(\mathbf{C}^n)$ is a compact homogeneous space via the action of the group $\mathrm{SL}(n)$. It is clear that $G_k(\mathbf{C}^n) = \mathrm{SL}(n)/P$, where P is a parabolic subgroup of $\mathrm{SL}(n)$ preserving some k-dimensional subspace of $\mathbf{C}^n$.

Now let G be an arbitrary complex semisimple group, acting transitively on a compact manifold M. Then M can be identified with the homogeneous space G/P, where P is a parabolic subgroup of G, that is, a subgroup containing a maximal soluble subgroup of G.

In the case $G = \mathrm{SL}(n)$ every compact homogeneous manifold can be expressed as a flag manifold (incomplete, in general) in $\mathbf{C}^n$. Here the Grassmannians correspond to the maximal parabolic subgroups.

There exists a Kähler metric χ on the manifold $M = G/P$ which is invariant under the action of the compact form K of G. The imaginary part of χ gives a symplectic form on M, regarded as a real manifold. This symplectic structure allows us to construct a moment mapping μ: $M \to \mathfrak{k}^*$, where $\mathfrak{k}$ is the Lie algebra of K (see [2], [3], [10], [16]). Using the mapping μ we can determine the orbits of a Cartan subgroup H in M, whose closures are isomorphic toric varieties, and thus define a decomposition of M into strata.

The classical notion of a matroid is connected with the action of $\mathrm{SL}(n, \mathbf{C})$ on the Grassmannian. Replacing $\mathrm{SL}(n, \mathbf{C})$ by another semisimple group, and the Grassmannian by another homogeneous space, we obtain new combinatorial geometries, which we call general matroids. Thus, for example, replacing the Grassmannian by an arbitrary flag manifold gives rise to supermatroids (see [7]) with special restrictions, which are described in §9.

All the necessary information about toric varieties and the moment mapping (see [1], [5], [10]) is contained in §3, and about the geometry of homogeneous manifolds of complex semisimple groups in §4.

In §5 we give the definition of strata in the general case, and in §6 we study their connection with the decomposition into Schubert cells.

Just as in the Grassmannian case, the strata on an arbitrary homogeneous space G/P give rise to convex polytopes, which we call hypersimplexes. All the edges of the hypersimplexes are parallel to roots of the Lie algebra $\mathfrak{G}$ of G. In the Grassmannian case this property is equivalent to the replacement axiom for matroids. In the general case it provides a possible definition for matroids connected with a pair (G, P), where G is a semisimple group and P a parabolic subgroup.

In §8 we define the (W, Q)-matroid for an arbitrary Coxeter group W and an arbitrary subset Q of generators, which agrees with the usual definition of a matroid in the case where $W = S_n$, and Q generates a maximal parabolic subgroup. In the case of an arbitrary parabolic subgroup of S_n, the notion of (W, Q)-matroid is closely related to the recently introduced notion of a greedoid (see [20], [21]). This connection will be described in more detail in future publications. It seems to us that the notion of a (W, Q)-matroid is of independent interest in combinatorics.

In §9 we consider examples of strata on homogeneous spaces of the semisimple groups $\mathrm{SO}(n)$, $\mathrm{Sp}(2n)$, and G_2.

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§1. Torus orbits and strata on the Grassmannian

1.1. Let $G_k(E)$ denote the manifold of all k-dimensional subspaces of the n-dimensional complex space E.

The group $\mathrm{GL}(E)$ of all linear transformations on E acts in a natural way on the Grassmannian $G_k(E)$. The essential part for us is the action of a Cartan subgroup $H \subset \mathrm{GL}(E)$. Let us fix a basis $\{e_1, \ldots, e_n\}$ in E, that is, we identify E with the coordinate space $\mathbf{C}^n$. Then we may take H to be the subgroup of all diagonal matrices with respect to the chosen basis. Let $H\cdot X$ denote the orbit of a point $X \in G_k(\mathbf{C}^n)$ under the action of H, and $\overline{H\cdot X}$ its closure in $G_k(\mathbf{C}^n)$. It can be shown that $\overline{H\cdot X}$ is a compact algebraic variety consisting of finitely many orbits of H, where $H\cdot X$ is the unique everywhere dense open orbit in $\overline{H\cdot X}$. Moreover, $\overline{H\cdot X}$ is a toric variety (see §3 for a definition).

Each orbit $H\cdot X$ in $G_k(\mathbf{C}^n)$ is isomorphic to $(\mathbf{C}^*)^r$ for some $r \leqslant n-1$. An orbit of dimension $n-1$ will be called *non-degenerate*. The fixed points under the action of H are the k-dimensional coordinate subspaces of $\mathbf{C}^n$.

In this section we study submanifolds of $G_k(\mathbf{C}^n)$ consisting of orbits of H whose closures are isomorphic as toric varieties.

1.2. Definition of stratum.

Let $I_n = \{1, \ldots, n\}$. Let $B(I_n)$ denote the set of subsets of I_n. We put $B_p(I_n) = \{J \in B(I_n) \mid |J| = p\}$. For any $J = \{i_1, \ldots, i_p\} \in B_p(I_n)$ let $\mathbf{C}^J$ denote the coordinate subspace of $\mathbf{C}^n$ spanned by $e_{i_1}, \ldots, e_{i_p}$.

We call two subspaces X and Y *equivalent* if $\dim(X \cap \mathbf{C}^J) = \dim(Y \cap \mathbf{C}^J)$ for all $J \in B(I_n)$.

Definition 1. The equivalence classes of subspaces in $G_k(\mathbf{C}^n)$ are called *strata.*

It is clear that every stratum Γ can be given by a function $s : B(I_n) \to \mathbf{Z}$ as follows: $\Gamma = \{X \in G_k(\mathbf{C}^n) \mid \dim(X \cap \mathbf{C}^J) = s(J)\}$. From now on we define a stratum in terms of a different function r, putting

$$\Gamma_r = \{X \in G_k(\mathbf{C}^n) \mid \dim(X/X \cap \mathbf{C}^{I_n \setminus J}) = r(J)\}.$$

It is sometimes convenient to define a stratum via the function r^*:

$$\Gamma^{r^*} = \{X \in G_k(\mathbf{C}^n) \mid \dim(\mathbf{C}^J/\mathbf{C}^J \cap X) = r^*(J)\}.$$

Both of these methods are equally valid, and as we shall see later (§2) they give rise to dual combinatorial geometries.

The functions s, r, and r^* that define the same stratum are related by the following formulae:

$$r(J) = k - s(I_n \setminus J) = |J| + r^*(I_n \setminus J) + k - n.$$

Example. Let $r_0(J) = \min(k, |J|)$. Then the stratum Γ_{r_0} in $G_k(\mathbf{C}^n)$ consists of those subspaces that are in general position with respect to all coordinate subspaces. Γ_{r_0} is the unique everywhere dense stratum in $G_k(\mathbf{C}^n)$. We call it the general stratum.

We call a function $r : B(I_n) \to \mathbf{Z}$ admissible for the Grassmannian $G_k(\mathbf{C}^n)$ if the set Γ_r is non-empty.

Proposition 1. *Suppose that the function $r : B(I_n) \to \mathbf{Z}$ is admissible for $G_k(\mathbf{C}^n)$. Then:*

1) $r(I_n) = k$,
2) $0 \leqslant r(J) \leqslant |J|$,
3) *if* $J \subseteq I$, *then* $r(J) \leqslant r(I)$,
4) $r(I) + r(J) \geqslant r(I \cap J) + r(I \cup J)$.

The proof is obvious.

1.3. The connection between strata and Schubert cells.

It is traditional to decompose the Grassmannian $G_k(\mathbf{C}^n)$ into Schubert cells. This is done by choosing a coordinate flag $0 \subset \mathbf{C}^{J_1} \subset \ldots \subset \mathbf{C}^{J_n} = \mathbf{C}^n$, where $|J_i| = i$. The Schubert cells corresponding to the given flag consist of subspaces in $G_k(\mathbf{C}^n)$ having a fixed dimension of intersection with each $\mathbf{C}^{J_i}$.

For each of the $n!$ coordinate flags in $\mathbf{C}^n$ we choose precisely one of the corresponding Schubert cells. We call the intersection of all these Schubert cells a small cell.

***Proposition* 2.** *The decomposition of the Grassmannian into small cells coincides with the decomposition into strata.*

The proof is almost obvious (see [17]).

1.4. Configurations.

Let F be a k-dimensional complex vector space. Every k-dimensional subspace of $\mathbf{C}^n$ can be defined as the image under a non-singular linear transformation $x : F \to \mathbf{C}^n$. Two transformations x_1 and x_2 have the same image if and only if there is a $g \in \mathrm{GL}(F)$ such that $x_1 = x_2 \circ g$.

With respect to the basis $\{e_1, \dots, e_n\}$, x has the form $x(v) = \sum_{1 \leqslant j \leqslant n} x^j(v) e_j$, where $x^1, \dots, x^n$ span F^*.

Let $C_{n,k}$ denote the manifold of collections of n vectors lying in a given k-dimensional space and spanning it. We call the points of $C_{n,k}$ modulo the action of $\mathrm{GL}(k)$ configurations. In this way we have a principal $\mathrm{GL}(k)$-bundle q: $C_{n,k} \to G_k(\mathbf{C}^n)$.

Let X be given by the configuration $x^1, \dots, x^n$ in F^*. Then

$$\dim X/(X \cap \mathbf{C}^{I_n \setminus J}) = rk(x^{j_1}, \dots, x^{j_p})$$

for any $J = \{j_1, \dots, j_p\} \subset I_n$.

Choosing a basis in F^*, we can identify $C_{n,k}$ with the manifold of $n \times k$ matrices of rank k. Thus the points of $G_k(\mathbf{C}^n)$ can be defined as $n \times k$ matrices of rank k up to right multiplication by a non-singular square matrix of order k. By using configurations we can determine the points of $G_k(\mathbf{C}^n)/H$. We introduce an action of H on $C_{n,k}$ as follows: $h(x^1, \dots, x^n) = (h_1 x^1, \dots, h_n x^n)$, where $(x^1, \dots, x^n) \in C_{n,k}$, $h = \mathrm{diag}(h_1, \dots, h_n) \in H$. Then the actions of H on $C_{n,k}$ and $G_k(\mathbf{C}^n)$ are compatible in the sense that $q \circ h = h \circ q$ for all $h \in H$. Hence the projection q: $C_{n,k} \to G_k(\mathbf{C}^n)$ induces a map $\tilde{q}$: $C_{n,k}/H \to G_k(\mathbf{C}^n)/H$.

Consider the open subset $C^0_{n,k}$ of $C_{n,k}$ consisting of collections of non-zero vectors. Then $C^0_{n,k}/H$ can be identified with the manifold of collections of n points in projective space $\mathbf{C}P^{k-1}$. We call a collection of n points in $\mathbf{C}P^{k-1}$ modulo the action of $\mathrm{PGL}(k)$ a projective configuration. Thus every projective configuration determines some orbit of H in $G_k(\mathbf{C}^n)$.

1.5. Plücker coordinates.

Let $P(\Lambda^k(E))$ be the projectivization of the k-th exterior power of E. We recall the construction of the Plücker embedding $p : G_k(E) \to P(\Lambda^k(E))$. Let $X \subset E$ be any k-dimensional subspace. Then $\Lambda^k(X)$ is a one-dimensional subspace of $\Lambda^k(E)$, in other words, a point of $P(\Lambda^k(E))$. We put $p(X) = \Lambda^k(X)$.

We choose a basis in $\Lambda^k(E)$ of the form

$$\{e_J = e_{j_1} \wedge \cdots \wedge e_{j_k} \mid J = \{j_1, \dots, j_k\} \in B_k(I_n),\ j_1 < \cdots < j_k\}.$$

Then $p(X)$ can be written in the form $p(X) = \sum_{J\in B_k(I_n)} p^J(X)e_J$, where the $p^J(X)$ are determined up to a scalar. The numbers $p^J(X)$ are called the Plücker coordinates of the subspace X.

If the subspace $X \in G_k(\mathbf{C}^n)$ is given by an $n \times k$ matrix, then $p^J(X) = p^{j_1,\dots,j_k}(X)$ is equal to the minor of this matrix consisting of the rows with indices $j_1, \dots, j_k$.

The image of $G_k(\mathbf{C}^n)$ is given in Plücker coordinates by the equations (see [14]):

$$\sum_{l=0}^{k} (-1)^l p^{i_1,\dots,i_{k-1},j_l} \cdot p^{j_0,\dots,\hat{j_l},\dots,j_k} = 0$$

for all $\{i_1, \dots, i_{k-1}\} \in B_{k-1}(I_n)$, $\{j_0, \dots, j_k\} \in B_{k+1}(I_n)$.

1.6. Affine coordinates.

Let $J = \{j_1, \dots, j_k\}$ be a k-element subset of I_n. Let C_J be the set of all $n \times k$ matrices such that the minor consisting of rows $j_1, \dots, j_k$ is the identity matrix. Then q induces an isomorphism from C_J onto an open everywhere dense subset of $G_k(\mathbf{C}^n)$. The manifold $q(C_J)$ is called an affine chart in $G_k(\mathbf{C}^n)$, and the entries of the matrix $q^{-1}(X) \in C_J$ are called the affine coordinates of the point $X \in G_k(\mathbf{C}^n)$. It is clear that $G_k(\mathbf{C}^n)$ is covered by the affine charts $q(C_J)$ as J runs over all k-element subsets of I_n.

1.7. The moment mapping in $G_k(\mathbf{C}^n)$.

With each subset $J \subset I_n$ we associate the point δ_J in $\mathbf{R}^n$ with coordinates

$$(\delta_J)_i = \begin{cases} 1, & i \in J, \\ 0, & i \notin J. \end{cases}$$

Consider the mapping $\mu(X) : G_k(\mathbf{C}^n) \to \mathbf{R}^n$ that sends a point $X \in G_k(\mathbf{C}^n)$ with Plücker coordinates $p^J(X)$ into the convex combination

$$\mu(X) = \sum_{J\in B_k(I_n)} |p^J(X)|^2 \cdot \delta_J \Big/ \sum_{J\in B_k(I_n)} |p^J(X)|^2. \tag{1}$$

It is clear from the formula that $\mu(G_k(\mathbf{C}^n))$ is the convex polytope in $\mathbf{R}^n$ given by

$$\sum_{1\leqslant i\leqslant n} \xi_i = k, \qquad 0 \leqslant \xi_i \leqslant 1.$$

Let us denote this by $\Delta_{n,k}$. This polytope was first considered in [4], where it was called a hypersimplex. The mapping μ was constructed in [4] for the real Grassmannian $G_k(\mathbf{R}^n)$ and the closure of a general orbit of $(\mathbf{R}^+)^n$ in $G_k(\mathbf{R}^n)$ was shown to be mapped homeomorphically onto $\Delta_{n,k}$.

In [2] and [3] a generalization of μ was constructed for an arbitrary compact Kähler manifold with an action of the torus $(\mathbf{C}^*)^n$. This mapping is called the *moment mapping* (see §3). A very important convexity theorem was also proved in these articles, greatly generalizing the results of [4].

Theorem 1. *Let $H \cdot X$ be the orbit of $X \in G_k(\mathbf{C}^n)$ under the action of H. Then $\mu(\overline{H \cdot X})$ is a compact polytope in $\mathbf{R}^n$ with vertex set* Vert $\mu(\overline{H \cdot X}) = \{\delta_J \mid p^J(X) \neq 0\}$. *The mapping μ induces a one-to-one correspondence between the p-dimensional orbits of H in $\overline{H \cdot X}$ and the open p-dimensional faces of $\mu(\overline{H \cdot X})$.*

All the assertions of Theorem 1, other than the description of Vert $\mu(\overline{H \cdot X})$, hold in greater generality, so we will not give a proof here (see [2], [3], 3.2, 4.3).

Definition 2. Two orbits $H \cdot X$ and $H \cdot Y$ are called *equivalent* if $\mu(H \cdot X) = \mu(H \cdot Y)$. The equivalence classes of orbits of H in $G_k(\mathbf{C}^n)$ are called *strata.*

Note that, for any stratum Γ, the image $\mu(\Gamma)$ is the interior of a convex polytope. We denote this by Δ_Γ.

Proposition 3. *Definitions* 1 *and* 2 *are equivalent.*

The proof involves the important combinatorial notion of a list.

The *list* of an arbitrary subspace $X \in G_k(\mathbf{C}^n)$ is the subset

$$L_X = \{J \in B_k(I_n) \mid X \cap \mathbf{C}^{I_n \setminus J} = \{0\}\}.$$

Now let Γ be a stratum in $G_k(\mathbf{C}^n)$ in the sense of Definition 1, and let $X \in \Gamma$. It is clear that the list L_X is uniquely determined by the function r defining the stratum (see 1.2). On the other hand, it is clear from the definition of the list that $L_X = \{J \in B_k(I_n) \mid p^J(X) \neq 0\}$. Hence, by Theorem 1, Γ coincides with some stratum in the sense of Definition 2.

In the other direction we must show that the function

$$r_X(J) = \dim(X / X \cap \mathbf{C}^{I_n \setminus J})$$

is uniquely determined by the list L_X. This follows from the next lemma.

Lemma 1. $r_X(J) = \max_{B \in L_X} |B \cap J|$.

Suppose that X is defined by the configuration of vectors $x^1, \ldots, x^n \in F^*$. Then $r_X(\{i_1, \ldots, i_p\}) = \mathrm{rk}(x^{i_1}, \ldots, x^{i_p})$. But $\mathrm{rk}(x^{i_1}, \ldots, x^{i_p})$ is equal to the maximal number of independent vectors among $x^{i_1}, \ldots, x^{i_p}$. Let $x^{i_{j_1}}, \ldots, x^{i_{j_q}}$ be some maximal independent subsystem of vectors. We complete this to a basis using vectors from the configuration. This can be done because $\mathrm{rk}(x^1, \ldots, x^n) = k$. Let B be the set of indices of vectors in this basis. Then clearly $\mathrm{rk}(x^{i_1}, \ldots, x^{i_p}) = |B \cap \{i_1, \ldots, i_p\}|$. On the other hand, it is clear that $r_X(J) \geqslant |B \cap J|$ for all B in L_X.

This proves Lemma 1 and Proposition 3.

As we shall see in §2, Lemma 1 has a purely combinatorial generalization relating to arbitrary matroids. We note that the stratum Γ in $G_k(\mathbf{C}^n)$ is defined by the list L_Γ. Specifically:

$$\Gamma = \{X \in G_k(\mathbf{C}^n) \mid L_X = L_\Gamma\}.$$

1.8. Non-degenerate strata.

A stratum Γ in $G_k(\mathbf{C}^n)$ consisting of non-degenerate orbits of H (that is, orbits of dimension $n-1$) is called *non-degenerate*. If Γ is non-degenerate, then the quotient group of H over the subgroup of scalar matrices acts freely on Γ.

***Proposition* 4.** *The following conditions are equivalent*:

a) *the stratum* $\Gamma \subset G_k(\mathbf{C}^n)$ *is degenerate*;

b) *there is a* $J \in B(I_n)$ *such that any* $X \in \Gamma$ *can be expressed in the form* $X \cap \mathbf{C}^J \oplus X \cap \mathbf{C}^{I_n \setminus J}$.

This proposition will be proved in §2.

It is clear that non-degenerate orbits are always defined by projective configurations.

Example. Suppose that the orbit $H \cdot X \subset G_3(\mathbf{C}^n)$ is defined by a projective configuration in $\mathbf{C}P^2$. Then $H \cdot X$ is non-degenerate if there exist four points in the configuration in general position.

1.9. Admissible polytopes and hypersimplexes.

A polytope Δ in $\mathbf{R}^n$ is called *admissible* for $G_k(\mathbf{C}^n)$ if $\Delta = \Delta_\Gamma$ for some stratum $\Gamma \subset G_k(\mathbf{C}^n)$. It follows from Theorem 1 that any face of an admissible polytope is again an admissible polytope.

A polytope Δ in $\mathbf{R}^n$ is called an *(n, k)-hypersimplex* if all its edges and vertices are edges and vertices of $\Delta_{n,k}$. Note that every edge of $\Delta_{n,k}$ is parallel to a vector $\delta_i - \delta_j$ for some $i, j \in I_n$.

Figure 1 shows all (4,2)-hypersimplexes up to the action of the symmetry group of the octahedron: the octahedron $\Delta_{4,2}$, its top half, and all its faces.

***Proposition* 5.** *An admissible polytope is an (n, k)-hypersimplex.*

Proof. By Theorem 1 every face, in particular an edge l, of an admissible polytope is an admissible polytope.

Let δ_I and δ_J be the vertices of l. Then $L_Y = \{I, J\}$ for some $Y \in G_k(\mathbf{C}^n)$. Suppose that l is not an edge of $\Delta_{n,k}$. Then $|I \cap J| \leqslant k-2$. Hence there exist $i, j \in I \setminus J$ and $p \in J \setminus I$. Then it follows from Lemma 1 that $r_Y(\{i, p\}) = r_Y(\{j, p\}) = 1 = r_Y(\{p\})$ and $r_Y(\{i, j, p\}) = 2$. But then $\{i, p\}$ and $\{j, p\}$ violate condition 4) of Proposition 1, which contradicts the admissibility of l.

Remark. Not every (n, k)-hypersimplex is an admissible polytope (see the example in 2.2).

1.10. Examples.

It is not difficult to describe the strata in $G_2(\mathbf{C}^n)$, since this reduces to the description of projective configurations in $\mathbf{C}P^1$. The strata are indexed by decompositions of I_n as a union of disjoint sets $A_0, A_1, \ldots, A_p$, where $p \geqslant 3$.

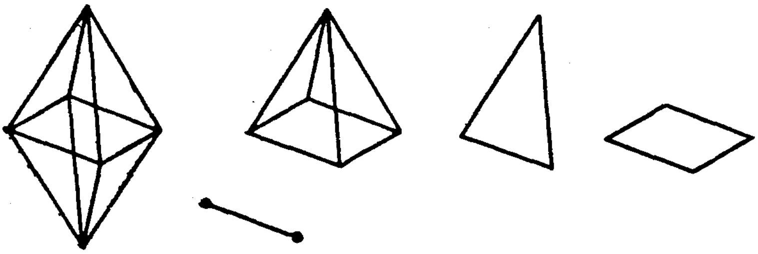

Fig. 1

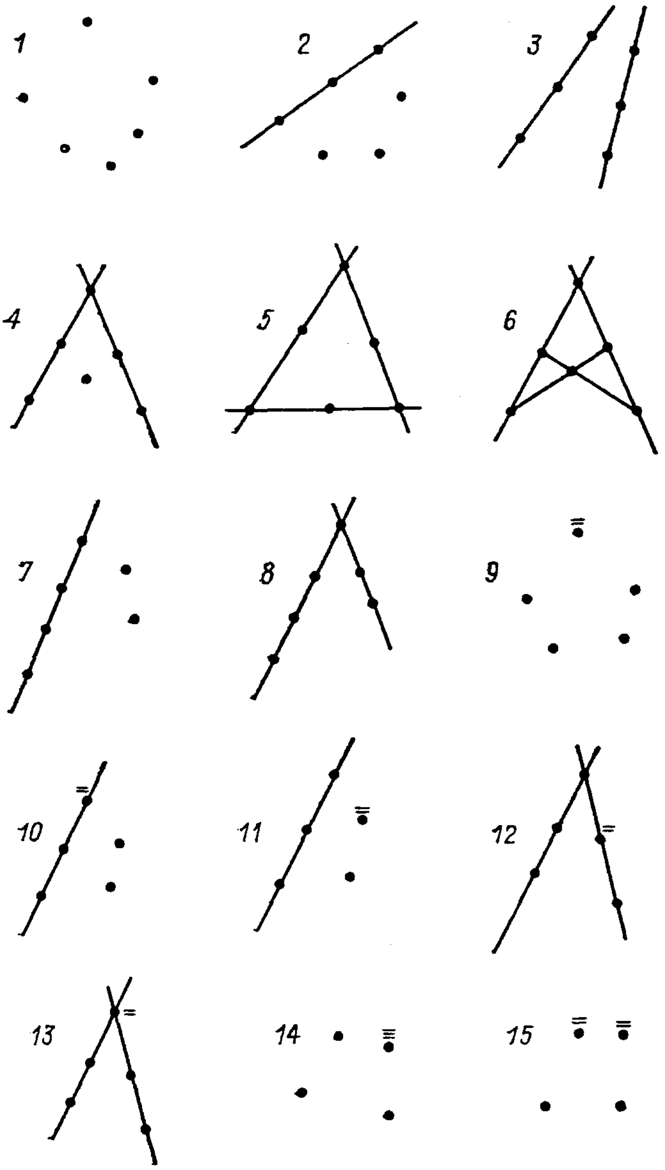

Fig. 2. Non-degenerate strata in $G_3(\mathbf{C}^6)$. The symbol = denotes coincidence of points

The function r_Γ is defined as follows:

$$r_\Gamma(J)=\begin{cases}0 & \text{if } J\subset A_0,\\ 1 & \text{if } J\subset A_0\cup A_i,\ i\neq 0,\\ 2 & \text{otherwise.}\end{cases}$$

We enumerate all non-degenerate strata in $G_3(\mathbf{C}^6)$ (degenerate strata correspond to lower-dimensional Grassmannians). To each non-degenerate stratum there corresponds a configuration type of six points in $\mathbf{C}P^2$. These are enumerated in Fig. 2. (Lines are drawn in the diagrams if they contain more than two distinct points of the configuration.)

1.11. Adjacency of strata.

Let Γ be a stratum in $G_k(\mathbf{C}^n)$. It turns out that its closure $\bar{\Gamma}$ is not always a union of strata, although this holds in $G_2(\mathbf{C}^n)$ and $G_3(\mathbf{C}^6)$.

Example 1. Consider the single orbit stratum Γ in $G_3(\mathbf{C}^7)$ defined by the projective configuration depicted in Fig. 3. It is clear that Γ is adjacent to the orbit defined by the configuration with supplementary condition $x^7 = 0$. But this orbit lies in a one-parameter family of orbits, forming a stratum in which no other orbit is adjacent to Γ.

Example 2. Consider the stratum Γ in $G_3(\mathbf{C}^8)$ obtained from the previous example by adding another point with index 8 at the intersection of the lines (64) and (135) (Fig. 4). The adjacent configuration in which the points 1 and 6, 5 and 7, and 4 and 3 are identified, defines a stratum of 6-dimensional orbits (Fig. 5). Here the orbit lies in $\bar{\Gamma}$ only if the duality relation $(|x^1x^5|\cdot|x^3x^8|)/(|x^5x^3|\cdot|x^1x^8|)= 1$ holds.

Adding one more point in general position to both configurations (see Figs. 4 and 5), we obtain an example in which all the strata are non-degenerate.

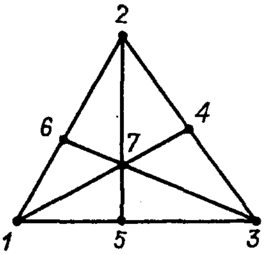

Fig. 3

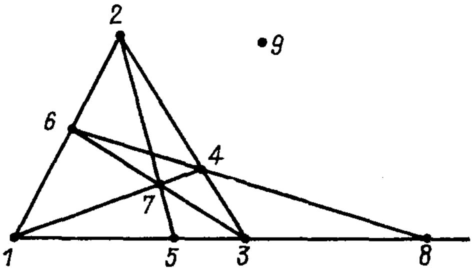

Fig. 4

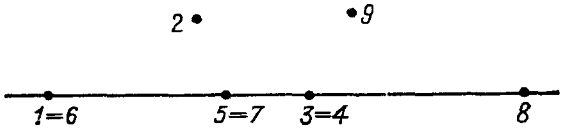

Fig. 5

§2. Matroids and strata on the Grassmannian

2.1. Matroids, the basic definitions.

Definition 1. A *matroid of rank* k is a pair (S, r), where S is a set and r is an integer-valued function on $B(S)$ such that:

1) $r(S) = k$;

2) $0 \leqslant r(A) \leqslant |A|$;

3) if $A \subseteq B$, then $r(A) \leqslant r(B)$;

4) $r(A)+r(B) \geqslant r(A \cup B)+r(A \cap B)$.

The term matroid is sometiems replaced by *combinatorial pregeometry* (see, for example, [8]). The function r is called the *rank* function of (S, r). We consider only finite sets S.

It follows from Proposition 1 of §1 that to every stratum Γ in $G_k(\mathbf{C}^n)$ there corresponds a matroid (I_n, r_Γ), where r_Γ is the admissible function defining Γ.

We say that two matroids (S_1, r_1) and (S_2, r_2) are *isomorphic* ($(S_1, r_1) \cong$ $\cong (S_2, r_2)$) if there is a bijection from S_1 onto S_2 transforming r_1 to r_2.

Example. A configuration of n vectors $x^1, \ldots, x^n$ in a vector space F^k over any field determines a matroid (I_n, r) with rank function $r(\{i_1, \ldots, i_p\}) = rk(x^{i_1}, \ldots, x^{i_p})$.

A matroid is said to be *representable* over a field F if it is isomorphic to a matroid defined by a configuration of vectors in a vector space over F. In particular, the description of all strata in Grassmannians reduces to the description of all matroids representable over $\mathbf{C}$. This is a very difficult combinatorial problem, which has not yet been solved.

We remark that not every matroid is representable over a field, or even over a division ring. (For examples, see [6], [7].)

We enumerate certain basic notions from matroid theory ([6]-[8]), and the corresponding facts about the geometry of orbits of the torus group on the Grassmannian.

A set $A \subseteq S$ is said to be *closed*, or a *flat* of the matroid, if $r(A \cup a) >$ $> r(A)$ for all $a \in S\backslash A$. The set of flats of a matroid (S, r) will be denoted by $\mathrm{Fl}(S, r)$. It follows from the properties of the rank function that $\mathrm{Fl}(S, r)$ is closed under intersections. We introduce an operation

$$V\colon A \vee B = \bigcap_{J\in \mathrm{Fl}(S,r),\, A,B\subseteq J} J$$

on $\mathrm{Fl}(S, r)$. Then $\mathrm{Fl}(S, r)$ becomes a geometrical lattice, that is,

$$r(A) + r(B) \geqslant r(A \cap B) + r(A \vee B).$$

Let $X \in G_k(\mathbf{C}^n)$. Let $R(X)$ denote the geometrical lattice of all subspaces of X obtained by intersecting X with the coordinate subspaces $\mathbf{C}^J$, and let $R^*(X)$ denote the lattice of subspaces of X^* that are dual to subspaces in $R(X)$. The following assertion is easy to verify:

Proposition 1. $R^*(X) \cong \mathrm{Fl}(I_n, r_\Gamma)$ *for any* $X \in \Gamma$.

A set $A \subseteq S$ is called *independent* if $r(A) = |A|$. A maximal independent set is called a *basis* of (S, r). The set of all bases of (S, r) is denoted by $\mathscr{B}(S, r)$.

Consider a stratum $\Gamma \subset G_k(\mathbf{C}^n)$. Then it is clear that its list L_Γ coincides with $\mathscr{B}(I_n, r_\Gamma)$. We proved in 1.7 that a stratum is uniquely determined by its list. It turns out that this assertion generalizes to arbitrary matroids: a matroid (S, r) is uniquely determined by its set of bases. Moreover, the properties of the set of bases can be used as a basis for the definition of a matroid.

Proposition 2. *Let* (S, r) *be a matroid of rank* k. *Then*

1) $|B| = k$ *for all* $B \in \mathscr{B}(S, r)$.

2) *Let* B, $B' \in \mathscr{B}(S, r)$. *For any* $b \in B \setminus B'$ *there is a* $b' \in B' \setminus B$ *such that* $(B \setminus b) \cup b' \in \mathscr{B}(S, r)$. (*Exchange axiom.*)

If a set $\mathscr{B} \subset B(S)$ *satisfies conditions* 1) *and* 2), *then it is the set of bases of a uniquely determined matroid* (S, r).

For a proof see [8].

2.2. Hypersimplexes and matroids.

Just as for a stratum on the Grassmannian, we can associate with every matroid (S, r) a polytope Δ_r. Let $\mathscr{L}(S)$ be the vector space of all real-valued functions on S. We define the polytope Δ_r to be the convex hull of the characteristic functions δ_B for all bases B of (S, r). We define a scalar product on $\mathscr{L}(S)$: $(f, g) = \sum_{a \in S} f(a) \cdot g(a)$. Then the function r on $B(S)$ is defined by the formula

$$r(J) = \max_{\xi \in \Delta_r} (\delta_J, \xi).$$

We remark that matroids are isomorphic if and only if the corresponding polytopes are congruent.

The question arises, which polytopes in $\mathscr{L}(S)$ correspond to matroids?

A polytope in $\mathscr{L}(S)$ with vertices of the form δ_B, where $B \in B_k(S)$, and edges parallel to vectors $\delta_a - \delta_b$ for $a, b \in S$, is called an (S, k)-*hypersimplex.*

Theorem 1. *A polytope* Δ *in* $\mathscr{L}(S)$ *corresponds to some matroid* (S, r) *of rank* k *if and only if* Δ *is an* (S, k)-*hypersimplex.*

The proof of Theorem 1 follows immediately from Lemma 1.

Lemma 1. *Let* $\mathscr{B}$ *be a collection of k-element subsets of* S, *and* $\Delta_{\mathscr{B}}$ *the polytope in* $\mathscr{L}(S)$ *consisting of the convex combination of the characteristic functions* δ_B *for all* $B \in \mathscr{B}$. *Then condition* 2) *of Proposition* 2 *is equivalent to the condition that* Δ *is an* (S, r)-*hypersimplex.*

Proof. Let l be an edge of $\Delta_{\mathscr{B}}$ with vertices δ_{B_1} and δ_{B_2}. There is a linear function χ on $\mathscr{L}(S)$ that is constant on l and such that its values at all other points of $\Delta_{\mathscr{B}}$ are less than that on l. Suppose that $\chi = \sum_{b\in S} \nu_b \delta_b^*$. We choose $b \in (B_1 \cup B_2)\setminus(B_1 \cap B_2)$ so that ν_b is minimal. We may assume without loss of generality that $b \in B_1$. Then it follows from condition 2) that there is an $a \in B_2$ such that $B = (B_1 \setminus b) \cup a \in \mathscr{B}$. Then $\chi(\delta_{B_1}) \leqslant \chi(\delta_B)$. Thus $B = B_2$ and l is parallel to $\delta_a - \delta_b$.

Conversely, suppose that $\Delta_{\mathscr{B}}$ is a hypersimplex. Consider two of its vertices δ_{B_1} and δ_{B_2} and an arbitrary $a \in B_1 \setminus B_2$.

Let $\Delta'_{\mathscr{B}}$ be that face of $\Delta_{\mathscr{B}}$ on which the linear function $\chi = \sum_{b \in B_1 \cup B_2} \delta_b^*$ attains its maximum. It is clear that $\Delta'_{\mathscr{B}}$ is a hypersimplex, and that $\operatorname{Vert} \Delta'_{\mathscr{B}} = \{\delta_B \mid B \subset B_1 \cup B_2\}$.

Consider the linear function δ_a^* for $a \in B_1 \setminus B_2$. Since $\delta_a^*(\delta_{B_1}) > \delta_a^*(\delta_{B_2})$, there is an edge $l \subseteq \Delta'_{\mathscr{B}}$ with δ_{B_1} as a vertex, on which δ_a^* is decreasing. Let δ_B be the second vertex of l. Since $\delta_a^*(\delta_B) < \delta_a^*(\delta_{B_1})$, we have $\delta_a^*(\delta_B) = 0$, that is, $a \notin B$. But l is parallel to $\delta_b - \delta_a$, so $B = (B_1 \setminus a) \cup b$. Here $b \in B_1 \cup B_2$, so $b \in B_2$. This proves the assertion.

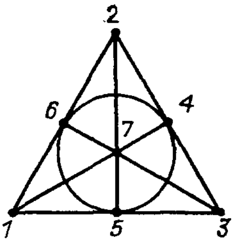

Fig. 6

Example. Consider the matroid corresponding to the configuration of all points in the projective plane over the field F_2 of two elements (Fig. 6). The configuration is not realizable over **C** or **R**. The corresponding hypersimplex Δ_F has full symmetry group $\mathrm{PGL}(3, F_2)$ and the following numbers of faces in each dimension:

Dimension of face	0	1	2	3	4	5	6
Number of faces	28	126	245	238	112	21	1

Note that to each face there corresponds a configuration of points in $\mathbf{R}P^2$, that is, each face is an admissible polytope. All the types of configurations obtained in this way are depicted in Fig. 7.

By taking configurations of points in projective space over a finite field F_q, we obtain a series of hypersimplexes $\Delta(m, F_q)$ with full symmetry group $\mathrm{PGL}(m, F_q)$.

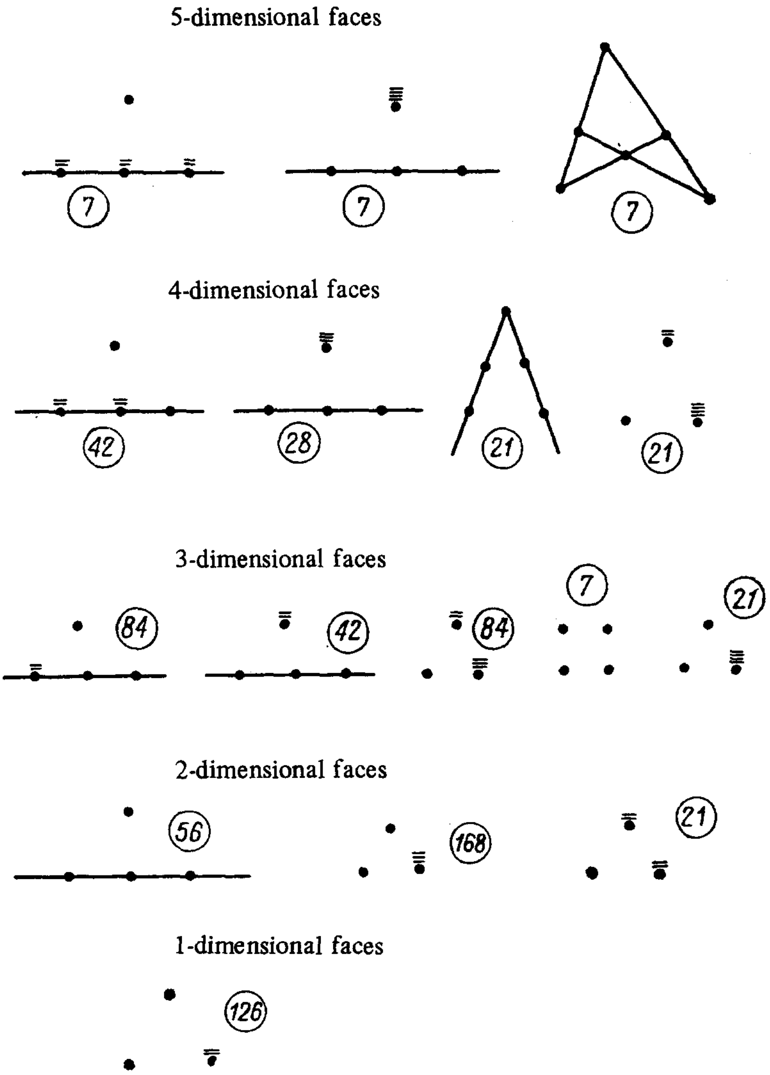

Fig. 7. The numbers in circles are the numbers of faces of a given type

2.3. Operations on matroids.

The *restriction* of a matroid (S, r) to $A \subseteq S$ is the matroid (A, r) obtained by restricting the rank function r of (S, r) to $B(A)$. The matroid (A, r) is called a submatroid of (S, r).

The *contraction* $(S, r)/A$ of a matroid (S, r) over $A \subseteq S$ is the matroid $(S\backslash A, \bar{r})$ with rank function $\bar{r}$ defined on $B(S\backslash A)$ as follows: $\bar{r}(J) = r(J \cup A) - r(A)$. The matroid $(S, r)/A$ is called a quotient matroid.

Suppose that Γ is a stratum in $G_k(\mathbf{C}^n)$, (I_n, r_Γ) is the corresponding matroid, $J \subseteq I_n$, and $r(J) = p$. Consider the strata $\Gamma_1 \subset G_p(\mathbf{C}^J)$ and $\Gamma_2 \subset G_{k-p}(\mathbf{C}^{I_n \setminus J})$ obtained by projecting Γ onto $\mathbf{C}^J$ along $\mathbf{C}^{I_n \setminus J}$ and by intersecting it with $\mathbf{C}^{I_n \setminus J}$ respectively. Then the matroids (J, r_{Γ_1}) and $(I_n \setminus J, r_{\Gamma_2})$ are isomorphic to the restriction of (I_n, r_Γ) to J and its contraction over J respectively.

Suppose that (S, r) is a matroid of rank k and $|S| = n$. The matroid (S, r^*) of rank $n-k$ with rank function $r^*(A) = |A| - k + r(S\backslash A)$ is called the *dual* matroid to (S, r). The duality between the strata $\Gamma \subset G_k(\mathbf{C}^n)$ and $\Gamma^* = \{X^\perp \in G_{n-k}((\mathbf{C}^n)^*)$, where $X \in \Gamma\}$ clearly corresponds to a duality of matroids.

Let (S_1, r_1) and (S_2, r_2) be given matroids. Then their product $(S_1, r_1) \times (S_2, r_2)$ is the matroid $(S_1 \cup S_2, r)$ with rank function $r(J) = r_1(J \cap S_1) + r(J \cap S_2)$.

If Γ_1 and Γ_2 are arbitrary strata in $G_p(\mathbf{C}^n)$ and $G_q(\mathbf{C}^m)$, then

$$\Gamma_1 \times \Gamma_2 = \{X \in G_{p+q}(\mathbf{C}^n \oplus \mathbf{C}^m) \mid X = X_1 \oplus X_2 \quad \text{for} \quad X_1 \in \Gamma_1, \quad X_2 \in \Gamma_2\}$$

is a stratum. Clearly $(I_n \cup I_m, r_{\Gamma_1 \times \Gamma_2}) = (I_n, r_{\Gamma_1}) \times (I_m, r_{\Gamma_2})$.

2.4. Dimension of a hypersimplex.

Let Δ_r be the hypersimplex corresponding to the matroid (S, r). We shall compute its dimension. To do this, we need a few more concepts from matroid theory.

A subset $U \subseteq S$ is called a *separator* of (S, r) if $(S, r) = (U, r) \times (S\backslash U, r)$, that is, $r(A) = r(A \cap U) + r(A\backslash U)$ for all $A \subseteq S$.

Proposition 3 (see [6]). *The union and the intersection of separators, and the complement of every separator in S, is also a separator. The collection of non-empty minimal separators forms a partition of S.*

Let $K(S, r)$ denote the number of minimal non-empty separators of (S, r). A matroid (S, r) is said to be *connected* if $K(S, r) = 1$.

***Proposition* 4.** *Let (S, r) be a matroid with collection of bases $\mathscr{B}$ and rank function r, and let Δ_r be the corresponding hypersimplex. Then* $\dim \Delta_r = -K(S, r) + |S|$.

Proof. Consider the subspace W of $\mathscr{L}(S)$ spanned by vectors $\alpha - \beta$, where α and β are vertices joined by an edge. It follows from Theorem 1 that every edge of Δ_r has the form $\delta_a - \delta_b$. We introduce an equivalence relation on $S : a \sim b$ if and only if $\delta_a - \delta_b$ is an edge of Δ_r. Consider the partition of S

into a disjoint union of equivalence classes $J_1, \ldots, J_q$. Clearly $\dim W = \dim \Delta_r = n-q$, and W is defined as a subspace by q equations

$$\sum_{b \in J_i} \xi_b = k_i \qquad (i = 1, \ldots, q).$$

Let $(J_i, r_i) = (J_i, r)$, let Δ_{r_i} be the hypersimplex corresponding to (J_i, r_i), and let $\mathscr{B}_i$ be its collection of bases. Clearly (J_i, r_i) is a matroid of rank k_i, and for any $B \in \mathscr{B}$ we have $B \cap J_i \in \mathscr{B}_i$. Hence $\Delta_r \subseteq \Delta_{r_1} \times \ldots \times \Delta_{r_q}$. If $\alpha_1 \times \ldots \times \alpha_q \in \Delta_r$ is a vertex, and β_i is a vertex of Δ_{r_i} joined by an edge to α_i, then $\alpha_1 \times \ldots \times \beta_i \times \ldots \times \alpha_q \in \Delta_r$ because Δ_r is a hypersimplex. From this it is easy to deduce that $\Delta_r = \Delta_{r_1} \times \ldots \times \Delta_{r_q}$, that is,

$$\mathscr{B} = \{B_1 \cup \ldots \cup B_q \mid B_i \in \mathscr{B}_i\}.$$

Thus $(S, r) = (J_1, r_1) \times \ldots \times (J_q, r_q)$. So $J_1, \ldots, J_q$ are separators of (S, r).

To the matroid (J_i, r_i) there corresponds a hypersimplex of maximal dimension. We show that (J_i, r_i) is connected. Indeed, suppose U is a separator of (J_i, r_i), $J_i \neq U$, $\emptyset \neq U$. Then the function $\chi(\xi) = (\delta_U, \xi)$ is constant on Δ_{r_i}, which contradicts the assumption that Δ_{r_i} has maximal dimension.

Thus $J_1, \ldots, J_q$ are the minimal separators of (S, r), $q = K(S, r)$, and $\dim \Delta_r = n-q$.

Note that Proposition 4 of §1 follows from Proposition 4 of this section.

2.5. Description of the faces of a hypersimplex.

Let (S, r) be a connected matroid. A non-empty subset $A \subset S$ is called *non-degenerate* if $(S, r)/A$ and (A, r) are connected.

***Theorem* 2.** *Let (S, r) be a connected matroid, and Δ_r its hypersimplex. The collection of faces of codimension 1 of Δ_r is in one-to-one correspondence with the set of non-degenerate subsets A of S. To the non-degenerate subset A there corresponds the face Δ_{r_A}, where $(S, r_A) = (A, r) \times (S, r)/A$.*

Proof. As we showed in the proof of the previous theorem, a face of Δ_r is defined by an equation $\sum_{b \in A} \xi_b = k_1$, where k_1 is the maximum value of the function (δ_A, ξ) on Δ_r. Clearly this face is the hypersimplex corresponding to some matroid (S, r_A). The set $\mathscr{B}_A$ of bases of (S, r_A) consists of elements of the form $B_1 \cup B_2$, where B_1 and B_2 are bases of (A, r) and $(S, r)/A$ respectively. Thus, $(S, r_A) = (A, r) \times (S, r)/A$. Since $\dim \Delta_{r_A} = |S| - 2$, we have $K(S, r) = 2$. Hence A is non-degenerate.

2.6. The exchange axiom and ordering on $B(I_n)$.

We introduce a partial order on $B_k(I_n)$ as follows. Let $A, B \in B_k(I_n)$, where $A = \{i_1, \ldots, i_k\}$, $i_1 < \ldots < i_k$ and $B = \{j_1, \ldots, j_k\}$, $j_1 < \ldots < j_k$. Then $A \leqslant B$ if and only if $i_p \leqslant j_p$ for all $p = 1, \ldots, k$.

Let S_n be the group of permutations of the elements of I_n. Then we can associate an ordering of $B_k(I_n)$ with each $s \in S_n$ by putting $A \overset{s}{\leqslant} B$ if and only if $s^{-1}(A) \leqslant s^{-1}(B)$. Clearly $\overset{e}{\leqslant}$ is just $\leqslant$.

Suppose that $\mathscr{B} \subset B_k(I_n)$. An element $B \in \mathscr{B}$ is called s-minimal in $\mathscr{B}$ if $B \overset{s}{\leqslant} J$ for all $J \in \mathscr{B}$. We say that $\mathscr{B}$ satisfies the minimality condition if there is an s-minimal element in $\mathscr{B}$ for all $s \in S_n$.

Proposition 5. *The exchange condition in Proposition* 2 *is equivalent to the minimality condition.*

We shall need this reformulation of the exchange condition later. The minimality condition is equivalent to the convergence condition for the greedy algorithm (see [21]).

§3. The moment mapping and toric varieties

3.1. Let M be a real symplectic variety, that is, a non-degenerate closed 2-form ω is defined on M. Suppose that a compact Lie group K with Lie algebra $\mathfrak{k}$ acts on M, preserving the form ω. Let TM denote the Lie algebra of vector fields on M, and $\Omega^1 M$ the set of 1-forms on M. Then there is a natural Lie algebra homomorphism $a\colon \mathfrak{k} \to TM$ and a dual map $a^*\colon \Omega^1 M \to \mathfrak{k}^*$.

Furthermore, the 2-form ω defines, at each point $X \in M$, an isomorphism between the tangent space $T_X M$ and the cotangent space $\Omega^1_X M$ as follows: $\omega_X(v)(v') = \omega_X(v, v')$ for all $v' \in T_X M$.

For any smooth mapping $\theta : M_1 \to M_2$, let $d\theta_X$ denote the differential of θ at X.

Definition 1. A mapping $\mu\colon M \to \mathfrak{k}^*$ is called a *moment mapping* if

1) μ is equivariant under the action of K;

2) $d\mu_X = a^* \circ \omega_X$ at each point $X \in M$.

Remark. From now on we shall always consider the case where M is a complex manifold with a K-invariant Kähler metric χ. Then by definition the imaginary part ω of χ is a symplectic form on M, regarded as a real manifold. Clearly ω is invariant under the action of K. Hence we can define a moment mapping in this case.

Example. Let $K = U(n)$ and $M = \mathbf{C}P^n$. Clearly there is a K-invariant Kähler form χ on M given by $\chi = \sum dz_i\, d\bar{z}_i / \sum z_i \bar{z}_i$ in homogeneous coordinates. We put $z_i = x_i + \sqrt{-1}y_i$. Then the 2-form $\omega = \operatorname{Im} \chi$ is given by $\omega = \sum dx_i \wedge dy_i / \sum |z_i|^2$. We identify $\mathfrak{k}^*$ with the set of Hermitian matrices of order $n+1$. Then the moment mapping μ sends a point $X \in \mathbf{C}P^n$ with coordinates $(z_0, z_1, \ldots, z_n)$ to the Hermitian matrix (z_{ij}), where $z_{ij} = z_i \bar{z}_j / \sum |z_i|^2$.

Let K be an arbitrary compact Lie group and M a symplectic manifold on which K acts transitively. Then the moment mapping μ: $M \to \mathfrak{k}^*$ enables us to identify M with some orbit of K in its coadjoint representation on $\mathfrak{k}^*$.

3.2. Convexity theorem.

Let M be a compact Kähler manifold. Suppose that a complex group H isomorphic to $(\mathbf{C}^*)^n$ acts on M so that the Kähler metric χ is invariant with respect to the compact form T of H. Suppose we are given a moment mapping $\mu : M \to \mathbf{R}^n$ (in this case the Lie algebra of T is isomorphic to $\mathbf{R}^n$).

***Theorem* 1.** *The image of M under the moment mapping is a convex polytope. The closure $\overline{H\cdot X}$ of a general orbit of H in M is mapped onto $\mu(M)$. The mapping μ induces a one-to-one correspondence between the set of orbits of H in $\overline{H\cdot X}$ and the set of faces of the polytope $\mu(M)$, whereby a q-dimensional (over $\mathbf{C}$) orbit of H is mapped onto an open q-dimensional (over $\mathbf{R}$) face of $\mu(M)$.*

For a proof see [2] and [3].

Example. Let $M = \mathbf{C}P^n$ and let H be the subgroup of diagonal matrices in $SL(n+1)$. Then $\mu(M)$ is an n-dimensional simplex in $\mathbf{R}^n$.

Remark. Theorem 1 also holds for compact Kähler manifolds with singularities.

3.3. Toric varieties.

A complex algebraic variety M is called *toric* if the group $H = (\mathbf{C}^*)^n$ acts on it in such a way that there is a single open everywhere dense orbit isomorphic to H.

We assume that there is a Kähler metric χ on the compact toric variety M, which is invariant under the action of the compact form T of H. Suppose we are given a moment mapping $\mu : M \to \mathbf{R}^n$. Then clearly all the assertions of Theorem 1 apply to M (see [5]).

A *fan* Σ in $\mathbf{R}^n$ is a collection of convex polyhedral cones in $\mathbf{R}^n$ such that Σ contains all the faces of each of its cones, and any two cones can intersect only in a common face. A fan Σ is called integral if all the one-dimensional cones in Σ are spanned by vectors with integer coordinates.

It is known (see [1]) that any toric variety is defined by an integral fan Σ_M. One can show that the fan Σ_M is dual to the polytope $\mu(M)$, that is, every cone of Σ_M in $\mathbf{R}^n$ consists of linear functions that attain their maximum on some face of $\mu(M)$.

The moment mapping enables us to reformulate many of the properties of compact toric varieties in terms of convex polytopes (see [5]).

A convex polytope in $\mathbf{R}^n$ is called *simple* if precisely n edges emanate from each vertex.

***Proposition* 1.** *A toric variety has no singularities if and only if its image under the moment mapping is a simple polytope.*

See [5] for a proof.

The *F-polynomial* of a simple n-dimensional polytope Δ is the generating polynomial: $F = \sum_{0 \leqslant i \leqslant n} F_i t^i$, where F_i is the number of i-dimensional faces of the polytope. We put $H(t) = F(t-1)$. The H-polynomial of $\mu(M)$ enables us to compute the Betti numbers of M.

***Proposition* 2.** *The polynomial H of a simple polytope is symmetric, that is, $H_i = H_{n-i}$. A smooth toric variety M has only even non-zero Betti numbers b_{2i}, where $b_{2i} = H_i$ for the H-polynomial of $\mu(M)$.*

A proof of Proposition 2 is given in [5]. It uses the following beautiful theorem. Let φ be a general linear function for a polytope Δ, that is, φ is not constant on any edge of Δ. The index $i_\varphi(\alpha)$ of a vertex α of the simple polytope Δ is the number of edges emanating from α on which φ is decreasing. Clearly $0 \leqslant i_\varphi(\alpha) \leqslant n$. Let h_i denote the number of vertices of index i.

***Proposition* 3.** *The H-polynomial of a simple polytope has the form*

$$H(t) = \sum_{0 \leqslant i \leqslant n} h_i t^i.$$

Example. Let G be a complex semisimple Lie group, and B a Borel subgroup. Let $M = G/B$ and let H be a Cartan subgroup of G. Then M admits a Kähler metric invariant with respect to the compact form T of H. We can define a moment mapping $\mu : M \to \mathfrak{h}_{\mathbf{R}}^*$, where $\mathfrak{h}_{\mathbf{R}}^*$ is the real form of the Lie algebra $\mathfrak{h}$ of H. Let $\Delta = \mu(M)$. In 5.1 we show that Δ is the convex hull of the points $W \cdot \lambda$, where $\lambda \in \mathfrak{h}_{\mathbf{R}}^*$ is a point in general position, and $W \cdot \lambda$ is the orbit of λ under the action of the Weyl group W.

Consider a general orbit $H \cdot X$ of H in M. Then $\mu(\overline{H \cdot X}) = \Delta$. Every vertex α of Δ lies in a Weyl chamber, and hence uniquely determines a system of simple roots $\sigma_1(\alpha), \ldots, \sigma_n(\alpha)$ in $\mathfrak{h}_{\mathbf{R}}^*$, defined by the condition $(\sigma_i(\alpha), \alpha) > 0$ for all $i = 1, \ldots, n$. It is easy to verify that all edges of Δ that contain α have as their other vertex a point of the form $r_{\sigma_i(\alpha)}(\alpha)$, where r_σ is reflection with respect to the root σ. Hence Δ is simple, and so $\overline{H \cdot X}$ is a smooth compact variety. We compute its Betti numbers, using Proposition 3. (This was done in [12].)

Let α be an arbitrary vertex of Δ, let $\sigma_1, \ldots, \sigma_n$ be the associated system of simple roots, and let $R^{(\pm)}$ be the set of positive (negative) roots in this system. Any other vertex of Δ has the form $w(\alpha)$ for some $w \in W$. Let $\varphi(\xi) = (\xi, \alpha)$ for all $\xi \in \mathfrak{h}_{\mathbf{R}}^*$. We compute the index of $w(\alpha)$. The edges emanating from $w(\alpha)$ are multiples of the roots $w(\sigma_j)$. Hence

$$i_\varphi(w(\alpha)) = |\{\sigma_j \mid (w(\sigma_j), \alpha) < 0\}| = |\{\sigma_j \mid w(\sigma_j) \in R^-\}|.$$

Thus the coefficient h_i of the H-polynomial is the number of elements of the Weyl group that send precisely i simple roots to negative roots.

§4. The geometry of compact homogeneous spaces

4.1. Parabolic subgroups.

Let G be a complex semisimple Lie group with Lie algebra $\mathfrak{G}$ and let H be a fixed Cartan subgroup of G. Let $\mathfrak{h}$ denote the Lie algebra of H and $\mathfrak{h}_{\mathbf{R}}$ its real part. We also fix a Borel subgroup B containing H. The choice of B defines a simple root system $\sigma_1, \ldots, \sigma_n$ in $\mathfrak{h}_{\mathbf{R}}$ for the Lie algebra $\mathfrak{G}$. A closed subgroup of G containing B is called a *standard parabolic subgroup.*

Any compact homogeneous space M with respect to the action of G has the form G/P, where P is some standard parabolic subgroup of G.

Let W be the Weyl group of G, that is, the subgroup of motions of $\mathfrak{h}^*_{\mathbf{R}}$ generated by the reflections $r_1, \ldots, r_n$ with respect to the roots $\sigma_1, \ldots, \sigma_n$. Let S be some subset of the simple roots. The subgroup $W_S \subset W$ generated by the elements r_i for all $s_i \in S$ is called a *parabolic* subgroup of W.

Let $N(H)$ be the normalizer of H in G. Then W is canonically isomorphic to $N(H)/H$. Let P be a standard parabolic subgroup of G. It can be shown that $(N(H) \cap P)/H$ is a parabolic subgroup of W. Every standard parabolic subgroup P can be uniquely recovered from the intersection $(N(H) \cap P)/H$. Thus there are one-to-one correspondences between standard parabolic subgroups in G, parabolic subgroups in W, and subsets of the simple roots. Let us agree to denote the subset of simple roots corresponding to a standard parabolic subgroup P by S_P, and the parabolic subgroup W_{S_P} by W_P.

4.2. Generalized Plücker coordinates.

Let $P \subset G$ be a standard parabolic subgroup. Consider a finite-dimensional irreducible representation ρ of G on a space V with principal weight $\lambda = \sum_{\sigma_i \in S_P} \omega_i$, where ω_i is the fundamental weight corresponding to the simple root σ_i. Then the homogeneous manifold $M = G/P$ can be identified with the orbit of a principal vector in the projectivization $P(V)$ of V.

Let $\mathcal{A}$ be the set of weights of ρ taken with multiplicity. We choose a weight basis $\{e_\alpha \mid \alpha \in \mathcal{A}\}$ in V. Then any point $X \in G/P$ determines, uniquely up to a scalar d, a collection of numbers $p^\alpha(X)$, where $X = d \cdot \sum_{\alpha \in \mathcal{A}} p^\alpha(X) e_\alpha$.

Let $W \cdot \lambda$ be the orbit of the principal weight λ in $\mathfrak{h}^*_{\mathbf{R}}$ under the action of W. Then $W \cdot \lambda$ can be identified with the set W^P of left cosets W/W_P. The points of $W \cdot \lambda$ lie at the vertices of some convex polytope $\Delta_P \in \mathfrak{h}^*_{\mathbf{R}}$. The other points of $\mathcal{A}$ lie inside Δ_P (see [2]).

The numbers $\{p^\alpha(X) \mid \alpha \in W \cdot \lambda\}$, defined up to a scalar, are called the *generalized Plücker coordinates* of $X \in M$.

Example 1. Let $G = \mathrm{SL}(n)$ and $M = G_k(\mathbf{C}^n)$. Then $W \cong S_n$, $W_P \cong S_k \times S_{n-k}$, and $V = \Lambda^k E$, where E is the canonical representation of $\mathrm{SL}(n)$. The set $\mathcal{A} = W^P$ can be identified with $B_k(I_n)$. The basis elements e_α are $e_{i_1} \wedge \cdots \wedge e_{i_k}$, where $\alpha = \{i_1, \ldots, i_k\}$, and $p^\alpha(X)$ are the ordinary Plücker coordinates of $X \in G_k(\mathbf{C}^n)$ (see 1.5).

Example 2. Let $G = \mathrm{SL}(n)$ and $P = B$. Then M is the manifold of complete flags in $\mathbf{C}^n$. In this case $W_P = \{1\}$, and $W^P \cong S_n$. The elements of $W \cdot \lambda$ can thus be identified with permutations. Hence the generalized Plücker coordinates of a flag are indexed by permutations. Let X be a flag of subspaces $X_1 \subset \ldots \subset X_n$, then $p^{(j_1,\ldots,j_n)}(X) = p^{j_1}(X_1), \ldots , p^{j_1,\ldots,j_n}(X_n)$ for any permutation $(j_1, \ldots, j_n) \in S_n$.

4.3. Kähler structures on *G/P*.

We choose a compact form K of G that contains the compact form T of H. Then there is a K-invariant Hermitian metric $\overline{\chi}$ on V. It induces a Kähler metric on the projective space $P(V)$. We denote the restriction to M by χ. The metric χ defines a Kähler structure on M.

It is easy to show that we can choose a weight basis $\{e_\alpha\}$ in V such that $\overline{\chi}(\sum p_1^\alpha e_\alpha, \sum p_2^\alpha e_\alpha) = \sum p_1^\alpha \overline{p_2^\alpha}$, where $\alpha \in \mathcal{A}$.

From now on we shall always use the weight basis chosen in this way.

4.4. Definition of stratum.

Let $X \in G/P$. The *list* of X is the subset $L_X \subset W \cdot \lambda$ defined by $L_X = \{\alpha \in W \cdot \lambda \mid p^\alpha(X) \neq 0\}$. Two points X and Y in M are said to be equivalent if $L_X = L_Y$.

Definition 1. The equivalence classes Γ in M are called *strata.*

The set $L_\Gamma = L_X$ for any $X \in \Gamma$ is called the list of Γ.

Note that the definition of the list L_X does not depend on the choice of weight basis in V, because the weights $\alpha \in W \cdot \lambda$ have no multiplicity.

§5. Definition of a stratum via the moment mapping

5.1. The moment mapping in the case $M = G/P$.

In 4.2 we constructed a Kähler form χ on $M = G/P$. The imaginary part of χ is a symplectic form on M and invariant under the action of K, in particular under the action of the compact part T of H. We identify the Lie algebra of T with $\mathfrak{h}_{\mathbf{R}}$. It is not hard to verify that the mapping $\mu \colon M \to \mathfrak{h}_{\mathbf{R}}^*$ given by $\mu(X) = \sum_{\alpha \in \mathcal{A}} | p^\alpha(X) |^2 \cdot \alpha / \sum_{\alpha \in \mathcal{A}} | p^\alpha(X) |^2$, where $X = d \cdot \sum_{\alpha \in \mathcal{A}} p^\alpha(X) e_\alpha$, is a moment mapping.

5.2. The images of orbits of *H*.

Let $X \in M$. Let $H \cdot X$ denote the orbit of X under the action of H, and $\overline{H \cdot X}$ its closure.

Proposition 1. *The image $\mu(\overline{H \cdot X})$ is a convex polytope with vertices at the points α for all $\alpha \in L_X$.*

Proof. Since $\overline{H \cdot X}$ is a compact Kähler variety (possibly with singularities), Theorem 1 of §3 applies to it. Hence $\mu(\overline{H \cdot X})$ is a convex polytope, and all its vertices are the images of fixed points of $\overline{H \cdot X}$ under the action of H.

Let v_α denote the line spanned by the weight vector e_α. Then the only fixed points of H in M are v_α, where $\alpha \in W\cdot\lambda$. Since $\mu(v_\alpha) = \alpha$, it remains to see when we have $v_\alpha \in \overline{H\cdot X}$. Clearly, if $p^\alpha(X) = 0$ then $p^\alpha(Y) = 0$ for all $Y \in \overline{H\cdot X}$. Suppose that $p^\alpha(X) \neq 0$. Note that, since $\mathcal{A} \subseteq \Delta_P$, there exists $h \in \mathfrak{h}_{\mathbf{R}}$ such that $\alpha(h) > \beta(h)$ for all $\beta \in \mathcal{A}$, $\beta \neq \alpha$. Consider the one-parameter subgroup $h_t = \exp ht$ of H. A simple calculation shows that $\lim_{t\to\infty} h_t(X) = v_\alpha$. Hence $v_\alpha \in \overline{H\cdot X}$.

We say that two orbits $H\cdot X$ and $H\cdot Y$ are equivalent if $\mu(H\cdot X) = \mu(H\cdot Y)$. Then immediately from Proposition 1 we have the following result.

Proposition 2. *A class of equivalent orbits of H in M is a stratum.*

§6. Strata and Schubert cells

6.1. Schubert cells in $M = G/P$.

Let C be a Borel subgroup of G containing H. The Schubert cells on $M = G/P$ associated with C are the orbits of C in M. In other words, a Schubert cell is a double coset in $C\backslash G/P$. It is known (see, for example, [9]) that there are as many Schubert cells as cosets W/W_P, and hence they have the form $C\cdot\alpha\cdot P$, where $\alpha \in W^P = W/W_P$. For brevity we denote the Schubert cell $C\cdot\alpha\cdot P$ by $C\cdot\alpha$.

6.2. The connection between strata and Schubert cells.

For each Borel subgroup $C \supset H$ we choose precisely one Schubert cell in M associated with C. We call the intersection of all the chosen cells a *thin cell* if it is non-empty.

Theorem 1. *The partition of M into thin cells is the same as the partition into strata.*

Proof. Suppose that two points X and Y lie in the same thin cell. We prove that $L_X = L_Y$. Suppose not; then there is an $\alpha \in W\cdot\lambda$ such that $p^\alpha(X) = 0$ but $p^\alpha(Y) \neq 0$. We choose a Borel subgroup $C \supset H$ with respect to which α has least weight. Then $p^\alpha(c(X)) = 0$ for all $c \in C$. Thus Y and X cannot lie in the same orbit of C, and hence not in the same thin cell. Conversely, suppose that $X, Y \in M$ and $L_X = L_Y$. Assume that X and Y lie in distinct thin cells. Then there is a Borel subgroup C, containing H, such that $X \in C\cdot\alpha$, $Y \in C\cdot\beta$, $\alpha \neq \beta$. Let $\sigma_1, \ldots, \sigma_n$ be the system of simple roots of the Lie algebra $\mathfrak{G}$ connected with the given Borel subgroup C. There is an $h \in \mathfrak{h}_{\mathbf{R}}$ such that $\sigma_i(h) < 0$ for all $i = 1, \ldots, n$ and $\alpha(h) \neq \beta(h)$. We put $h_t = \exp h\cdot t$. Then $\lim_{t\to\infty} h_t c h_t^{-1}$ exists and lies in H for all $c \in C$. Hence $\lim_{t\to\infty} h_t(X) = v_\alpha$, and $\lim_{t\to\infty} h_t(Y) = v_\beta$. But it follows from the proof of Proposition 1 of §5 that $\lim_{t\to\infty} h_t(X) = v_\gamma$, where $\gamma \in L_X$ is such that $\gamma(h) > \delta(h)$ for $\delta \in L_X$, $\delta \neq \gamma$. Since $L_X = L_Y$, we have $\gamma = \alpha = \beta$, contrary to hypothesis.

§7. Polytopes corresponding to strata

7.1. We study in more detail the combinatorial properties of the polytopes corresponding to toric varieties in G/P.

A polytope Δ in $\mathfrak{h}_{\mathbf{R}}^*$ is said to be *admissible* for G/P if it is the image of the closure of some orbit $H\cdot X$ in $M = G/P$ under the moment mapping $\mu\colon M \to \mathfrak{h}_{\mathbf{R}}^*$. It follows from Proposition 1 of §5 that $\operatorname{Vert}\Delta \subset W\cdot\lambda$ for any admissible polytope Δ, where $W\cdot\lambda$ is the orbit of the principal weight λ of the representation ρ under the action of the Weyl group, and $\operatorname{Vert}\Delta$ is the set of vertices of Δ.

Definition 1. A polytope Δ with vertices in $W\cdot\lambda$ is called a *(G, P)-hypersimplex* if all its edges are parallel to roots of the Lie algebra $\mathfrak{G}$ of G.

Definition 1 is a generalization of the definition of hypersimplex for the Grassmannian $G_k(\mathbf{C}^n)$ given in 2.4.

Theorem 1. *Every admissible polytope for G/P is a (G, P)-hypersimplex.*

Proof. Let Δ be an admissible polytope. Then any edge l of Δ is also admissible. Consider an orbit $H\cdot X \subset M$ such that $\mu(\overline{H\cdot X}) = l$. Let H_X be the subgroup of H that fixes X, and $\mathfrak{h}_X$ the corresponding Lie subalgebra. Clearly $\dim\mathfrak{h}_X = \dim\mathfrak{h} - 1$, and the subspace $\mathfrak{h}_X$ of $\mathfrak{h}$ is defined by the equation $\gamma(h) = \beta(h)$ for all $h \in \mathfrak{h}_X$, where β and γ are the vertices of l.

On the other hand, to a point $X \in G/P$ there corresponds a coset gP. Then the condition $h \in \mathfrak{h}_X$ can be expressed in the form $(\exp ht)\cdot g \in g\cdot P$. Since $\exp ht \in P$ we have

$$(1) \qquad (\exp ht)\cdot g(\exp ht)^{-1} \in gP$$

for any $t \in C$.

Let $\mathfrak{P}$ be the Lie algebra of P, R a root system of $\mathfrak{G}$, g_α a root vector in $\mathfrak{G}$ of weight α, and $R^P = \{\alpha \in R \mid g_\alpha \in \mathfrak{P}\}$. Then every $g \in G$ can be expressed in the form $g = \prod_{\alpha\in R\setminus R^P}(\exp g_\alpha)\cdot p_0$, where $p_0 \in P$ and g_α is some vector, possibly zero. We put $g_0 = \prod_{\alpha\in R\setminus R^P}\exp g_\alpha$. Then condition (1) is equivalent to the following condition on g_0: $(\exp ht)\cdot g_0\cdot(\exp ht)^{-1} = g_0$.

This can be rewritten as

$$\begin{aligned}\prod_{\alpha\in R\setminus R^P}(\exp ht)\cdot\exp g_\alpha\cdot(\exp ht)^{-1} &= \prod_{\alpha\in R\setminus R^P}\exp\,[ht,\ g_\alpha] \\ &= \prod_{\alpha\in R\setminus R^P}\exp\,\alpha(h)t\cdot g_\alpha \\ &= \prod_{\alpha\in R\setminus R^P}\exp\,g_\alpha.\end{aligned}$$

Since this equality must hold for any $t \in C$, it is clear that $\alpha(h) = 0$ for $\alpha \in R\setminus R^P$, $g_\alpha \neq 0$. Since R has no repeated vectors and $\dim\mathfrak{h}_X = \dim\mathfrak{h} - 1$, only one vector g_α is distinct from 0. It follows that the weight of $\beta-\gamma$ is a multiple of α, which proves Theorem 1.

Theorem 1 enables us to state a necessary condition for a polytope with vertices in $W\cdot\lambda$ to be admissible. As we showed in 2.2, in the case $M = G_k(\mathbf{C}^n)$ this condition is equivalent to the exchange axiom for bases of a matroid.

§8. The general (W, Q)-matroid

8.1. Definition of a (W, Q)-matroid.

Let W be a Coxeter group. We recall that a Coxeter group is a group with a finite set of generators R, subject to the relations: $r^2 = 1$ for all $r \in R$, $(r_ir_j)^{m(r_ir_j)} = 1$ for all $r_i, r_j \in R$, where $m(r_i, r_j) \in \mathbf{N} \cup \infty$.

Let $w \in W$. The minimal number of factors in a factorization $w = r_1, \dots, r_l$, where $r_i \in R$, is called the length of w and denoted by $l(w)$. The Bruhat partial order on W is defined as follows: $w_1 \leqslant w_2$ if there exist s_1 and $s_2 \in W$ such that $w_2 = s_1w_1s_2$ and $l(w_2) = l(s_1)+l(w_1)+l(s_2)$.

With each $w \in W$ we associate a new ordering on W thus: $w_1 \overset{w}{\leqslant} w_2$ if $w^{-1}w_1 \leqslant w^{-1}w_2$. Clearly the Bruhat ordering introduced above coincides with $\overset{1}{\leqslant}$.

Let L be an arbitrary subset of W. An element $s \in L$ is called w-minimal in L if $s \overset{w}{\leqslant} u$ for all $u \in L$. We shall say that L satisfies the minimality condition if there is a w-minimal element in L for all $w \in W$.

Definition 1. A *flag W-matroid* is a pair (W, L), where W is a Coxeter group and L is a subset of W satisfying the minimality condition.

The set L is called the set of bases of the flag matroid (W, L).

Let Q be an arbitrary subset of R. Let W_Q denote the subgroup of W generated by Q. The subgroups W_Q are called parabolic subgroups of W. Let W^Q denote the set of left cosets W/W_Q. Let $\alpha \in W^Q$; then α satisfies the minimality condition as a subset of W. For any class $\alpha \in W^Q$, let α_w denote the w-minimal element of α. We introduce a partial ordering $\overset{w}{\leqslant}$ on W^Q by putting $\alpha \overset{w}{\leqslant} \beta$ if $\alpha_w \overset{w}{\leqslant} \beta_w$. Then the minimality condition makes sense for subsets $L \subseteq W^Q$.

Definition 2. A *(W, Q)-matroid* is a triple (W, Q, L), where W is a Coxeter group, Q is a subset of generators, and L is a subset of W^Q satisfying the minimality condition.

The set L is called the set of bases of the given (W, Q)-matroid.

A flag W-matroid (W, L) is said to be W_Q-invariant if $W_Q\cdot L = L$.

Proposition 1. *Let W be a Coxeter group and $Q \subset R$. The natural projection from W onto W^Q induces a one-to-one correspondence between W_Q-invariant flag W-matroids and (W, Q)-matroids.*

8.2. Examples.
Example 1. Let $W = S_n$ be the group of permutations of $I_n = \{1, \ldots, n\}$, let R be the set of transpositions $(i, i+1)$, $i = 1, \ldots, n$, and let $Q = R \setminus (k, k+1)$. Then $W_Q = \{w \in W \mid w(I_k) = I_k\}$. We show that in this case the definition of a (W, Q)-matroid coincides with that of an ordinary matroid.

The set W^Q of left cosets W/W_Q can be naturally identified with the set $B_k(I_n)$ of all k-element subsets of I_n. Here the Bruhat ordering in W^Q takes the following form: for all $A, B \in B_k(I_n)$, $A = \{a_1, \ldots, a_k\}$, $B = \{b_1, \ldots, b_k\}$, where $a_1 < \ldots < a_k$ and $b_1 < \ldots < b_k$, $A \leqslant B$ if $a_i \leqslant b_i$ for all $i = 1, \ldots, k$ (see 2.6). Any element $w \in W$ defines a new linear ordering in $I_n : w(1) < \ldots < w(n)$, which defines an ordering $\overset{w}{\leqslant}$ in $B_k(I_n)$.

Proposition 2. *Suppose that $L \subseteq B_k(I_n) = W^Q$. A triple (W, Q, L) is a (W, Q)-matroid if and only if L is the set of bases of some matroid (I_n, r) of rank k.*

The proof follows immediately from Proposition 5 of §2.

Example 2. Let $W = S_n$ as before, and let

$$Q = Q_l = \{(l+1, l+2), \ldots, (n-1, n)\}.$$

In this case the notion of (W, Q_l)-matroid is closely related to that of greedoid (see [20], [21]).

Suppose we are given some set S of letters (an alphabet). We consider only the case $|S| < \infty$ and identify S with I_n. A word is an arbitrary sequence of letters from S. The length of a word α is denoted by $|\alpha|$. An arbitrary collection $\mathscr{L}$ of words is called a language. A language $\mathscr{L}$ is called a *greedoid* if the following conditions are satisfied:

1) $\varnothing \in \mathscr{L}$;

2) any word $\alpha \in \mathscr{L}$ contains no repeated letters;

3) for any $\alpha \in \mathscr{L}$, if $\alpha = \beta\gamma$, then $\beta \in \mathscr{L}$;

4) if $\alpha, \beta \in \mathscr{L}$, and $|\alpha| < |\beta|$, then there is an $x \in \beta$ such that $\alpha x \in \mathscr{L}$.

It follows from axiom 4) that all the maximal words of a greedoid have the same length, which we call the rank of $\mathscr{L}$. The maximal words of $\mathscr{L}$ are called the bases.

Let $\mathscr{L}$ be a greedoid of rank l with alphabet I_n. Then the set L of its bases is naturally identified with some subset of the set W^{Q_l} of left cosets. On the other hand, from each subset $L \subseteq W^{Q_l}$ we can construct a language $\hat{L}$, consisting of all words obtained from words in L by discarding their ends.

Proposition 3. a) *Let L be the set of bases of an (W, Q_l)-matroid. Then $\hat{L}$ is a greedoid of rank l.* b) *Let $\mathscr{L}$ be a greedoid of rank l with alphabet I_n such that $i \in \mathscr{L}$ for all $i \in I_n$. Then the set of bases L of the greedoid is the set of bases of some (W, Q_l)-matroid.*

Example 3. Let $J_n = \{1, \ldots, n, 1^*, \ldots, n^*\}$, and let $*$ be the involution in J_n such that $(i)^* = i^*$, $(i^*)^* = i$. A subset $A \subseteq J_n$ is called admissible if $A \cap A^* = \varnothing$. Let $R(J_n)$ denote the set of all admissible subsets of J_n, and $R_k(J_n)$ the set of all k-element admissible subsets.

Suppose that $L \subseteq R_k(J_n)$. The pair (J_n, L) is called a symplectic matroid of rank k if the following conditions hold:

1) for any $A, B \in L$ and $a \in A \backslash B$ there is a $b \notin A$ such that either $(A \setminus \{a\}) \cup \{b\} \in L$ or $(A \setminus \{a, b^*\}) \cup \{a^*, b\} \in L$;

2) for any $A, B \in L$ and $b \in B \backslash (A \cup A^*)$ there is an $a \in A$ such that $(A \setminus \{a\}) \cup \{b\} \in L$.

The set L is called the set of bases of (J_n, L).

Let W be the group of permutations of J_n that commute with $*$, that is, the Weyl group of the Lie algebras $\mathrm{sp}(2n)$ and $o(2n+1)$. Then R consists of the permutations $r_i = (i, i+1)\cdot(i^*, (i+1)^*)$ $(i = 1, \ldots, n-1)$ and the transposition (n, n^*). Let $Q = R \setminus \{r_k\}$. Then $W_Q = \{w \in W \mid w(I_k) = I_k\}$ and W^Q is naturally identified with $R_k(J_n)$. The linear ordering $1 < \ldots < n < < n^* < \ldots < 1^*$ on J_n induces a partial ordering on $R_k(J_n)$, just as in Example 1. Under the identification of $R_k(J_n)$ with W^Q, this coincides with the Bruhat ordering. As in Example 1, there is an ordering on $R_k(J_n)$ associated with each $w \in W$.

***Proposition* 4.** *Let $L \subseteq R_k(J_n) = W^Q$. The triple (W, Q, L) is a (W, Q)-matroid if and only if L is the set of bases of some symplectic matroid of rank k.*

It turns out that a symplectic matroid can be defined in terms of a rank function by analogy with an ordinary matroid (see 2.1). Here the role of the Boolean algebra $B(I_n)$ is played by the set $R(J_n)$ of admissible subsets, which becomes a lattice after adjoining a maximal element 1. With each symplectic matroid (J_n, L) we associate the rank function $r : R(J_n) \to \mathbf{Z}$, where $r(A) = \max\limits_{B \in L} |A \cap B|$.

A symplectic matroid (J_n, L) is called *loop-free* if, for each $a \in J_n$, there is an $A \in L$ such that $a \in A$.

***Proposition* 5.** a) *Let (J_n, L) be a loop-free symplectic matroid. Then its rank function satisfies the following conditions*:

1) $0 < r(A) \leqslant 1$ *for all* $A \in R(J_n) \backslash \varnothing$;

2) $r(A) \leqslant r(B)$ *for* $A \subseteq B$;

3) $r(A) + r(B) \geqslant r(A \cap B) + r(A \cup B)$ *for all* $A, B \in R(J_n)$ *such that* $A \cup B \in R(J_n)$.

b) *Conversely, suppose that* $r : R(J_n) \to \mathbf{Z}$ *satisfies* 1)–3). *Then r is the rank function of some loop-free symplectic matroid* (J_n, L).

Remark. The lattice $R(J_n) \cup 1$ is dual to the lattice of faces of the n-dimensional cube. This lattice can be defined axiomatically [22]. Other interesting non-distributive lattices can apparently be associated with the matroids of other Coxeter groups.

Example 4. A symplectic matroid of rank k is called orthogonal if, for any $A \in L$ and $a \in A$ such that $(A \setminus \{a\}) \cup \{a^*\} \in L$, there is a $b \in A \cup A^*$ with $(A \setminus \{a\}) \cup \{b\} \in L$ and $(A \setminus \{a\}) \cup \{b^*\} \in L$.

Let W be the subgroup of permutations of J_n commuting with $*$ such that $|w(I_n) \backslash I_n|$ is even for $w \in W$ (the Weyl group of the Lie algebra $o(2n)$). In this case $R = \{r_1, \ldots, r_n\}$, where $r_1, \ldots, r_{n-1}$ are as in Example 3, and $r_n = (n^*, n-1) \cdot (n, (n-1)^*)$. Let $Q = R \backslash r_k$. Then $W_Q = \{w \in W | \, w(I_k) = I_k\}$ if $k \neq n-1$, while $W_Q = \{w \in W \mid w(I') = I'\}$ if $k = n-1$, where $I' = \{1, \ldots, n-1, n^*\}$. The set W^Q of left cosets can be identified with $R_k(J_n)$ for $k < n-1$, and with $R_n^+(J_n)$ (respectively $R_n^-(J_n)$) for $k = n$ (respectively $k = n-1$), where

$$R_n^{+(-)}(J_n) = \{X \in R_n(J_n) | \; |X \setminus I_n| = 0(1) \bmod 2\}.$$

The set $R_{n-1}(J_n)$ is identified with W^Q for $Q = R \setminus \{r_{n-1}, r_n\}$. Under these identifications the Bruhat ordering on $R_k(J_n)$ is induced (just as in Examples 1 and 3) by the partial ordering on J_n:

$$1 < \ldots < n, \; n^* < \ldots < 1^*, \; 1 < \ldots < n-1 < n^*,$$

(n and n^* are incomparable).

***Proposition* 6.** *Let $L \subseteq R_k(J_n) = W^Q$. A triple (W, Q, L) is a (W, Q)-matroid if and only if L is the set of bases of some orthogonal matroid of rank k.*

8.3. Let W be the Weyl group of the Lie algebra of G. Let $Q \subseteq \{\sigma_1, \ldots, \sigma_n\}$, where $\sigma_1, \ldots, \sigma_n \in \mathfrak{h}_{\mathbf{R}}^*$ is a simple root system. Consider the point $\omega_Q \in \mathfrak{h}_{\mathbf{R}}^*$ defined by

$$\frac{(\omega_Q, \sigma_i)}{(\sigma_i, \sigma_i)} = \begin{cases} 1 & \text{for} \quad \sigma_i \notin Q, \\ 0 & \text{for} \quad \sigma_i \in Q. \end{cases}$$

Since W_Q is the subgroup fixing ω_Q, we can define a mapping $\bar{\mu}$: $W^Q \to \mathfrak{h}_{\mathbf{R}}^*$ that sends $w \cdot W_Q$ to $w(\omega_Q)$. With any subset $L \subseteq W^Q$ we associate a polytope Δ_L, the convex combination of points in $\bar{\mu}(L)$. Let P be the standard parabolic subgroup of G such that $S_P = Q$ (see 4.2). Then $\omega_Q = \lambda$, and the moment mapping μ sends $M = G/P$ onto Δ_{W^Q}.

The mapping $\bar{\mu}$ identifies the elements of W^Q with the orbits $W \cdot \omega_Q$. This identification enables us to define the partial ordering $\overset{w}{\leqslant}$ geometrically. Let C_w be the convex cone in $\mathfrak{h}_{\mathbf{R}}^*$ consisting of vectors $y = \sum_{i=1}^{n} m_i w(\sigma_i)$ such that $m_i \geqslant 0$ for all $i = 1, \ldots, n$. We define an ordering $\overset{w}{\leqslant}$ on $\mathfrak{h}_{\mathbf{R}}^*$ by putting $x \overset{w}{\leqslant} y$ if $y - x \in C_w$. The restriction of this ordering to $W \cdot \omega_Q$ agrees with the ordering $\overset{w}{\leqslant}$ on W^Q.

***Theorem* 1.** *Let W be the Weyl group of a semisimple group G, and let P be a standard parabolic subgroup of G. Then the following conditions are equivalent*:

1) *(W, S_P, L) is a (W, S_P)-matroid*;

2) *Δ_L is a (G, P)-hypersimplex.*

Proof. 1) $\Rightarrow$ 2). Suppose that (W, S_P, L) is a (W, S_P)-matroid, and that Δ_L is not a hypersimplex. Then there is an edge l with vertices α and β that is not parallel to any root. Consider a linear function χ on $\mathfrak{h}_{\mathbf{R}}^*$ which is constant on l and takes greater values on the other points of Δ_L. There is a unique simple root system $\tilde\sigma_1, \ldots, \tilde\sigma_n$ such that $\chi(\tilde\sigma_i) > 0$. Since the Weyl group acts transitively on simple root systems, there is a $w \in W$ sending $\{\sigma_1, \ldots, \sigma_n\}$ to $\{\tilde\sigma_1, \ldots, \tilde\sigma_n\}$. Then for any $\gamma \neq \alpha$, $\gamma \in L$, the vector $\alpha - \gamma$ has at least one negative coefficient with respect to the basis $\tilde\sigma_1, \ldots, \tilde\sigma_n$. The same holds if we replace α by β. Hence L contains no w-minimal element, contrary to hypothesis.

2) $\Rightarrow$ 1). Suppose that Δ_L is a hypersimplex and let w be any element of W. We put $\tilde\sigma_1 = w(\sigma_1), \ldots, \tilde\sigma_n = w(\sigma_n)$. We choose a linear function χ on $\mathfrak{h}_{\mathbf{R}}^*$ such that $\chi(\tilde\sigma_i) > 0$ for all $i = 1, \ldots, n$. Let α be a vertex of Δ_L on which χ attains its minimum. Let $\beta_1, \ldots, \beta_n$ be the vertices of Δ_L that are joined to α by edges. Since every edge of Δ_L is parallel to a root, we have $\beta_i - \alpha = k_i\gamma_i$, where γ_i is a root and $k_i \geqslant 0$. Then for any $\beta \in L$ we have $\beta - \alpha = \sum_{i=1}^{n} a_i k_i \gamma_i$, where $a_i \geqslant 0$.

Since $\chi(\alpha)$ is the minimum of χ, we have $\chi(\gamma_i) \geqslant 0$. Hence all the roots are γ_i-positive relative to the simple root system $\tilde\sigma_1, \ldots, \tilde\sigma_n$. This implies that $\beta - \alpha = \sum b_i\tilde\sigma_i$, where $b_i \geqslant 0$. Thus $\beta \overset{w}{\geqslant} \alpha$, so α is a w-minimal element.

The next result follows from Theorem 1 of this section and Theorem 1 of §7.

***Theorem* 2.** *Let Γ be an arbitrary stratum in G/P with list L_Γ. Then (W, S_P, L_Γ) is a (W, S_P)-matroid.*

Example 1. Consider the polytope Δ_W corresponding to an open everywhere dense stratum on G/B. The polytope Δ_W was studied in [23] and [12]. It can be shown that there is a bijection between the set of its faces and the set of all left cosets of all parabolic subgroups of W: $\bigcup_{Q \subseteq R} W^Q$, where the dimension of the face corresponding to $w \cdot W_Q$ is $|Q|$. In combinatorics Δ_W is called a Coxeter complex (see [24]).

For $W = S_3$, Δ_W is a regular hexagon, and for $W = S_4$, Δ_W is a semiregular polyhedron in $\mathbf{R}^3$ with 24 vertices, 8 hexagonal and 6 square faces.

Example 2. Let (J_n, L) be a symplectic matroid of rank k. Then the hypersimplex Δ_L has as vertices some of the vertices of the cube $E_n = \{\xi \in \mathbf{R}^n \mid |\xi_i| \leqslant 1\}$, and its edges are either edges of the cube or diagonals of its 2-dimensional faces. The symplectic matroid constructed on Δ_L will be orthogonal if and only if all the edges of Δ_L are diagonals of two-dimensional faces of the cube.

8.4. Non-degeneracy of (W, Q)-matroids and strata.

Let (W, Q, L) be a (W, Q)-matroid. A subgroup $\overline{W} \subseteq W$ containing some parabolic subgroup of W is called a separator of (W, Q, L) if $L \subseteq \overline{W}\cdot\alpha$ for some $\alpha \in L$.

***Proposition* 7.** *Let $\overline{W}$ be a separator of the (W, Q)-matroid (W, Q, L), $L \subseteq \overline{W}\cdot w\cdot W_Q$, $W_{\bar{Q}} = \overline{W} \cap wW_Q w^{-1}$, and $\bar{L} = \{\alpha w^{-1} \cap \overline{W} \mid \alpha \in L\}$. Then $(\overline{W}, \bar{Q}, \bar{L})$ is a $(\overline{W}, \bar{Q})$-matroid.*

A (W, Q)-matroid is called non-degenerate if it has no separators other than W. A stratum $\Gamma \subset G/P$ is called non-degenerate if Γ consists of orbits of a torus H of maximal dimension.

***Proposition* 8.** *A stratum $\Gamma \subset G/P$ is non-degenerate if and only if the corresponding (W, S_P)-matroid is non-degenerate. Every degenerate stratum $\Gamma \subset G/P$ is isomorphic to some stratum $\Gamma' \subset G'/P'$, where G' is a regular semisimple subgroup of G and $P' = P \cap G'$.*

§9. Examples of strata

9.1. Strata on flag manifolds.

Let $F_{k_1,\ldots,k_m}(E)$ $(k_1 < \ldots < k_m)$ denote the flag manifold of subspaces $(X_1 \subset \ldots \subset X_m)$ of dimensions $k_1, \ldots, k_m$ respectively of a complex n-dimensional space E. In particular, $G_k(E) = F_k(E)$. We choose a basis $\{e_1, \ldots, e_n\}$ in E, thereby identifying E with $\mathbf{C}^n$, and a diagonal subgroup H of $\mathrm{GL}(n)$ with respect to this basis. We consider various ways of defining strata in this case.

It follows from Theorem 1 of §6 that any stratum Γ in $F_{k_1,\ldots,k_m}(\mathbf{C}^n)$ is an intersection of the Schubert cells for all the coordinate flags. We recall that the Schubert cell for a coordinate flag $\mathbf{C}^{I_1} \subset \ldots \subset \mathbf{C}^{I_n}$, $I_j \in B_j(I_n)$, consists of flags $(X_1, \ldots, X_m)$ such that $\dim X_i \cap \mathbf{C}^{I_j}$ is fixed for all $i = 1, \ldots, m$, $j = 1, \ldots, n$.

Hence any stratum Γ on a flag manifold is determined by the dimension of the intersection of subspaces in the flag with all the coordinate subspaces.

Just as in the Grassmannian case, we shall define a stratum Γ by the functions $r_1, \ldots, r_m : B(I_n) \to \mathbf{Z}$ given by $r_i(J) = \dim(X_i/X_i \cap \mathbf{C}^{I_n \setminus J})$ for any $(X_1, \ldots, X_m) \in \Gamma$. Consider the embedding

$$F_{k_1,\ldots,k_m}(\mathbf{C}^n) \hookrightarrow G_{k_1}(\mathbf{C}^n) \times \ldots \times G_{k_m}(\mathbf{C}^n).$$

In 1.4 we described a principal fibration $q\colon C_{n,k} \to G_k(\mathbf{C}^n)$. Let q denote the natural fibration $C_{n,k_1} \times \ldots \times C_{n,k_m} \to G_{k_1}(\mathbf{C}^n) \times \ldots \times G_{k_m}(\mathbf{C}^n)$. We shall describe $q^{-1}(F_{k_1,\ldots,k_m}(\mathbf{C}^n))$.

Definition 1. A sequence of collections of n vectors

$$\{x_j^i,\ i = 1, \ldots, n,\ j = 1, \ldots m\}$$

in spaces $F_1, \ldots, F_m$ of dimensions $k_1, \ldots, k_m$ is said to be *concordant* if $\sum_{1\leqslant j\leqslant n} \lambda_j x_p^j = 0$ implies that $\sum_{1\leqslant j\leqslant n} \lambda_j x_i^j = 0$ for all $i \leqslant p$.

We denote the submanifold of all concordant sequences in $C_{n,k_1} \times \ldots \times C_{n,k_m}$ by $CF_{n,k_1,\ldots,k_m}$. It is not hard to verify that $q^{-1}(F_{k_1,\ldots,k_m}(\mathbf{C}^n)) = CF_{n,k_1,\ldots,k_m}$.

The fibration q: $CF_{n,k_1,\ldots,k_m} \to F_{k_1,\ldots,k_m}(\mathbf{C}^n)$, just as in the Grassmannian case, descends to a fibration

$$\tilde{q}\colon CF_{n,k_1,\ldots,k_m}/H \to F_{k_1,\ldots,k_m}(\mathbf{C}^n)/H.$$

Whereas in the Grassmannian case a stratum is a matroid, it is clear from the above that a stratum on a flag manifold can be defined as a sequence of matroids.

Definition 2. A sequence of matroids $(S, r_1), \ldots, (S, r_m)$ with collections of closed sets $\mathrm{Fl}_1, \ldots, \mathrm{Fl}_m$ is said to be *concordant* if $\mathrm{Fl}_1 \subseteq \ldots \subseteq \mathrm{Fl}_m$.

Proposition 1. *Let Γ be a stratum in $F_{k_1,\ldots,k_m}(\mathbf{C}^n)$, defined by a sequence of functions $r_1, \ldots, r_m$. Then the matroids $(I_n, r_1), \ldots, (I_n, r_m)$ form a concordant sequence.*

The proof is obvious.

For any $A \subseteq S$ let $\bar{A}^i$ denote the closure of A in the matroid (S, r_i), that is, $\bar{A}^i = \bigcap_{C \in \mathrm{Fl}_i,\ A \subseteq C} C$.

Lemma 1. *Let $(S, r_1), \ldots, (S, r_m)$ be a concordant sequence of matroids, of ranks $k_1, \ldots, k_m$. Then*:

a) $\bar{A}^i \subseteq \bar{A}^j$ *for any* $A \subseteq S$ *and* $j \leqslant i$;

b) $r_1 \leqslant \ldots \leqslant r_m$.

Proof. a) follows from the definition.

b) Suppose that there exist i and j with $i < j$ such that $r_i(J) > r_j(J)$ for some $J \subseteq S$. Among all such J we choose one J_0 with the minimum number of elements. Note that $J_0 \neq \varnothing$. Then for any $b \in J_0$, $r_i(J_0\backslash b) = r_j(J_0\backslash b)$. Hence $b \in \overline{(J_0 \setminus b)}^j$ and $b \notin \overline{(J_0 \setminus b)}^i$, which contradicts a).

Let $(X_1, \ldots, X_m) = X$ be a flag in $\mathbf{C}^n$, and let p_i^J be the Plücker coordinates of X_i. The numbers $p^{J_1,\ldots,J_m}(X) = \prod_{i=1}^{m} p_i^{J_i}$, where $J_i \in B_{k_i}(I_n)$, $J_1 \subset \ldots \subset J_m$, are called the generalized Plücker coordinates of the flag. From the generalized Plücker coordinates we can uniquely recover the Plücker coordinates of each X_i.

The *list* of a flag $X \in F_{k_1,\ldots,k_m}(\mathbf{C}^n)$ is the set

$$L_X = \{(J_1, \ldots, J_m) \mid p^{J_1,\ldots,J_m}(X) \neq 0\}.$$

It follows from §4 that every stratum Γ in $F_{k_1,\dots,k_m}(\mathbf{C}^n)$ can be defined by a list L_Γ:

$$\Gamma = \{X \in F_{k_1,\dots,k_m}(\mathbf{C}^n) \mid L_X = L_\Gamma\}.$$

For a concordant sequence of matroids $(S, r_1), \dots, (S, r_m)$ we define a list of bases $\mathscr{B}$ as follows:

$$\mathscr{B} = \{(B_1, \dots, B_m) \mid B_1 \subset \dots \subset B_m,\ r_i(S) = r_i(B_i) = |B_i|\}.$$

Hence, just as we could associate a hypersimplex with every matroid in 2.2, we associate with every concordant sequence of matroids a certain polytope $\Delta_{\mathscr{B}}$ in $\mathscr{L}(S)$, specifically

$$\operatorname{Vert} \Delta_{\mathscr{B}} = \{\delta_{B_1} + \dots + \delta_{B_m} \mid (B_1, \dots, B_m) \in \mathscr{B}\}.$$

Let $(S, r_1), \dots, (S, r_m)$ be a concordant sequence of matroids of ranks $k_1, \dots, k_m$ respectively. We define a function $r = r_1 + \dots + r_m$ on S. Then r is monotone, $r(\emptyset) = 0$, and $r(A \cup B) + r(A \cap B) \leqslant r(A) + r(B)$. Thus the pair (S, r) is by definition a *supermatroid* (see [8]). It is clear that two distinct concordant sequences of matroids give rise to different supermatroids.

Suppose we are given a *supermatroid* (S, r) and a sequence of numbers $k_1 < \dots < k_m$ such that $r(S) = \sum_{i=1}^{m} k_i$. We shall be interested in the question of when (S, r) can arise from a concordant sequence of matroids of ranks $k_1, \dots, k_m$. We put

$$\mathscr{B}_{k_1,\dots,k_m}(I_n, r) = \{(B_1, \dots, B_m) \mid B_i \in B_{k_i}(I_n),\ B_1 \subset \dots \subset B_m,\ r(B_i) = \\ = \sum_{1 \leqslant j \leqslant i} k_j + k_i(m - i)\}.$$

We choose some linear ordering of the elements of S, thereby identifying S with I_n.

***Proposition* 2.** *Let (I_n, r) be a supermatroid, and let $k_1, \dots, k_m$ be a sequence of numbers such that $r(I_n) = k_1 + \dots + k_m$. Then the following conditions are equivalent:*

a) *There is a concordant sequence of matroids $(I_n, r_1), \dots, (I_n, r_m)$ of ranks $k_1, \dots, k_m$ such that $r = r_1 + \dots + r_m$.*

b) *$\mathscr{B}_{k_1,\dots,k_m}(I_n, r)$ is a general matroid (see Example 2 of 8.3).*

Proof: a) $\Rightarrow$ b). Let $\mathscr{B}_i$ be the set of bases of (I_n, r_i). Then clearly

$$\mathscr{B}_{k_1,\dots,k_m}(I_n, r) = \{(B_1, \dots, B_m) \mid B_i \in \mathscr{B}_i, B_1 \subset \dots \subset B_m\}.$$

Let s be any linear ordering on I_n. Then each collection of bases $\mathscr{B}_i$ contains an s-minimal element C_i (see 2.6). We show that $C_1 \subset \dots \subset C_m$. Suppose not; then $C_i \not\subset C_{i+1}$ for some $i < m$. Let $A = C_i \cap C_{i+1}$ and let c be an s-minimal element in $C_i \setminus C_{i+1}$. Then $C_{i+1} \setminus C_i = D_0 \cup D_1$, where each element of D_0 is less than c (in the s-ordering), and each element of D_1 is greater than c. Then it follows from the s-minimality of C_i that for any $b_0 \in D_0$ and $d \in C_i \setminus C_{i+1}$ we have $(C_i \setminus d) \cup b_0 \notin \mathscr{B}_i$. Hence $D_0 \in \bar{A}^i$.

Similarly, for any $b_1 \in D_1$ we have $(C_{i+1}\setminus b_1) \cup c \notin \mathscr{B}_{i+1}$. Hence $c \in \overline{(C_{i+1}\setminus D_0)}^{i+1}$. Then $c \in \overline{(C_{i+1}\setminus D_0)}^{i} = \overline{(A \cup D_0)}^{i}$, since $\overline{(C_{i+1}\setminus D_0)}^{i+1} \subseteq \overline{(C_{i+1}\setminus D_0)}^{i}$. But $D_0 \subset \bar{A}^i$ and so $c \in \bar{A}^i$, which is impossible, since $A \cup c$ is an independent subset of (S, r_i). Hence $C_1 \subset \dots \subset C_m$. But then $(C_1, \dots, C_m)$ is an s-minimal element of $\mathscr{B}_{k_1,\dots,k_m}(I_n, r)$.

b) ⇒ a). We put $\mathscr{B}_i = \{B \in B_{k_i}(I_n) \mid$ there exists

$$(B_1, \dots, B_{i-1}, B, B_{i+1}, \dots, B_m) \in \mathscr{B}_{k_1,\dots,k_m}(I_n, r)\}.$$

Since $\mathscr{B}_i$ has an s-minimal element for all $s \in S_n$, $\mathscr{B}_i$ is the set of bases of some matroid (I_n, r_i) (see 2.6). Let Fl_i be the set of all closed sets of (I_n, r_i). Then for all $A \in \mathrm{Fl}_i$ and all $b \in I_n \setminus A$ there is a $B \in \mathscr{B}_i$ such that $b \notin B$ and $r_i(A) = |A \cap B|$. Let $|A| = p$. Let $s \in S_n$ be such that $s(I_p) = A$ and $s(p+1) = b$. Let $(C_1, \dots, C_m)$ be an s-minimal element of $\mathscr{B}_{k_1,\dots,k_m}(I_n)$. Then $C_i \overset{s}{\leqslant} B$, and it follows that $r_i(A) = |A \cap C_i|$ and $b \in C_i$. Then clearly $r_j(A) = |A \cap C_j|$ and $b \in C_j$ for any $j \geqslant i$. Since $b \in I_n \setminus A$ was chosen arbitrarily, $A \in \mathrm{Fl}_j$ for all $j \geqslant i$ by the definition of a closed set. We have thus shown that $\mathrm{Fl}_1 \subset \dots \subset \mathrm{Fl}_m$, that is, b) ⇒ a).

9.2. Strata for the Grassmannians of the orthogonal and symplectic groups. a) Let $G = \mathrm{SO}(2n)$. Then G acts on a $2n$-dimensional space E, preserving a non-degenerate scalar product $(\ ,\ )$. We choose a basis $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ such that $(e_i, e_j) = (f_i, f_j) = 0$ and $(e_i, f_j) = \delta_{ij}$, thereby identifying E with $\mathbf{C}^{2n}$. We fix the Cartan subgroup H of diagonal matrices with respect to this basis. Consider the Grassmannian $I_k(\mathbf{C}^{2n})$ of k-dimensional isotropic subspaces of $\mathbf{C}^{2n}$ $(k \leqslant n)$. For $k = n$ we take $I_k(\mathbf{C}^{2n})$ to be a connected component of the Grassmannian of isotropic subspaces.

We recall that J_n is the set consisting of $2n$ elements $1, \dots, n, 1^*, \dots, n^*$ with involution $* : J_n \to J_n$. A subset $A \subseteq J_n$ is called admissible if $A \cap A^* = \emptyset$ (see Example 3 in 8.3). Let $R(J_n)$ denote the set of all admissible subsets of J_n, and $\bar{R}(J_n)$ the set of complements of admissible sets. For any $A \subseteq J_n$ let $\mathbf{C}^A$ denote the coordinate subspace of $\mathbf{C}^{2n}$ spanned by the basis vectors e_i, f_j for all $i \in A$, $j^* \in A$. If $A \in R(J_n)$, then $\mathbf{C}^A$ is isotropic.

***Proposition* 3.** *Any stratum* Γ *in* $I_k(\mathbf{C}^{2n})$ *is defined by a function* $r_\Gamma : R(J_n) \to \mathbf{Z}$ *as follows*:

$$\Gamma = \{X \in I_k(\mathbf{C}^{2n}) \mid \dim(X / X \cap \mathbf{C}^{J_n \setminus A}) = r_\Gamma(A)\}.$$

Proof. By Theorem 1 of §6 every stratum is an intersection of Schubert cells $\cap\, C \cdot \alpha(C)$ over all Borel subgroups C that contain the given Cartan subgroup H. All the subgroups C are indexed by coordinate flags

$$\mathbf{C}^{D_1} \subset \dots \subset \mathbf{C}^{D_n} \subset \mathbf{C}^{D_n \cup D_1^*} \subset \dots \subset \mathbf{C}^{D_n \cup D_n^*},$$

where $D_i \in R(J_n)$, $|D_i| = i$, and the corresponding Schubert cells $C\cdot\alpha$ are determined by the dimensions of intersection of $X \in C\cdot\alpha$ with $\mathbf{C}^{D_i}$ and $\mathbf{C}^{D_n \cup D_i^*}$. Proposition 3 follows easily from this.

It can be shown that, for $A \notin R(J_n) \cup \bar{R}(J_n)$, dim $(X \cap \mathbf{C}^A)$ can change at the boundaries of Γ.

The Grassmannian $I_k(\mathbf{C}^{2n})$ can be realized as the orbit of a principal vector in $P(\Lambda^k(\mathbf{C}^{2n}))$. Thus $X \in I_k(\mathbf{C}^{2n})$ is represented in the form

$$X = c\cdot \sum_{J\in B_k(J_n)} p^J(X)\, e_J, \text{ where } e_J = e_{j_1} \wedge \dots \wedge e_{j_k} \text{ for } J = \{j_1, \dots, j_k\}.$$

The generalized Plücker coordinates (see 4.2) in this case are defined up to a multiplier as a collection of numbers $\{p^J(X) \mid J \in R_k(J_n)\}$. Thus the list L_X of $X \in I_k(\mathbf{C}^{2n})$ consists of admissible k-element subsets of J_n.

Any k-dimensional subspace of $\mathbf{C}^{2n}$ is determined by a configuration of $2n$ vectors $x^1, \dots, x^n, \tilde{x}_1, \dots, \tilde{x}^n$ in the k-dimensional space F^*, as described in 1.3. It turns out that the configuration $(x^1 \dots, x^n, \tilde{x}^1, \dots, \tilde{x}^n)$ determines an isotropic subspace of $\mathbf{C}^{2n}$ if and only if

$$(2)\quad \sum_{1\leqslant i\leqslant n} x^i\cdot\tilde{x}^i = 0, \text{ where } x^i\cdot\tilde{x}^i = x^i \otimes \tilde{x}^i + \tilde{x}^i \otimes x^i \text{ is a vector in } S^2(F^*).$$

If $CI_{2n,\,k}$ denotes the set of collections of $2n$ vectors in k-dimensional space that satisfy (2), then there is a GL(k)-bundle $q\colon CI_{2n,\,k} \to I_k(\mathbf{C}^{2n})$, as in the Grassmannian case.

b) The case G = Sp($2n$) is entirely analogous to the case of SO($2n$). It differs only in that the symmetric form (,) is replaced by the skew form $\langle\,,\rangle$. Any k-dimensional null subspace with respect to the form $\langle\,,\rangle$ can be determined by a configuration of $2n$ vectors $x^1, \dots, x^n, \tilde{x}^1, \dots, \tilde{x}^n$ in k-dimensional space, satisfying the equation

$$\sum_{1\leqslant i\leqslant n} x^i \wedge \tilde{x}^i = 0.$$

c) Let G = SO($2n+1$). Then G is the group of linear transformations of a space E that preserve the non-degenerate bilinear form (,). As G/P, where P is a maximal parabolic subgroup, we take the Grassmannian $I_k(E)$ of k-dimensional isotropic subspaces of E. We choose a basis

$$\{e_1, \dots, e_n, f_1, \dots, f_n, g_0\}$$

of E such that $(e_i, e_j) = (f_i, f_j) = (g_0, e_i) = (g_0, f_j) = 0$, $(g_0, g_0) = 1$ and $(e_i, f_j) = \delta_{ij}$. We take H to be the subgroup of diagonal matrices with respect to this basis.

It can be shown that a stratum Γ in $I_k(E)$ is determined by the function $r_\Gamma\colon R(J_n) \to \mathbf{Z}$, just as in Proposition 3. The proof uses the fact that $\dim(X \cap \mathbf{C}^A) = \dim(X \cap (\mathbf{C}^A \oplus \mathbf{C}g_0))$ for any $X \in I_k(E)$ and $A \in R(J_n)$. The Plücker coordinates and the list L_X of an isotropic subspace X are defined similarly.

Note also that any k-dimensional isotropic subspace X in $\mathbf{C}^{2n+1}$ is determined by a configuration of $2n+1$ vectors $z, x^1, \ldots, x^n, \tilde{x}^1, \ldots, \tilde{x}^n$ in $\mathbf{C}^k$ such that $\sum\limits_{1\leqslant i\leqslant n} x^i\cdot\tilde{x}^i+2z\cdot z=0$.

Note that the vectors $\pm z$ are uniquely determined by the collection $x^1, \ldots, x^n, \tilde{x}^1, \ldots, \tilde{x}^n$.

9.3. Grassmannians and strata for G_2.
There are three standard parabolic subgroups in G_2. We describe the corresponding compact homogeneous spaces.

Let $\mathbf{Q}$ be the Cayley algebra over $\mathbf{C}$. We choose a basis

$$(1, e_0, e_1, e_2, e_3, f_1, f_2, f_3\}$$

in $\mathbf{Q}$ such that the following conditions hold:

$$e_0\cdot f_q=-f_q\cdot e_0=-f_q;\quad e_0\cdot e_q=-e_q\cdot e_0=e_q;\quad e_0^2=1;\quad e_q^2=f_q^2=0;$$
$$e_p\cdot f_q=(1-e_0)/2\cdot\delta_{pq};\quad e_i\cdot e_j=-e_j\cdot e_i=f_k;\quad f_i\cdot f_j=-f_j\cdot f_i=e_k,$$

where (i, j, k) is a cyclic permutation of $(1, 2, 3)$, $p\neq 0$ and $q\neq 0$.

Let E be the subspace of $\mathbf{Q}$ spanned by $e_0, \ldots, e_3, f_1, \ldots, f_3$. Then G_2 is the group of automorphisms of $\mathbf{Q}$, and E is a faithful irreducible representation of G_2 [13].

Let P_1 and P_2 be the maximal parabolic subgroups of G_2 corresponding to the simple roots with indices 1 and 2 respectively (see Fig. 8). Then the homogeneous spaces G_2/P_1 and G_2/P_2 can be realized as the sets of one-dimensional (respectively, two-dimensional) subalgebras of E with trivial multiplication. The homogeneous space G_2/B, where B is the Borel subgroup $P_1\cap P_2$ of G, is realized as the space of flags (X_0, X_1) such that $X_0\subset X_1\subset E$ and $X_1\cdot X_1=0$. Note that $G_2/P_1\cong I_1(\mathbf{C}^7)$.

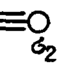

Fig. 8

It is easy to verify that the points of G_2/P_i are defined by configurations of seven vectors $z, x_1, x_2, x_3, y_1, y_2, y_3$ in $\mathbf{C}^i$ satisfying the conditions: $\sum\limits_{1\leqslant i\leqslant 3} x_i\cdot y_i+2z\cdot z=0$; $x_i\wedge x_j=y_k\wedge z$; $\sum\limits_{1\leqslant i\leqslant 3} x_i\wedge y_i=0$; and $y_i\wedge y_j=x_k\wedge z$, where (i, j, k) is a cyclic permutation of $(1, 2, 3)$.

Let $\mathscr{K}_3=\{1, 2, 3, \tilde{1}, \tilde{2}, \tilde{3}\}$. We call any one-element subset of $\mathscr{K}_3$ admissible, and also any two-element subset of the form $\{p, \tilde{j}\}$, where $p\neq j$. The admissible subsets are precisely those corresponding to coordinate subalgebras $\mathbf{C}^J\subset\mathbf{C}^7=E$ with trivial multiplication, that is, subalgebras spanned by basis vectors. It can be shown that any stratum Γ in G_2/P_i is determined by a function r_Γ defined on the admissible subsets of $\mathscr{K}_3$. Specifically, $\Gamma=\{X\in G_2/P_i \mid \dim(X/X\cap\mathbf{C}^{\mathscr{K}_3\setminus I})=r_\Gamma(I)\}$.

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