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MAXIMAL COHEN-MACAULAY MODULES AND TATE-COHOMOLOGY OVER GORENSTEIN RINGS

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The main theme of this article is :

Why should one consider Maximal Cohen-Macaulay Modules ?

Although there has been a lot of work and success lately in the theory of such modules, of which this conference witnessed, it has remained mysterious - at least to the present author - why these modules provide such a powerful tool in studying the algebra and geometry of singularities for example.

We try to give one answer here, at least for the case of Gorenstein rings. Their role is special as over such rings "maximal Cohen-Macaulay" and "being a syzygy module of arbitrarily high order" are synonymous.

It turns out, that these modules, in a very precise sense, describe all stable homological features of such rings.

The motif was the observation that maximal Cohen-Macaulay modules - at least up to projective modules - carry a natural triangulated structure which implies that there is a naturally defined cohomology-theory attached to these modules - the Tate-cohomology.

To be more specific let us explain the essential points in the case of a local hypersurface ring R :

It was observed by D.Eisenbud, [Eis] , that any finitely generated module over R admits a minimal free resolution which becomes eventually periodic of period two.

Maximal Cohen-Macaulay modules over R - without free summands - are characterized as having a resolution periodic right from the start. Furthermore, the periodic part of the resolution comes from a "matrix factorization" of the defining equation and these matrix factorizations behave like "free complexes modulo two", exhibiting the forementioned

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*) Supported by a "Heisenberg-Stipendium" of the DFG.

triangulated structure if one considers such matrix factorizations "up to homotopy".

Now one may proceed as follows : trace back a minimal resolution of a module until the periodic tail is reached, turn around and extend by periodicity to a complex which is then unbounded, acyclic and consists only of free modules of finite rank.

If we started with a maximal Cohen-Macaulay module, except for possible free summands no information is lost as the original module can be recovered from the image of a differential in this complex.

If we began with an arbitrary finitely generated R-module, the loss is the non-periodic part of the resolution, which is a finite free complex. In return we obtain a complete resolution - and a maximal Cohen-Macaulay module as its 0^{th} syzygy module - canonically attached to the (resolution of the) module we started with. More generally, one could have taken any complex of modules with bounded, finitely generated cohomology in the beginning to obtain still such a complete resolution as well as a maximal Cohen-Macaulay module from "its" minimal resolution.

Checking the necessary details, which is rather straightforward, one has hence the following essentially equivalent data :

- Acyclic projective complexes up to homotopy, (that is, the "complete resolutions"),.
- Maximal Cohen-Macaulay modules up to projective modules,
- Right bounded complexes with bounded, finitely generated cohomology, (the objects which are resolved), modulo finite complexes of finitely generated projective modules, (the "heads" of the resolutions), all this up to homotopies and quasi-isomorphisms.

The first and third of these categories carry natural triangulated structures compatible with the associations mentioned above. Hence they induce a natural such structure on the middle one, which turns out to be the "triangulation" observed before.

These three equivalent structures give rise to a cohomology theory, unbounded and stabilizing in the positive range the usual Ext-modules. (In case of hypersurfaces, this cohomology is furthermore by construction obviously periodic of period two.)

Carrying out this program, one observes immediately that the only fact needed on the ring - aside being noetherian - is that it has finite injective dimension as a module over itself, or, equivalently, that it admits a projective dualizing module.

In particular this means that commutativity is no way essential.

Hence we develop the theory for not necessarily commutative rings which are (strongly) Gorenstein in the sense that they are noetherian and of finite injective dimension as modules over itself on both sides.

To define maximal Cohen-Macaulay modules (MCM for short) over such rings we use (one of) their defining properties in the commutative case: they will be those modules which are acyclic with respect to the duality defined by $\mathrm{RHom}_R(-,R)$.

Having extended the theory to include non-commutative "Gorenstein rings", quite different looking "classical" results become special cases of this same theory, for example :

- the theory of MCM's over a hypersurface ring as the starting point,
- the theory of integral group representations as initiated by J.Tate, (which has as common intersection with the above the case of cyclic groups),
- the monadic description of certain (derived) categories of coherent sheaves of modules on varieties such as projective space - originally developed by Bernstein-Gelfand-Gelfand and Beilinson, [BGG],[Gel] - or projective complete intersections defined by quadrics, obtained independently by Kapranov and the author. A more detailed account is contained in [BEH;App.] , this volume. (Here the intersection with the first theory consists of course of the projective quadrics and the general machine can be used - as was already implicitely done by R.G.Swan, [Sw], - to determine the higher algebraic K-groups of smooth such quadrics.)

The presentation of the general theory here is not unambiguous. Once the main idea is clear, one could essentially take any book on homological algebra, look at the chapter on Tate-cohomology and generalize all its statements to the general case by a rather obvious dictionary.

As this is so, we chose a different path.

We develop first the general theory in the framework of derived or triangulated categories to obtain the equivalence of categories sketched above and stated in precise form as a theorem in section 4.

This we consider the main result.

From there, all the essential properties which are known for the Tate-cohomology of finite groups follow more or less immediately in general and we try to make clear how it connects with the classical theory in section 6.

But before that, in section 5 , we give some rather concrete applications to modules. It is shown that any module admits a MCM-approximation, that is a surjection from a maximal Cohen-Macaulay module onto the given one, such that the kernel is of finite projective dimension. Such a MCM is then uniquely determined up to direct summands and yields a right adjoint of the inclusion of MCMs into the category of all finitely generated modules modulo projective modules. The same method - which is essentially the duality theorem for Gorenstein rings - yields then a method to construct for any complex a complete resolution by "symmetrization". This will be used then in section 7 to prove a duality theorem for Tate-cohomology. It contains the classical theorem for integral group representations and our aim in section 9 is to show that in fact it contains also Serre's duality theorem for projective space. For this we extend the theory of Bernstein-Gelfand-Gelfand mentioned before in a straightforward manner to "linear superspaces" and show that there the theory of maximal Cohen-Macaulay modules is essentially the same as the theory of coherent sheaves on the dual projective superspace. This is preceeded in section 8 by some easy examples, like minimal primitive quotients of enveloping algebras of semi-simple Lie-algebras or graded Lie-algebras - but for these rings the meaning of a maximal Cohen-Macaulay module - with respect to representation theory say - is not clear yet.

Concerning other applications, even for the most natural generalization from hypersurfaces to complete intersections the general structure of the Tate-cohomology groups is not known. We offer only a result that bounds the support of these groups : It is clear from the construction that the support is in the non-regular locus, but more precisely, if there exists a Noether normalization, the corresponding Noether different will annihilate all Tate-cohomology.

Hence, going back to hypersurfaces, we reverse the order of ideas and look at some peculiar features of Tate-cohomology of cyclic groups and try to generalize them at least to hypersurfaces.

The result is the Herbrand-difference in section 10 which generalizes the classical notion of Herbrand-quotients for cyclic groups. We develop its properties to convince the reader that it may serve as a suitable intersection form on ("co"-)primitive algebraic cycles in case of homogeneous polynomials whose underlying projective hypersurface is regular. Apart from rather simple examples like quadrics or cubic surfaces - where we regain the E_6-lattice algebraically - the use of this theory in connection with the description of algebraic cycles and the (variational) Hodge-conjecture has still to be further pursued.

A final comment on the methods : although the theory is non-commutative, all the inspiration comes from the theory of commutative Gorenstein rings. This has two disadvantages. First, some of the proofs can possibly be simplified by using more sophisticated non-commutative techniques. Secondly, some pre-requisites which are obvious in the commutative case - for the duality for example - seemed not to have been treated yet, so that we have to go through the general machinery once more. This makes in particular section 7 a little bit cumbersome.

On the other hand, this enables us to avoid the use of more special structure - even in the case of group rings all results are established without using the co-algebra structure and cup-products.

Many people stimulated the work presented.
It will become apparent, I hope, how much this work was influenced by M.Auslander. Large parts here can be seen as special cases or applications of his "stable module theory". He essentially also ignited the recent interest in the theory of maximal Cohen-Macaulay modules and always insisted that it should be important. His remarks - sometimes sibylline at least for the author - always brought about fruitful ideas.

i am also very grateful to L.Avramov who pointed out to me that what was termed "stable cohomology theory" then is indeed a natural extension of Tate's classical theory for finite groups. Without him, I would rather have missed this beautiful and clear connection.

G.-M.Greuel, H.Knörrer and F.-O.Schreyer helped through various discussions along the way and conversations with J.Steenbrink made me become aware of the geometric content of the theory.

I am particularly indepted to the organizers of this conference, G.-M. Greuel and G.Trautmann, for the opportunity to publish this work in these proceedings, although I was hindered from attending the actual conference. Also they showed incredible patience awaiting the final version of this manuscript.

0. - *Notations and Conventions*

0.1. All rings considered will be associative with unit, left and right noetherian, all modules will be unitary - unless explicitely stated otherwise.

If S is any ring, Mod-S ,(mod-S), denotes the category of all (finitely generated) right S-modules, $P(S)$ the full subcategory of all finitely generated projective S-modules.

If $S = \oplus S_i$ is a graded ring, Mod-$S.$, mod-$S.$, $P(S.)$, denote the corresponding categories of graded S-modules with degree-preserving S-linear maps as morphisms.

0.2. By $D^*(S)$ for $*$ in $\{\ ,+,-,b\}$ we denote the <u>derived categories</u> of Mod-S , whose objects are "all" complexes of S-modules with <u>finitely generated cohomology modules</u>, non-zero only in the range indicated by "*". (In Hartshorne's notation, [Ha], this would be : $D^*_{\text{mod-}S}(\text{Mod-}S)$.)

Correspondingly, $K^*(S)$ denotes the <u>homotopy categories</u> of those complexes of S-modules whose cohomology modules are again finitely generated.

Each $D^*(S)$ or $K^*(S)$ is considered a <u>triangulated category</u> with respect to its natural triangulated structure.

In general, for terminology, notations and results on derived or triangulated categories, we use [Ver] as basic reference.

In particular, let us mention explicitely the following conventions :

- <u>Complexes</u> will usually be indexed upstairs and will have differentials of degree $+1$: $X = (X^\cdot, d_X^\cdot) = (X^i,\ d_X^i : X^{i-1} \longrightarrow X^i)_{i\in\mathbb{Z}}$.
- <u>Finitely generated S-modules</u> are tacitely identified with those complexes whose only non-zero term - if any - is in degree zero. (in other words, the canonical embedding $\text{mod-}S \longrightarrow D^*(S)$ gets no own name.)
- In any triangulated category D the <u>translation functor</u> will be denoted T - rather than T_D - or $-[1]$. In categories derived from complexes, it shifts those one place "to the left", that is against the direction of the differential, changes the sign of the differential but leaves morphisms "unchanged".
- Deviating from [Ver] and following [BBD;1.1.1.], <u>distinguished triangles</u> will be displayed linearly :
$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow[(1)]{d} (X[1]) \quad ,$$
where "(1)" indicates the morphism of degree one if the target $X[1]$ is not explicitely mentioned.

0.3. We will mostly deal with $D^b(S)$ - which by definition now is the derived category of all complexes of S-modules with finitely generated <u>total</u> cohomology. Without further notice it will be identified with its full triangulated subcategory of "<u>projective resolutions</u>" , $K^{-,b}(P(S))$, which is the homotopy category of all those bounded-above complexes of finitely generated projective S-modules whose cohomology is bounded. An isomorphism $F \longrightarrow X$ - in $D^b(S)$ - from an object F in $K^{-,b}(P(S))$ is called a <u>projective resolution</u> of X .

1. - *Perfect Complexes and the Stabilized Derived Category*

1.1. A complex of S-modules is perfect, if it is isomorphic in $D(S)$ to a finite complex of finitely generated projective S-modules.

Perfect complexes form an essential, (i.e. closed under isomorphisms), full and triangulated subcategory of $D^b(S)$, denoted $D^b_{perf}(S)$. The terminology "perfect" is borrowed from [SGA VI;I,2.1], - where it would be "P(S)-perfect" -, but has nothing to do with the notion of a "perfect" module or ideal as defined by grade-conditions - e.g. in [Kap;p.126]. In the sense used here, a module - considered as a complex - is perfect if and only if its projective dimension is finite.

1.2. An intrinsic characterization of perfect complexes is given by

Lemma 1.2.1.: Let S be a ring, X an object in $D^b(S)$. Then the following conditions are equivalent :

(i) X is perfect ,

(ii) There is an integer $i(X)$ such that for any $i \geqq i(X)$ and any finitely generated S-module M

$$\mathrm{Ext}^i_S(X,M) =_{def} \mathrm{Hom}_{D^b(S)}(X,T^iM) = 0 .$$

(iii) For any exact functor $F: D^b(S) \longrightarrow D$ into another triangulated category D for which $F(S) = 0$, - where S is considered a (complex of) right module(s) over itself -, one has $F(X) = 0$.

Proof: It follows from the definition that a complex X is perfect iff in (any of) its projective resolution(s) $F \longrightarrow X$ all the syzygy modules $\mathrm{Cok}(d^i_F)$ "far enough back" - $i << 0$ - are themselves projective modules. But as F has a priori bounded cohomology, this happens iff the augmented complex $(F^{\leq i} \longrightarrow \mathrm{Cok}(d^i_F))$ is contractible for some i, iff condition (ii) is satisfied.

The equivalence of (i) and (ii) shows that $D^b_{perf}(S)$ is a thick triangulated subcategory of $D^b(S)$ - in the sense of [Ver;I.2.1.1.]. Condition (iii) on the other hand characterizes the thick hull of the single object S considered as a subcategory of $D^b(S)$. It is clear that this hull contains $D^b_{perf}(S)$, which is already "thick" and contains S, whence one has in fact equality. This proves the equivalence of (i) and (iii).

It follows from the characterization of $D^b_{perf}(S)$ as a thick subcategory of $D^b(S)$ that there exists a quotient, which is again triangulated and unique up to natural equivalence, satisfying the universal property that any exact functor F on $D^b(S)$ as in (iii) above factors over it.

To give the child a name, we make the

Definition 1.2.2.: The triangulated quotient category

$$\underline{D^b(S)} = D^b(S)/D^b_{perf}(S)$$

will be called the stabilized derived category of S. The projection functor $D^b(S) \longrightarrow \underline{D^b(S)}$ is the identity on objects, which hence still are complexes, and associates to a morphism f in $D^b(S)$ - or to an actual morphism of complexes - its class $\underline{f}$ in $\underline{D^b(S)}$.

1.3. Remarks: (a) "Stabilized" as by the Lemma above the morphisms in $\underline{D^b(S)}$ satisfy :

$$\mathrm{Hom}_{\underline{D^b(S)}}(X,T^iY) = \mathrm{Hom}_{D^b(S)}(X,T^iY) = \mathrm{Ext}^i_S(X,Y)$$

for all $i >> 0$ - depending on the complexes X and Y.

(b) In practice, perfect complexes reflect those homological features of (complexes of) S-modules which are induced from "regular" rings in the following sense :

Assume S can be represented as a left module of finite flat dimension over some ring P, for example if P is of finite right global dimension (= "regular"). Then the image of the (left) derived functor

$$\underline{\underline{L}}(-\otimes_P S) : D^b(P) \longrightarrow D^b(S)$$

has as its thick hull $D^b_{perf}(S)$.

This follows as the assumptions just guarantee that the image of $D^b(P)$ under the (derived) tensor-product consists only of perfect complexes and as it certainely contains (an object isomorphic to) S. Now one may conclude as in the proof of (i) <==> (iii) above.

2. - *The Category of Modules modulo Projectives*

2.1. Recall - from [He 1;§3] or [A-B;1.43] for example - the following

Definition 2.1.1.: The (projectively) stabilized category of finitely generated S-modules, denoted $\underline{\text{mod-}S}$, is obtained by factoring out all projective modules from mod-S.

This means that $\underline{\text{mod-}S}$ has the same objects as mod-S but that its morphisms are given by

$$\underline{\mathrm{Hom}}_S(M,N) =_{def} \mathrm{Hom}_{\underline{mod\text{-}S}}(M,N) = \frac{\mathrm{Hom}_S(M,N)}{\{f : M \longrightarrow N \mid f \text{ factors over a projective } S\text{-module}\}}$$

for all finitely generated S-modules M and N .

$\underline{\text{mod-S}}$ is still an additive category and two S-modules M and N are isomorphic in $\underline{\text{mod-S}}$ if and only if they are <u>stably equivalent</u> (by projectives) in mod-S : $M \oplus P \simeq_S N \oplus Q$ for some finitely generated projective S-modules P and Q - [A-B;1.44].

2.2. On $\underline{\text{mod-S}}$ there is the "<u>loop-space functor</u>" Ω_S defined, introduced by Eckmann-Hilton and studied in general by A.Heller, [He 1] . It is obtained by choosing for any finitely generated S-module M a surjection from a finitely generated projective S-module $p_M : P_M \longrightarrow M$ and setting :

(2.2.1.) $\Omega_S M = \mathrm{Ker}(p_M)$.

With these notations one has :

<u>Lemma 2.2.2.</u>: The composition

$$\text{mod-S} \longrightarrow D^b(S) \longrightarrow \underline{D^b(S)}$$

factors uniquely over the canonical projection functor mod-S $\longrightarrow$ $\underline{\text{mod-S}}$ and yields hence a naturally defined functor

$$\iota_S : \underline{\text{mod-S}} \longrightarrow \underline{D^b(S)} .$$

It transforms the loop-space functor Ω_S into the inverse of the translation functor on $\underline{D^b(S)}$.

Only the last assertion needs a proof. By definition of Ω_S there is a distinguished triangle

$$\iota_S(\Omega_S M) \longrightarrow \iota_S(P_M) \longrightarrow \iota_S(M) \xrightarrow{d} \iota_S(\Omega_S M)[1]$$

in which $\iota_S(P_M)$ is obviously perfect. Hence the morphism d becomes an isomorphism in $\underline{D^b(S)}$.

3. - *Complete Resolutions and the Category of Acyclic Projective Complexes*

3.1. Adopting classical terminology - see for example [C-E;XII.3] - we make the following

Definition 3.1.1.: Assume given a S-module M over some ring S. Then a complete resolution of M (over S) is an acyclic complex (A,d_A) of finitely generated projective S-modules such that

$$\mathrm{Cok}(d_A^0 : A^{-1} \longrightarrow A^0) = M \quad .$$

To abbreviate notations, the complex $A_- = (A^{\leqq 0}, d_A|_{A^{\leqq 0}})$ with its natural induced augmentation onto M is called the associated projective resolution of M, whereas the complex $A_+ = (A^{\geqq 1}, d_A|_{A^{\geqq 1}})[1]$ with its induced natural co-augmentation from M into it is the associated projective co-resolution of M.

Remark that M is necessarily finitely generated if it admits a complete resolution and that a finitely generated module admits a complete resolution if and only if it admits a projective co-resolution.

The complete resolution A itself is obtained as the translated mapping-cone of $-d^0 : A_- \longrightarrow A_+$, that is $A = C(-d^0)[1]$, so that $-d^0$ serves as the connecting homomorphism from the associated projective resolution to the associated projective co-resolution.

3.2. Instead of considering right away all those S-modules which admit a complete resolution - a seemingly hopeless task in general - we rather introduce the "category of complete resolutions" as independent notion :

APC(S) denotes the homotopy category of (unbounded) acyclic complexes of finitely generated projective S-modules.

(In Verdier's notation, [Ver], this would be : $K^{\infty,\emptyset}(P(S))$.)
It is a full subcategory of $K(S)$, closed under translation and forming mapping-cones, hence inherits a triangulated structure from $K(S)$.

3.3. On APC(S) we define now the following functors :
First, for any complex (X,d_X) in $K(S)$ set

(3.3.1.) $$\Omega_i(X) = \mathrm{Cok}(d_X^{-i} : X^{-i-1} \longrightarrow X^{-i}) \quad \text{for any integer } i \ .$$

and call it the <u>i-th syzygy module</u> of X . As obviously

$$\Omega_0(X[1]) = \mathrm{Cok}(-d_X^1 : X^0 \longrightarrow X^1) \simeq \Omega_{-1}(X) \quad ,$$

one obtains

(3.3.2.) $\quad \Omega_i(X[j]) \simeq \Omega_{i-j}(X)$

for all integers i,j and all complexes X .(Remark that this isomorphism of functors is not "canonical" : for odd j it depends - at least - on the placement of $\pm\mathrm{id}$ in either even or odd degrees.)

If $f : X \longrightarrow Y$ is a morphism of complexes of finitely generated projective S-modules which is <u>zero-homotopic</u>, then, for any i , the induced morphism of S-modules $\Omega_i(f) : \Omega_i(X) \longrightarrow \Omega_i(Y)$ <u>factors over a finitely generated projective S-module</u> - namely even over <u>both</u> X^{-i+1} and Y^{-i} . This shows :

<u>Lemma 3.3.3.</u>: Each Ω_i defines a functor from <u>APC(S)</u> into <u>mod-S</u> . It transforms the <u>inverse</u> of the translation functor on <u>APC(S)</u> into the loop-space functor Ω_S on <u>mod-S</u> .

(The reader may excuse the confusing use of Ω's here.)

3.4. Next we will set up functors from <u>APC(S)</u> into <u>$D^b(S)$</u> . For this recall that for any complex X in $K(S)$ its <u>naive filtration</u>, $(\sigma_{\leqq k}X)_{k\in\mathbb{Z}}$, is given by

$$(3.4.1.)\quad \begin{array}{ccccccccccccc} \sigma_{\leqq k+1}X = X^{\leqq k+1} & \equiv & (& \cdots\longrightarrow & X^i & \longrightarrow & \cdots & \longrightarrow & X^k & \longrightarrow & X^{k+1} & \longrightarrow & 0\cdots) \\ \downarrow & & & \cdots & \Big\| \mathrm{id}_{X^i} & & \cdots & & \Big\| \mathrm{id}_{X^k} & & \downarrow & & \downarrow\cdots \\ \sigma_{\leqq k}X = X^{\leqq k} & \equiv & (& \cdots\longrightarrow & X^i & \longrightarrow & \cdots & \longrightarrow & X^k & \longrightarrow & 0 & \longrightarrow & 0\cdots) \end{array}$$

(Remark that the notation "$\sigma_{\leqq k}$" for these "tronqués bêtes à droite" is in accordance with [SGA 4;XVII,1.1.16] and [BBD] , but contrary to [Ha;I§7] .)

One obviously has an equality of functors

(3.4.2.) $\quad \sigma_{\leqq k}\circ T^i = T^i\circ\sigma_{\leqq k+i} \qquad$ for all k and i .

Coming back to acyclic complexes of finitely generated projective S-modules, the following is easily established.

Lemma 3.4.3.: Let A and B be objects of $\underline{APC(S)}$. Then one has

(i) In $D(S)$ (the class of) the obvious morphism of complexes $\sigma_{\leq k}A \longrightarrow (\Omega_{-k}A)[-k]$ becomes an isomorphism, or - equivalently -

(i') $(\sigma_{\leq k}A)[k]$ is a projective resolution of $\Omega_{-k}A$.

(ii) For any two integers $k \leq l$, the mapping-cone over the natural morphism $\sigma_{\leq l}A \longrightarrow \sigma_{\leq k}A$ is *perfect*.

(iii) If $f : A \longrightarrow B$ is a morphism of complexes which is *homotopic to zero*, all the induced morphisms $\underline{\sigma_{\leq k}f} : \underline{\sigma_{\leq k}A} \longrightarrow \underline{\sigma_{\leq k}B}$ in $\underline{D^b(S)}$ are *zero*.

For (iii) just consider the commutative diagram of morphisms of complexes

$$\begin{tikzcd}
\sigma_{\leq k}A \arrow[r] \arrow[d, "\sigma_{\leq k}f"'] & (\Omega_{-k}A)[-k] \arrow[d, "(\Omega_{-k}f)[-k]"] \\
\sigma_{\leq k}B \arrow[r] & (\Omega_{-k}B)[-k]
\end{tikzcd}$$

whose horizontal arrows become isomorphisms in $D^b(S)$ - hence a fortiori in $\underline{D^b(S)}$ - by (i) . But, as observed above, f zero-homotopic implies that $\Omega_{-k}f$ factors over a projective module, and so the class of $(\Omega_{-k}f)[-k]$ is zero in $\underline{D^b(S)}$.

These elementary facts show :

Lemma 3.4.4.: The naive truncations $(\sigma_{\leq k})_{k\in\mathbb{Z}}$ define a directed system of functors

$$\cdots \xrightarrow{\simeq} \underline{\sigma}_{\leq k} \xrightarrow{\simeq} \underline{\sigma}_{\leq k-1} \xrightarrow{\simeq} \cdots$$

from $\underline{APC(S)}$ into $\underline{D^b(S)}$, whose transition morphisms are all isomorphisms. In particular, its *inverse limit* $\underline{\sigma}_{\leq} = \varprojlim_{k} \underline{\sigma}_{\leq k}$ exists and is an *exact functor of triangulated categories*

$$\underline{\sigma}_{\leq} : \underline{APC(S)} \longrightarrow \underline{D^b(S)} \quad .$$

4. - *Maximal Cohen-Macaulay Modules over Gorenstein Rings*

4.1. From now on, assume for the given ring S - which is still supposed to be noetherian on both sides - that :

<u>S is of finite injective dimension both as a left or a right module over itself</u>.

By a result of A.Zaks, [Z], if both the left or right injective dimension of S are finite, they are the same and we will call this common value the <u>injective</u> or <u>virtual dimension</u> of S, and will abbreviate it as vdim S .(Consequently we will occasionally say that S is a "ring of finite injective (virtual) dimension", if vdim $S < \infty$.)

Such rings of finite virtual dimension could (and will) be called (<u>strongly</u>) <u>Gorenstein</u> in view of the well-known "commutative"

<u>Theorem 4.1.1.</u>: Assume that S is a (noetherian) <u>commutative</u> ring of <u>finite Krull dimension</u>. Then the following are equivalent :

(i) The injective dimension of S is finite.
(ii) For any prime $\mathbf{p}$ of S , the localization $S_{\mathbf{p}}$ is Gorenstein.
(iii) S admits a canonical module which is projective.

Furthermore, under these conditions, the injective dimension of S equals its Krull dimension.

(i) ==> (ii) is true for any noetherian commutative ring, but
(ii) ==> (i) needs the finite Krull dimension. For this and
(ii)<==> (iii) see, for example, [Aus 1;§1.4,Thm.2] and [FGR;5.5,5.6] .

<u>Remark</u>: We opt here for the term <u>strongly Gorenstein</u>, as in the commutative case it is indeed more restrictive than the usual definition which requires only (ii) above, but <u>not</u> the finite Krull dimension - see [Ha;p.296] for example.

On the other hand, M.Auslander introduced the notion of a (non-commutative) Gorenstein ring - see [FGR;p.47] - which, in general, does not imply "strongly Gorenstein". For example, as J.E.Roos, [Ro], pointed out, there are even rings of finite global dimension which are not Gorenstein in M.Auslander's sense.

Nevertheless, using J.E.Roos' results, many interesting (non-commutative) examples satisfy both definitions, namely those rings S which admit a filtration such that the associated graded ring is commutative, Gorenstein and of finite Krull dimension.

Let us also record the obvious fact that being strongly Gorenstein is left-right symmetric :
<u>A ring S is strongly Gorenstein if and only if this holds for S^{op}</u>, the opposite ring of S .

4.2. We now come to the main object(s) of our study :

<u>Definition 4.2.1.</u>: Let S as above be a ring of finite injective dimension. Then a finitely generated S-module is <u>maximal Cohen-Macaulay</u> (MCM for short), if and only if

$$\mathrm{Ext}^i_S(M,S) = 0 \quad \text{for} \quad i \neq 0 .$$

The full subcategory of maximal Cohen-Macaulay modules in mod-S is denoted MCM(S) , and, accordingly, its image in $\underline{\text{mod-S}}$ by $\underline{\text{MCM(S)}}$.

Again, the terminology is borrowed from commutative algebra, as over a local, commutative Gorenstein ring it coincides with the usual notion. (Following M.Auslander - Aus 1;3.2.2 , A-B;Ch.3 - one also could identify maximal Cohen-Macaulay modules as those of "<u>G</u>(orenstein)<u>-dimension zero</u>". But, only dealing with (strongly) Gorenstein rings, we believe "MCM" to be more suggestive.)

(To keep the definition "coordinate-free" one may replace the module S in the above definition by any <u>faithfully projective</u> (right) S-module P : An object M in mod-S is MCM if and only if

$$\mathrm{Ext}^i_S(M,P) = 0 \quad \text{for} \quad i \neq 0 ,$$

see for example [Ba;II.1.2.] .)

The analogy to the commutative case is supported as maximal Cohen-Macaulay modules in general share the following elementary properties :

<u>Lemma 4.2.2.</u>: Let S be a ring which is strongly Gorenstein. Then
(i) Any finitely generated <u>projective</u> S-module is MCM, that is, P(S) is a full subcategory of MCM(S) .
(ii) If $0 \longrightarrow M_1 \xrightarrow{f} M_2 \longrightarrow M_3 \longrightarrow 0$ is an exact sequence in mod-S , then
- M_2 , M_3 in MCM(S) implies that M_1 is MCM,
- M_1 , M_3 in MCM(S) implies that M_2 is MCM,

- M_1, M_2 in MCM(S) implies that M_3 is MCM <u>iff</u> $\mathrm{Hom}_S(f,S)$ is surjective.

(iii) A module M is MCM over S if and only if $M^* = \mathrm{Hom}_S(M,S)$ is MCM as a right module over S^{op}. Furthermore, MCM's are <u>reflexive</u> : $M = M^{**}$, and a sequence of such is short exact in mod-S, if and only if the dual sequence in mod-S^{op} is exact.
In other words : The functor $\mathrm{Hom}_S(-,S)$ induces an <u>exact duality</u> between MCM(S) and MCM(S^{op}).

(iv) Any module in mod-S admits a finite resolution by MCM's of length at most equal to vdim S .(In such a resolution all but the last module can be chosen to be finitely generated projective.)

Everything, except perhaps the first assertions in (iii), is obvious from the definition and left to the reader.

That M is maximal Cohen-Macaulay over S iff M^* is so over S^{op} follows immediately by dualizing a projective resolution of M : By the very definition of MCM, this yields a projective co-resolution - 3.1.- of M^* in mod-S^{op} and M^* is hence a syzygy module of arbitrarily high order. But as the (left) injective dimension of S is finite, this shows that M^* is necessarily $\mathrm{Hom}_S(-,S)$-acyclic, that is MCM. Furthermore, dualizing once again, it follows that M is reflexive.

4.3. <u>Remarks</u>: (a) Again, the duality-statement in (iii) above should rather be "coordinate-free" : Any <u>invertible S-bimodule</u> ω_S may serve as a <u>dualizing module</u> : The functor $\mathrm{Hom}_S(-,\omega_S)$ still defines an exact duality between MCM(S) and MCM(S^{op}) ,(cf. [Ba;II.5.]).

An obvious advantage of such a "coordinate-free" description is that it behaves better functorially. As an example, it is left to the reader to convince himself of the fact - not needed in the sequel - that being "strongly Gorenstein" or "MCM" is invariant under Morita-equivalence.

(b) In D.Quillen's terminology, [Qu 1;§2], property (ii) above can be rephrased as follows :

MCM(S) is an <u>exact subcategory</u> of mod-S , in which <u>any</u> epimorphism is <u>admissible</u> and in which the <u>admissible monomorphisms</u> are exactly those morphisms f , which are monomorphic in mod-S and <u>whose dual f^* is epimorphic</u> in mod-S^{op} .

In particular, there are defined <u>algebraic K-groups</u> for MCM(S) and property (iv) of the Lemma shows - by Thm.3, Cor.3 of (loc.cit.) - that <u>the K-groups of</u> mod-S <u>and</u> MCM(S) <u>are the same</u>.

4.4. Now we can state the main result :

<u>Theorem 4.4.1.</u>: Let S be a left and right noetherian ring, of finite injective dimension as a module over itself on either side. Then

(1) The syzygy-functor Ω_0 , defined in (3.3.), induces an equivalence of categories - denoted by the same symbol - :

$$\Omega_0 : \underline{\mathrm{APC}(S)} \longrightarrow \underline{\mathrm{MCM}(S)} \quad .$$

(2) The restriction of ι_S , defined in (2.2.), to $\underline{\mathrm{MCM}(S)}$ yields an equivalence of categories - again denoted by the same symbol - :

$$\iota_S : \underline{\mathrm{MCM}(S)} \longrightarrow \underline{D^b(S)} \quad .$$

(3) The triangulated structures induced on $\underline{\mathrm{MCM}(S)}$ by either Ω_0 or ι_S are the "same" (in the sense that the identity on $\underline{\mathrm{MCM}(S)}$ becomes an exact isomorphism of triangulated categories), and - with respect to these structures - both functors are <u>exact equivalences of triangulated categories</u>, transforming the corresponding translation functor T into an inverse of the loop-space functor Ω_S , (2.2.), restricted to $\underline{\mathrm{MCM}(S)}$.

Before going into the proof, let us resume the situation "graphically" : There is a diagram of categories and functors, commutative up to natural isomorphisms of functors, whose rows are "exact sequences" of categories

$$\begin{tikzcd}
 & 0 \arrow[rr] & & P(S) \arrow[rr] \arrow[dl, equal] & & \mathrm{MCM}(S) \arrow[rr] \arrow[dl] & & \underline{\mathrm{MCM}(S)} \arrow[rr] \arrow[dl] \arrow[dd, "\simeq\ \iota_S"] & & 0 & \\
0 \arrow[rr] & & P(S) \arrow[rr] \arrow[dr] & & \text{mod-}S \arrow[rr] \arrow[dr] & & \underline{\text{mod-}S} \arrow[rr] \arrow[dr, "\iota_S"'] & & 0 & & \underline{\mathrm{APC}(S)} \arrow[ulll, "\Omega_0"', "\simeq"] \arrow[dlll, "\simeq", "\iota_S\Omega_0 \simeq \underline{\sigma}_{\leq 0}"'] \\
 & 0 \arrow[rr] & & D^b_{perf}(S) \arrow[rr] & & D^b(S) \arrow[rr] & & \underline{D^b(S)} \arrow[rr] & & 0 &
\end{tikzcd}$$

(the unlabeled morphisms are the canonical embeddings.)

Proof of the Theorem :

(i) Ω_0 takes its values in MCM(S) .

Assume given an acyclic complex A of finitely generated projective S-modules. Then

$$\mathrm{Ext}^i_S(\Omega_0 A, S) = \mathrm{Ext}^{i+j}_S(\Omega_{-j} A, S)$$

for all $i > 0$, $j \geqq 0$. Now take $j > \mathrm{vdim}\, S - i$.

(ii) Ω_0 is surjective on objects.

Let M be a maximal Cohen-Macaulay module over S , $\mathbf{P}(M^*) \longrightarrow M^*$ a projective resolution of M^* in mod-S^{op} . As in the proof of the Lemma above,

$$0 \longrightarrow M^{**} \longrightarrow \mathrm{Hom}_{S^{op}}(\mathbf{P}(M^*), S^{op}) = \mathbf{P}(M^*)^*$$

will be a projective co-resolution of M^{**} in mod-S . Now $M \simeq M^{**}$, as M is reflexive, and hence extending the projective co-resolution of M^{**} by a projective resolution of M yields a desired pre-image.

(iii) Ω_0 is a full functor.

Let $f : M \longrightarrow N$ be a S-linear map of maximal Cohen-Macaulay modules over S . Extend it to a morphism $f^{\cdot} : \mathbf{P}(M) \longrightarrow \mathbf{P}(N)$ between chosen projective resolutions and analogously $f^* : N^* \longrightarrow M^*$ to a morphism $(f^*)^{\cdot} : \mathbf{P}(N^*) \longrightarrow \mathbf{P}(M^*)$. Connecting $\mathbf{P}(M)$ and $\mathbf{P}(M^*)^*$ as well as $\mathbf{P}(N)$ and $\mathbf{P}(N^*)^*$ to complete resolutions of M and N respectively, $f^{\cdot}$ and $((f^*)^{\cdot})^*$ fit together to yield a morphism of these complete resolutions. By construction, this provides a pre-image of f , hence Ω_0 is full.

It remains to be seen that Ω_0 is faithful, to complete the proof of assertion (1) .
Instead of proving this - and assertion (2) - directly, we rather show :

$\underline{\sigma_{\leqq}}$: APC(S) $\longrightarrow$ $D^b(S)$ is an equivalence of categories.

This will readily imply the claim, as by (3.3.),(3.4.) there are natural isomorphisms of functors

$$\underline{\sigma}_{\leqq} \xrightarrow{\simeq} \underline{\sigma}_{\leqq 0} \xrightarrow{\simeq} \iota_S \circ \Omega_0 \quad ,$$

whence $\underline{\sigma}_{\leqq}$ an equivalence gives that Ω_0 is faithful - and therefore also an equivalence by the above - so that finally also ι_S will be an equivalence of categories, establishing (2) . So we prove :

(iv) $\underline{\sigma_{\leq}}$ is essentially surjective.

By (3.4.), it will suffice to find for any complex X in $D^b(S)$ an integer k, an object A in APC(S) and a morphism $X \longrightarrow \sigma_{\leq k}A$ of complexes whose mapping cone is perfect. Also, replacing if necessary X by any of its resolutions, we may assume that X itself is already a (bounded-above) complex of finitely generated projective S-modules. But then, all syzygy modules of X which sit "far enough back" have to be maximal Cohen-Macaulay. More precisely, using the same argument as in (i) above, $\sigma_{\leq k}(X) = \mathrm{Cok}(d_X^k) = \mathrm{Im}(d_X^{k+1} : X^k \longrightarrow X^{k+1})$ is certainely MCM for $k \leq \min(i : H^i(X) \neq 0) - \mathrm{vdim}\, S$, and, $\sigma_{\leq k}(X)[k]$ is a resolution of this module which then can be extended to an acyclic complex of projectives by the argument already used in (ii) above.

(v) $\underline{\sigma_{\leq}}$ is fully faithful.

We use Verdier's criterion - [Ver;I.5.3.] - : It suffices to prove that for a given perfect complex Y in $D^b(S)$ and an object A in APC(S) there exists an integer k such that all morphisms in $D^b(S)$ from $\sigma_{\leq k}A$ into Y are zero. As $\sigma_{\leq k}A$ is a bounded above complex of projective modules, morphisms in $D^b(S)$ from $\sigma_{\leq k}A$ into Y are in bijection with homotopy classes of (actual) morphisms of complexes and it is hence to show that any such morphism of complexes is indeed zero-homotopic.

Furthermore, it is enough to prove this assertion in case $Y = P[-i]$, P a finitely generated projective S-module and i an integer, as these objects generate - up to isomorphisms in $D^b(S)$ - any perfect complex by forming mapping-cones.

Now, in this particular case, take any $k > i$ and let $f^{\cdot}$ be a complex-morphism from $\sigma_{\leq k}A$ to $P[-i]$:

$$\begin{tikzcd}
\cdots \arrow[r] & A^{i-1} \arrow[r, "d^i"] \arrow[d] & A^i \arrow[r, "d^{i+1}"] \arrow[d, "f^i"] & A^{i+1} \arrow[r] \arrow[d] & ? \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & P \arrow[r] & 0 \arrow[r] & 0 \cdots
\end{tikzcd}$$

The S-linear map f^i factors hence necessarily over $\mathrm{Cok}(d^i)$, let $g : \mathrm{Cok}(d^i) \longrightarrow P$ be the induced map. It remains to show that g can be further factored over the <u>inclusion</u> (by choice of k !) of $\mathrm{Cok}(d^i) = \Omega_{-i}A$ into A^{i+1}. But from the exact sequence

$$0 \longrightarrow \Omega_{-i}A \longrightarrow A^{i+1} \longrightarrow \Omega_{-i-1}A \longrightarrow 0$$

it follows that the obstruction for this lies in $\mathrm{Ext}^1_S(\Omega_{-i-1}A,P)$, which group vanishes as P is finitely generated projective and $\Omega_{-i-1}A$ is still maximal Cohen-Macaulay by the argument in (i) above.
(3) finally is just (2.2.2.) and (3.3.3.) reformulated. qed

4.5. The proof shows how to find (quasi-)inverses of Ω_0 or ι_S :

Assume chosen projective resolutions $\mathbf{P}(M) \xrightarrow{p} M$ and $\mathbf{P}(M^*) \xrightarrow{q} M^*$ for any maximal Cohen-Macaulay module M over S . Then - by (ii) above and (3.1.) - a <u>complete resolution of M</u>, denoted $\mathbf{CR}(M)$, is obtained by translating the mapping-cone of the composition

$$d_M : \mathbf{P}(M) \xrightarrow{p} M \xrightarrow{\simeq} M^{**} \xrightarrow{q^*} \mathbf{P}(M^*)^* \quad ,$$
(4.5.1.)
$$\mathbf{CR}(M) = C(d_M)[-1] \quad ,$$

and one has by construction $\Omega_0(\mathbf{CR}(M)) = M$.

An <u>inverse of ι_S</u> involves :

- choosing for any complex X in ($\underline{D^b(S)}$) a projective resolution $F \longrightarrow X$,
- truncating this resolution at some
  $$k \leq \min(i : H^i(X) = H^i(F) \neq 0) - \mathrm{vdim}\, S \quad ,$$
  to obtain $\sigma_{\leq k}(F)$ - for which then $\Omega_k(\sigma_{\leq k}F) = \Omega_k(F)$ is MCM,
- extending this truncated complex to an acyclic complex $\sigma_{\leq k}(F)^{\#}$ of finitely generated projective S-modules and finally
- taking the 0^{th}-syzygy module of this extension :

$$\iota_S^{-1}(X) = \Omega_0(\sigma_{\leq k}(F)^{\#}) \quad ,$$
(4.5.2.)
$$X \dashleftarrow F \xrightarrow{\sigma_{\leq k}} \sigma_{\leq k}(F) \xleftarrow{\sigma_{\leq k}} \sigma^{\bullet}_{\leq k}(F)^{\#} \xrightarrow{\Omega_0} \iota_S^{-1}(X) \quad .$$

Obviously, to find a representative of $\iota_S^{-1}(X)$ like this will be rather tedious in practice and we will give an often more useful device in the next section - at least in case that X is (in $\underline{D^b(S)}$ isomorphic to) just a single S-module - by exhibiting a <u>left adjoint</u> of the embedding <u>MCM(S)</u> $\longrightarrow$ <u>mod-S</u> .
But first we use the description here to investigate the behaviour of the just established equivalences with respect to <u>duality</u> and then to find the <u>distinguished triangles</u> for the induced exact structure on <u>MCM(S)</u> .

4.6. Let ω_S be a <u>dualizing module</u> for S as in Remark 4.3.(a). As ω_S is by definition finitely generated projective as an S-module,

- the functor $\mathrm{Hom}^{\cdot}_S(-,\omega_S)$ on complexes transforms objects in <u>APC(S)</u> into objects of <u>$\mathrm{APC}(S^{op})$</u>,
- the functor $\mathrm{Hom}_S(-,\omega_S)$ preserves the properties of being "MCM" or "projective" - apply (4.2.2.iii) mutatis mutandis,
- the derived functor $\mathrm{RHom}_S(-,\omega_S)$ maps $D^b(S)$ into $D^b(S^{op})$, as with S also ω_S is of finite injective dimension, and preserves the property of "perfection".

Having observed all this, it follows that these functors induce exact dualities on each of the three categories involved in Theorem 4.4.1.. More precisely :

<u>Proposition 4.6.1.</u>: With notations as before, there is the following commutative diagram of triangulated categories and exact functors, whose rows are given by the equivalences of (4.4.1.) for S and S^{op} respectively and whose vertical arrows are <u>exact dualities</u> induced by the functors above :

$$\begin{tikzcd}
\underline{\mathrm{APC}(S)}^{op} \arrow[r, "\Omega_0^{op}"] \arrow[d, "{\mathrm{Hom}^{\cdot}_S(-,\omega_S)[-1]}"'] & \underline{\mathrm{MCM}(S)}^{op} \arrow[r, "\iota_S^{op}"] \arrow[d, "{\mathrm{Hom}_S(-,\omega_S)}"] & \underline{D^b(S)}^{op} \arrow[d, "{\mathrm{RHom}_S(-,\omega_S)}"] \\
\underline{\mathrm{APC}(S^{op})} \arrow[r, "\Omega_0"] & \underline{\mathrm{MCM}(S^{op})} \arrow[r, "\iota_{S^{op}}"] & \underline{D^b(S^{op})}
\end{tikzcd}$$

<u>Proof</u>: The commutativity of the right-hand square expresses just the fact that maximal Cohen-Macaulay modules are by definition $\mathrm{Hom}_S(-,\omega_S)$-acyclic.

For the square on the left, the foregoing construction of $\mathbf{CR}(-)$ as an inverse of the syzygy functor Ω_0 shows that for any maximal Cohen-Macaulay module M a complete resolution of its ω_S-dual $\mathrm{Hom}_S(M,\omega_S)$ can be obtained as

$$\mathbf{CR}(\mathrm{Hom}_S(M,\omega_S)) \cong C(\mathrm{Hom}_S(d_M,\omega_S))[-1]$$
, by definition of $\mathbf{CR}(-)$ and the fact that on $\mathrm{MCM}(S)$, $\mathrm{Hom}_{S^{op}}(\mathrm{Hom}_S(-,\omega_S),\omega_S) \cong \mathrm{id}$.

$$\cong C(\mathrm{Hom}_S(d_M,\omega_S)[-1])$$
, by definition of mapping-cones.

$$\cong \mathrm{Hom}^{\cdot}_S(C(d_M),\omega_S)$$
, by a (non-canonical) isomorphism of complexes.

$$= \mathrm{Hom}^{\cdot}_S(C(d_M)[-1],\omega_S)[-1] \qquad \text{obviously}$$
$$= \mathrm{Hom}^{\cdot}_S(\mathbf{CR}(M),\omega_S)[-1] \qquad \text{by definition of } \mathbf{CR}(M)\ .$$

Applying Ω_0 then yields the result.
(In less formal terms, the shift by [-1] on the left is caused by the fact that connecting a projective resolution of M with the dual of one for M^* "naively", creates a complex indexed by the integers with zero "doubled". Re-indexing "correctly" - as in (3.1.) - introduces the shift if one dualizes.)

4.7. Next we want to describe the triangulated structure on $\underline{\mathrm{MCM}(S)}$ directly. In general, such a structure on an additive category is determined by its <u>distinguished triangles</u> and we will use the Theorem to describe those in the induced structure on $\underline{\mathrm{MCM}(S)}$.

For this, let $f : M \longrightarrow N$ be any S-linear map of maximal Cohen-Macaulay S-modules. Choose an embedding $i : M \longrightarrow Q$ of M into a finitely generated projective S-module such that its cokernel is still MCM. (This means just that Q serves as the first term in a projective co-resolution of M .) Then define a <u>mapping-cone</u> $C(f)$ of f as the push-out - or amalgamated sum - of f and i , so that there is a commutative diagram of short exact sequences of S-modules :

$$\text{(4.7.1.)} \qquad \begin{tikzcd} 0 \arrow[r] & M \arrow[r, "i"] \arrow[d, "f"'] & Q \arrow[r, "p"] \arrow[d] & \mathrm{Cok}(i) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & N \arrow[r, "i'"] & C(f) \arrow[r, "p'"] & \mathrm{Cok}(i) \arrow[r] & 0 \end{tikzcd} \ .$$

Remark that $\mathrm{Cok}(i)$ represents $T(M)$, the translate of M , as by (2.2.1.) $\Omega_S(\mathrm{Cok}(i))$ is represented by M and by (4.4.1.(3)) the translation functor T on $\underline{\mathrm{MCM}(S)}$ is an inverse of Ω_S . Now call

$$\text{(4.7.2.)} \qquad M \xrightarrow{f} N \xrightarrow{i'} C(f) \xrightarrow{-p'} TM = \mathrm{Cok}(i)$$

a <u>typical triangle</u>. Then the distinguished triangles in $\underline{\mathrm{MCM}(S)}$ are given by all those sequences of morphisms in $\underline{\mathrm{MCM}(S)}$ which are isomorphic to the image of a typical triangle under the projection functor $\mathrm{MCM}(S) \longrightarrow \underline{\mathrm{MCM}(S)}$.
That this yields in fact exactly all distinguished triangles in $\underline{\mathrm{MCM}(S)}$

is most easily seen by applying Ω_0 to the distinguished triangles in APC(S) .

4.8. Remark: Starting with (4.7.1.-2.) to define distinguished triangles on MCM(S) , it is possible - but lengthy - to verify directly the axioms - (TR 1) up to (TR 4) in [Ver;I.1.1.] - defining a triangulated category.

The key-fact, from this point of view, is that MCM(S) is an exact subcategory of mod-S , (see Remark 4.3.(b)), which is Frobenius :

This means - following A.Heller, [He 1;§3], - that with respect to the given exact structure there exists an object - namely S in MCM(S) - serving at the same time both as a (small, admissible) projective generator - which then yields by the usual procedure (admissible) projective resolutions - and as a (small, admissible) injective cogenerator - which hence allows the construction of (admissible) projective co-resolutions.

From these assumptions - exact and Frobenius - it follows already that MCM(S) , the stabilized category obtained by dividing out - in the sense of (2.1.) - the projective-injective objects, is triangulated in a natural way, the distinguished triangles being defined as above.
For example, the most laborious axiom of a triangulated structure, namely (TR 2) of (loc.cit.), which says that distinguished triangles can be rotated, was already anticipated by A.Heller, [He 1;Thm.5.3.], where it was (just ?) a "remarkable property" - triangulated structures were not defined yet ! (That "Frobenius" was the right categorical notion in our context was pointed out to me by D.Happel, who also described other examples arising from the representation theory of artinian algebras - see [Hap].)

Continuing in this abstract setting, one may wonder for which morphisms of triangulated categories - as the embedding $D^b_{perf}(S) \longrightarrow D^b(S)$ or the derived tensor-product $\underline{\underline{L}}(-\otimes_P S) : D^b(P) \longrightarrow D^b(S)$ of the Remark 1.3.(b) - an "excision theorem" like (4.4.1.(2)) can be established.
As a first step in this direction, one has of course to know how to recognize the "excised" subcategory, which is MCM(S) as a full subcategory of $D^b(S)$ in our case. Here, this can be done as follows - see [BBD] for the no(ta)tions used :

Both $D^b(S)$ and $D^b(S^{op})$, the derived category of left S-modules, carry natural <u>t-structures</u> whose <u>hearts</u> are precisely (equivalent to) the abelian categories mod-S , respectively S-mod = mod-S^{op} , which were "derived".

Now a "<u>dualizing complex</u>" - which in this case is just S or, more generally, any dualizing module ω_S ,(4.3.(a)), - establishes an exact duality

$$\mathrm{RHom}_S(-,\omega_S) : D^b(S)^{op} \longrightarrow D^b(S^{op}) \quad ,$$

as already used in (4.6.). Pulling back the natural t-structure from $D^b(S^{op})^{op}$ onto $D^b(S)$ by this duality establishes hence a second such structure on $D^b(S)$, whose "heart" - equivalent to $(S\text{-mod})^{op}$ by construction - consists of all those complexes X of (right) S-modules for which the ω_S-dual $\mathrm{RHom}_S(X,\omega_S)$ has its only non-vanishing cohomology in degree zero.

The intersection in $D^b(S)$ of the hearts of these two t-structures is by the very definition equivalent to its full subcategory of all maximal Cohen-Macaulay modules over S , that is MCM(S), - endowed already with its exact structure.

It seems hence reasonable to expect that duality theory is at the heart of the matter - and at least for "rings with dualizing modules", that is the (not necessarily commutative) generalization of Cohen-Macaulay rings, one may indeed extend (4.4.1.(2)). The prize to pay is some more categorical machinery as one needs a more complicated stabilized version of <u>MCM(S)</u> - one has then also formally to invert the still existing "loop-space functor", (2.2.), on the category of MCM's modulo projectives.

4.9. We close this section with a remark on K-groups.
Following D.Quillen [Qu 1], we denote $K_i'(S) = K_i(\text{mod-}S)$ the algebraic K-groups of all finitely generated (right) S-modules and $K_i(S) = K_i(P(S))$ the groups obtained from all finitely generated projective S-modules. On the other hand, as for any triangulated category - [SGA V;VIII] -, there is defined the Grothendieck-group of <u>MCM(S)</u> , which will be denoted $\underline{K}_0(S)$. Then one has :

- a short exact sequence : $K_0(S) \longrightarrow K_0'(S) \longrightarrow \underline{K}_0(S) \longrightarrow 0$, by [SGA V;VIII.3.1.] and (4.4.1.(2)) , as well as
- group-homomorphisms $K_i(S) \longrightarrow K_i'(S)$ for any i by [Qu 1;§7].

Hence the following question seems rather natural :

Given a ring S which is strongly Gorenstein, does there exist a "<u>stabilized</u>" higher algebraic K-theory for S , in the sense that there are naturally defined groups $\underline{K}_i(S) = K_i(\underline{MCM(S)})$ for $i > 0$ which fit the short exact sequence and natural group-homomorphisms just mentioned into a long exact sequence

$$\cdots \longrightarrow \underline{K}_{i+1}(S) \longrightarrow K_i(S) \longrightarrow K_i'(S) \longrightarrow \underline{K}_i(S) \longrightarrow \cdots \qquad ?$$

As will be seen later - (   ) -, in some special cases such groups can be defined, but starting from rather different interpretations of $\underline{MCM(S)}$ and its equivalent companions.

5. - *Maximal Cohen-Macaulay Approximations*

5.0. From now on, it will be assumed - if not explicitely stated otherwise - that "<u>modules</u>" are finitely generated and that "<u>complexes of modules</u>" have finitely generated, bounded cohomology.

The aim of this section is to investigate "how the subcategory of maximal Cohen-Macaulay modules is embedded into the category of all finitely generated modules".

As a more practical aspect of the general theory developed so far, we will give three (essentially equivalent) presentations of an arbitrary module in terms of a maximal Cohen-Macaulay module and a module of finite projective dimension. This extends (and follows) results of M.Auslander, [Aus 1], and M.Auslander-M.Bridger, [A-B].

5.1. We start with the following Lemma which shows that over a ring S ,which is noetherian and of finite injective dimension, maximal Cohen-Macaulay modules and modules of finite projective (or injective) dimension are "orthogonal" with respect to Ext_S^* .

<u>Lemma 5.1.1.</u>: Let S be a ring which is strongly Gorenstein. Then

(i) A S-module M is MCM iff $Ext_S^i(M,U) = 0$ for $i \neq 0$ and all S-modules U of finite projective dimension.

(ii) A S-module U is of finite projective dimension iff $Ext_S^i(M,U) = 0$ for all $i \neq 0$ and all MCM S-modules M .

(iii) A S-module U is of finite projective dimension iff it is of finite injective dimension.

(iv) A S-module is MCM and of finite projective dimension iff it is projective.

(v) Any S-linear map from a MCM to a module of finite projective dimension factors over a projective module.

Proof: In (i), (resp.(iv)), the "if"-part follows from the definition of MCM (and (4.2.2.i)), the "only if"-part by induction on the projective dimension of U (resp. M).
In (ii) again, the "only if"-part is obtained by induction on the projective dimension of U, whereas the "if"-part follows from (iv) and (4.2.2.iv) : Take M to be the "last" module in a finite resolution of U, in which all modules are MCM and all except perhaps M are projective. Then M is necessarily both MCM and of finite projective dimension, hence itself projective.
Using (4.2.2.iv) once more, it is clear that (ii) and (iii) are equivalent.
Finally, assertion (v) follows from (i) as well as (ii).

The next result shows that in fact maximal Cohen-Macaulay modules together with those of finite projective (injective) dimension "span" the category mod-S :

Theorem 5.1.2.: (The Syzygy-Theorem for Gorenstein Rings)
Let S be a ring which is strongly Gorenstein.

(1) Any finitely generated S-module N admits a presentation

(5.1.3) $0 \longrightarrow U \longrightarrow M \xrightarrow{p} N \longrightarrow 0$, where

- U is of finite projective dimension,
- M is maximal Cohen-Macaulay.

(2) Such a presentation is unique up to (projectively) stable equivalence : If $N = M_1/U_1$ is a second presentation of the same kind, there exist finitely generated projective S-modules P and Q, such that $M \oplus P \simeq M_1 \oplus Q$ and $U \oplus P \simeq U_1 \oplus Q$.

(3) The surjection $p : M \longrightarrow N$ is universal with respect to the following property : Whenever $f : M' \longrightarrow N$ is a morphism from some MCM S-module M' to N, it factors over p and this factorization is unique in mod-S .

An essentially equivalent form of this Theorem is :

<u>Theorem 5.1.4.</u>: Let S still be strongly Gorenstein. Then

(1) Any finitely generated S-module N occurs in an exact sequence

$$(5.1.5) \quad 0 \longrightarrow N \xrightarrow{i} V \longrightarrow L \longrightarrow 0 \quad , \quad \text{where}$$

- V is of <u>finite projective dimension</u>,
- L is <u>maximal Cohen-Macaulay</u>.

(2) Such a presentation is <u>unique up to</u> (projectively) <u>stable equivalence</u> in the same sense as above.

(3) The injection $i : N \longrightarrow V$ is universal with respect to the following property : Whenever $j : N \longrightarrow V'$ is a morphism from N to some S-module V' of finite projective dimension, it factors over i and this factorization is unique in $\underline{\text{mod-}S}$.

(4) Comparing with the representation as in (5.1.3), L is isomorphic to $T(M)$, the translate of M , in $\underline{MCM(S)}$.

<u>Proof of (5.1.2.) and (5.1.4)</u>: The asserted uniqueness in (2) as well as the claimed universal mapping property in (3) - which statements are easily seen to be equivalent anyway - follow immediately from the Lemma above. One also may deduce these properties - together with (5.1.4.(4)) - from Theorem (4.4.1) : Converting as usual the exact sequences (5.1.3.) and (5.1.5.) into distinguished triangles in $D^b(S)$ and then projecting them into the stabilized derived category $\underline{D^b(S)}$, U resp. V become zero-objects, whence one obtains isomorphisms in $\underline{D^b(S)}$:

$$U \cong 0 \longrightarrow M \xrightarrow[\simeq]{p} N \longrightarrow 0 \cong U[1] \quad \text{and}$$

$$V \cong 0 \longrightarrow L \xrightarrow[\simeq]{} N[1] \longrightarrow 0 \cong V[1] \; .$$

Furthermore, as $\iota_S : \underline{MCM(S)} \longrightarrow \underline{D^b(S)}$ is an exact equivalence, this shows that M resp. L are unique up to isomorphism in $\underline{MCM(S)}$: M represents $\iota_S^{-1}(N)$ and L represents $\iota_S^{-1}(N[1]) = (\iota_S^{-1}(N))[1] = T(M)$.

Hence it remains to verify the existence of the claimed representations. For this let $\mathbf{F} \longrightarrow N$ be a projective resolution of N .

As S is of finite injective dimension, the complex $\mathbf{F}^* = \mathrm{Hom}_S(\mathbf{F},S)$ has still bounded cohomology. Now let $\varphi: \mathbf{G} \longrightarrow \mathbf{F}^*$ be a projective resolution (over S^{op} of course) of $\mathbf{F}^*$ - remark that although $\mathbf{F}^*$ is a complex of projective S^{op}-modules, it is in general not bounded above!

Dualizing again, one obtains a quasi-isomorphism from $\mathbf{F}^{**} = \mathbf{F}$ to a <u>bounded-below</u> complex $\mathbf{G}^*$ of finitely generated projective S-modules :

(5.1.6.) :

$$\begin{tikzcd}
\cdots F^{-k-1} \arrow[r] \arrow[d] & F^{-k} \arrow[r] \arrow[d] & \cdots \arrow[r] & F^{-1} \arrow[r] \arrow[d] & F^{0} \arrow[r] \arrow[d] & 0 \arrow[d] & \equiv \mathbf{F} \cong \mathbf{F}^{**} \arrow[d, "\equiv \varphi^*"'] \\
0 \arrow[r] & (G^{k})^* \arrow[r] & \cdots \arrow[r] & (G^{1})^* \arrow[r, "\delta^0"] & (G^{0})^* \arrow[r, "\delta^1"] & (G^{-1})^* \arrow[r, "\delta^2"] & \cdots \equiv \mathbf{G}^*
\end{tikzcd} ,$$

where we have set $\delta^j = \mathrm{Hom}_S(d_{\mathbf{G}}^{-j+1}, S)$ for all j .

(Remark that the largest k , for which $G^k \neq 0$, can be chosen to be

(5.1.7.) $\quad k = \max(i : \mathrm{Ext}_S^i(N,S) \neq 0)$.)

As φ^* is a quasi-isomorphism, the only cohomology of $\mathbf{G}^*$ is the given module N in degree zero. Now set:

- $M = \mathrm{Ker}(\delta^1)$ and $U = \mathrm{Im}(\delta^0)$, so that $N = M/U$,
- $V = \mathrm{Cok}(\delta^0)$ and $L = \mathrm{Im}(\delta^1)$, so that $N = \mathrm{Ker}(V \xrightarrow{\delta'} L)$, where δ' is the surjection induced from δ^1 .

It follows from the construction that U and V are of finite projective dimension, as they are resolved respectively by the "negative (non-positive) tail" of $\mathbf{G}^*$, whereas M and L are necessarily MCM as they allow respectively the "non-negative (positive) head" of $\mathbf{G}^*$ as projective co-resolutions. Hence we have found the desired representations of N .

Remark also that this construction yields another, direct proof of (5.1.4.(4)): M and L occur in the short exact sequence

(I) $\quad 0 \longrightarrow M = \mathrm{Ker}(\delta^1) \longrightarrow (G^0)^* \longrightarrow \mathrm{Im}(\delta^1) = L \longrightarrow 0$

which shows, using (4.7.), that $L = M[1]$ in <u>MCM(S)</u> . qed

5.2. <u>Remarks</u>: (a) In a presentation as (5.1.3.), U cannot be an arbitrary module of finite projective dimension as it embeds into the MCM M . More precisely, the definition of U shows that it occurs in the short exact sequence

(II) $\quad 0 \longrightarrow U = \mathrm{Im}(\delta^0) \longrightarrow (G^0)^* \longrightarrow \mathrm{Cok}(\delta^0) = V \longrightarrow 0$,

whence it is already the first syzygy module of the module V of finite projective dimension. Applying $\mathrm{Ext}_S^{\cdot}(-,S)$ to (5.1.3.) ,(5.1.4.), one obtains furthermore

$$\mathrm{Ext}_S^i(U,S) = \mathrm{Ext}_S^{i+1}(N,S) = \mathrm{Ext}_S^{i+1}(V,S) \quad \text{for} \quad i > 0 \quad .$$

(b) As communicated to us by G.Evans, he and P.Griffith independently obtained (5.1.2.) in the commutative case.

5.3. The four exact sequences (5.1.3.), (5.1.5.), (I) and (II) fit into the following commutative diagram :

$$
(5.3.1.) \qquad
\begin{tikzcd}
 & 0 & 0 & & & \\
0 \arrow[r] & N \arrow[u] \arrow[r, "i"] & V \arrow[u] \arrow[r, "\delta'"] & L \arrow[r] & 0 & \text{(5.1.5.)} \\
0 \arrow[r] & M \arrow[u, "p"] \arrow[r, "i'"] & (G^0)^* \arrow[u, "p'"'] \arrow[r] \arrow[ur, phantom, "(*)"] & L \arrow[u, equal] \arrow[r] & 0 & \text{(I)} \\
 & U \arrow[u] \arrow[r, equal] \arrow[ur, phantom, "(*)"] & U \arrow[u] & & & \\
 & 0 \arrow[u] & 0 \arrow[u] & & & \\
 & \text{(5.1.3.)} & \text{(II)} & & &
\end{tikzcd}
,
$$

where i' is the natural inclusion induced by δ^1 and p' the natural surjection induced by δ^0 .

To interpret this diagram in terms of triangulated categories, consider its image in $D^b(S)$. It is then just the "display" of the octahedron constructed - by choice - either over the composable monomorphisms

$$U = \mathrm{Im}(\delta^0) \longrightarrow M = \mathrm{Ker}(\delta^1) \xrightarrow{i'} (G^0)^* \quad ,$$

or the composable epimorphisms

$$(G^0)^* \xrightarrow{p'} V = \mathrm{Cok}(\delta^0) \xrightarrow{\delta'} L = \mathrm{Im}(\delta^1) \quad .$$

The distinguished triangles of the octahedron are given by the four exact sequences above, its two "visible" commutative faces are marked by (*) , its "top" is L , its "bottom" is U and the "equator" is represented by the square (i', p', i, p) , - see [BBD;1.1.7.-8.] . Accordingly, we will call (5.3.1.) "the" canonical representation or octahedron associated to a module N over a strongly Gorenstein ring S . It is unique up to (projectively) stable equivalence in the sense that to one of the "triangles" (*) one may add one and the same projective module to each of the three vertices.

Remark that the "equator" is a bi-cartesian square which yields hence a "Mayer-Vietoris"-sequence

$$(5.3.2.) \qquad 0 \longrightarrow M \xrightarrow{p\oplus i'} N \oplus (G^0)^* \xrightarrow{i-p'} V \longrightarrow 0 \qquad ,$$

exhibiting the given module N - up to adding a projective module - as an <u>extension</u> of a module of finite projective dimension by a MCM. Again this representation of N is unique in <u>mod-S</u> , proving the so-called "Approximation Theorem" of M.Auslander - [Aus 1;Ch.3,Prop.8], [A-B;4.27,2.41] :

<u>Corollary 5.3.3.</u>: With the notations of (5.1.2.), there exist a finitely generated projective S-module Q and an exact sequence

$$0 \longrightarrow M \longrightarrow N \oplus Q \longrightarrow V \longrightarrow 0 \qquad ,$$

in which

- M is a maximal Cohen-Macaulay module,
- V is of finite projective dimension,

so that in <u>mod-S</u>

- the morphism $M \longrightarrow N \oplus Q \cong N$ is universal with respect to morphisms <u>from</u> MCM's to N and
- the morphism $N \cong N \oplus Q \longrightarrow V$ is universal with respect to morphisms from N <u>into</u> modules of finite projective dimension.

<u>Proof</u>: Take $Q = (G^0)^*$ and rewrite (5.1.2.), (5.1.4.) .

5.4. Following a suggestion of M.Auslander, we give the objects just constructed the following names :

<u>Definition 5.4.1.</u>: Given a S-module N and its canonical representation (5.3.1.), the occurring maximal Cohen-Macaulay module M (with the projection p onto N) will be called a <u>MCM-approximation</u> of N . Accordingly, the module V (with the embedding i of N into it) is a <u>hull of finite projective dimension</u> of N .

Choosing MCM-approximations (hulls of finite projective dimension) for any S-module yields a functor, denoted $\mathbf{M}$ (resp. $\mathbf{H}$) from <u>mod-S</u> to <u>MCM(S)</u> (resp. <u>fpd(S)</u> , the full subcategory of <u>mod-S</u> spanned by all modules of finite projective dimension).

In terms of these functors, the universal mapping properties (5.1.2.(3)) and (5.1.4.(3)) express of course just that **M** yields a <u>right adjoint</u> of the embedding <u>MCM(S)</u> $\longrightarrow$ <u>mod-S</u> , whereas **H** defines a <u>left adjoint</u> of the embedding <u>fpd(S)</u> $\longrightarrow$ <u>mod-S</u> .

Summarizing the above results "categorically", one may say that <u>mod-S</u> is <u>almost obtained by glueing</u> <u>MCM(S)</u> and <u>fpd(S)</u> along the pairs of adjoint functors :

$$(5.4.2.) \quad \underline{\mathrm{MCM}(S)} \underset{\mathbf{M}}{\overset{\text{inclusion}}{\rightleftarrows}} \underline{\text{mod-}S} \underset{\text{inclusion}}{\overset{\mathbf{H}}{\rightleftarrows}} \underline{\mathrm{fpd}(S)}$$

if compared to the notion of "<u>glueing triangulated categories</u>" as it is defined in BBD;1.4.3. - to which we also refer for terminology :

- <u>MCM(S)</u> and <u>fpd(S)</u> are <u>full</u> subcategories of <u>mod-S</u> (this corresponds to axiom 1.4.3.5. of (loc.cit.)) ,
- <u>MCM(S)</u> is <u>left-orthogonal</u> to <u>fpd(S)</u> and <u>fpd(S)</u> is <u>right-orthogonal</u> to <u>MCM(S)</u> , (which is (5.1.1.(i) and (ii)) rephrased and corresponds to axiom 1.4.3.3.),
- the embedding of <u>MCM(S)</u> into <u>mod-S</u> ,(it corresponds to $j_!$), admits **M** as its <u>right adjoint</u> (j^* of axiom 1.4.3.2.),
- the embedding of <u>fpd(S)</u> into <u>mod-S</u> ,(it corresponds to i_*), admits **H** as its <u>left adjoint</u> (i^* of axiom 1.4.3.1.),

and, most important,

- the presentations in (5.3.1.) describe the desired unique decomposition of an object in <u>mod-S</u> into its "components" in <u>MCM(S)</u> and <u>fpd(S)</u> , (more precisely, (5.3.3.) corresponds to the distinguished triangle $j_!j^* \longrightarrow id \longrightarrow i_*i^* \xrightarrow{(1)}$ in 1.4.3.4.).

What is lacking from a "complete" glueing is essentially the existence of the other adjoints ($i^!$, j_*) : even by adding projectives not any module can in general be embedded into a MCM nor represented as a quotient of a module of finite projective dimension by a MCM.

5.5. After this "meta-mathematical" digression, consider the following case(s) where a MCM-approximation can be neatly described :

<u>Proposition 5.5.1.</u>: Let S be strongly Gorenstein, N a S-module such that $\mathrm{Ext}^i_S(N,S) = 0$ <u>except</u> for a single value $i = n$.
Then a(ny) MCM-approximation of N is given by

$$(5.5.2.) \quad \mathbf{M}(N) = \mathrm{Hom}_S(\Omega_n \mathrm{Ext}^n(N,S),S) \quad ,$$

where $\Omega_n \mathrm{Ext}^n_S(N,S)$ denotes the corresponding syzygy module in some projective resolution of $\mathrm{Ext}^n_S(N,S)$ as left S-module.

The proof is a literal repetition of the proof of (5.1.2.), observing that - with its notations - $\mathbf{F}^*$ is, by the special assumption, quasi-isomorphic to $\mathrm{Ext}^n_S(N,S)[-n]$, and hence $\mathbf{G}$ a projective resolution of this module. But then $\mathbf{M}(N)$ is represented by

$$\mathrm{Ker}(\mathrm{Hom}_S(d^0_{\mathbf{G}},S)) = \mathrm{Hom}_S(\mathrm{Cok}(d^0_{\mathbf{G}}),S) = \mathrm{Hom}_S(\Omega_0\mathbf{G},S) \quad \text{and}$$

$\Omega_0\mathbf{G} = \Omega_n \mathrm{Ext}^n_S(N,S)$ by definition.

Applying this to the case of a commutative local Gorenstein ring we get

Corollary 5.5.2.: Let R be a local commutative Gorenstein ring of dimension r with residue class field k. Then $\mathbf{M}(k)$ can be obtained as the R-dual of the r-th syzygy-module in a (minimal) free resolution of k over R.

Sticking with the case of the Corollary, $\mathbf{M}(k)$ has a well-defined rank and choosing indeed a minimal resolution to obtain the MCM-approximation, its rank is given by

$$(5.5.3.) \qquad \mathrm{rank}_R \mathbf{M}(k) = \sum_{i=0}^{r-1} (-1)^{r-i-1} b_i \quad ,$$

where b_i denotes the i-th Betti-number of k over R.
Now, as in general the Betti-numbers of a local Gorenstein ring grow rather fast, the rank of this maximal Cohen-Macaulay module will accordingly be quite large - and the ranks of its syzygy modules even larger, unless R is a hypersurface ring, (see [Her],[Eis]).
Hence we ask :

Is the number given in (5.5.3.) the minimal possible rank for a non-free MCM which occurs as the syzygy module of some artinian R-module?

Considering more generally the case of not necessarily commutative but local rings S which are strongly Gorenstein, one still has minimal resolutions, and an object in mod-S admits a unique (up to S-isomorphism) minimal representative in mod-S, characterized by the property that it does not contain a free summand. In particular, we can - and will - in this case "minimize" the functors $\mathbf{M}$ or $\mathbf{H}$, (as well as, for example, complete resolutions). This enables one to define several new invariants for any S-module N : The rank of $\mathbf{H}(N)$ and the number of indecomposable summands of $\mathbf{H}(N)$ or $\mathbf{M}(N)$.

5.6. The reader may have observed that the proof of (5.1.2.-4.). in particular (5.1.6.), yields an extension of the construction of complete resolutions of MCM's, as given in (4.5.), to arbitrary S-modules. Furthermore, essentially the same argument provides a <u>functorial</u> way to obtain such complete resolutions for arbitrary objects in $D^b(S)$.

In other words, the projection functor $D^b(S) \longrightarrow \underline{D^b(S)}$ factors naturally over the equivalence $\underline{\sigma_{\leqq}} : \underline{APC(S)} \longrightarrow \underline{D^b(S)}$, completing the picture given in (4.4.).

Nevertheless, the result is quicker to state than to formalize :

<u>Complete resolutions can be obtained by "symmetrizing" projective resolutions with respect to duality</u>.

This shall mean :

- Assume given a complex X of S-modules - with bounded and finitely generated cohomology by our general assumption -, and consider it as an object in $D^b(S)$.
- Choose a <u>projective resolution</u> $\mathbf{P}(X) \longrightarrow X$ of X and <u>dualize it</u> to obtain $\mathbf{P}(X)^* = \mathrm{Hom}^{\cdot}_S(\mathbf{P}(X),S)$.
- By construction, this is a <u>bounded below</u> complex of finitely generated projective S^{op}-modules with bounded cohomology, (4.6.), hence a "<u>projective co-resolution</u>" of $\mathrm{RHom}_S(X,S)$, considered as an object in the full triangulated subcategory $K^{+,b}(P(S^{op}))$ of $D^b(S^{op})$ in analogy to (0.3.). Therein, this object is unique up to homotopy-equivalence.
- Now choose a <u>projective resolution</u> $\varphi_X : \mathbf{P}(\mathbf{P}(X)^*) \longrightarrow \mathbf{P}(X)^*$ of this co-resolution - it corresponds to the morphism φ of (5.1.6.) - and <u>dualize again</u>, this time of course with respect to $\mathrm{Hom}^{\cdot}_{S^{op}}(-,S)$, to obtain
- the <u>Norm-map</u> of X - called so here in analogy to the classical case of finite groups, see [C-E;XII.1.] for example - :

$$(5.6.1.) \quad \mathbf{N}(X) : \mathbf{P}(X) \xrightarrow[\simeq]{\text{can.}} \mathbf{P}(X)^{**} \xrightarrow{\varphi_X^*} \mathbf{P}(\mathbf{P}(X)^*)^* \quad ,$$

which by construction is a quasi-isomorphism from a projective resolution of X to a projective co-resolution of X .
(Considered in $D^b(S)$, the morphism $\mathbf{N}(X)$ is hence nothing but the usual "incarnation" of the duality-isomorphism for Gorenstein rings : $X \xrightarrow{\simeq} \mathrm{RHom}_{S^{op}}(\mathrm{RHom}_S(X,S),S)$.)

- Coming to an end, set - in accordance with (4.5.) - :

(5.6.2.) $\mathbf{CR}(X) = C(\mathbf{N}(X))[-1]$,

and call this translated mapping cone over the Norm-map a <u>complete resolution</u> of X .

This complete resolution of X gives by its definition rise to a distinguished triangle in $K(P(S))$, the homotopy-category of <u>all</u> complexes of finitely generated projective (right) S-modules, cf. (0.3.), :

(5.6.3.) $\mathbf{CR}(X) \xrightarrow{\mathbf{r}(X)} \mathbf{P}(X) \xrightarrow{\mathbf{N}(X)} \mathbf{P}(\mathbf{P}(X)^*)^* \xrightarrow{\mathbf{i}(X)} \mathbf{CR}(X)[1]$,

which is the "typical" distinguished triangle associated to the mapping cone over $\mathbf{N}(X)$ "rotated once to the left".
The occurring morphism $\mathbf{r}(X)$ is just the <u>restriction</u> of the complete resolution to the given projective resolution, whereas $\mathbf{i}(X)$ is the morphism "<u>inflating</u>" the chosen co-resolution to a complete resolution. (For the role of the shift, see once again the remark preceeding (4.7.) as well as (3.1.).)

To abbreviate notations, we will set in the sequel :

(5.6.4.) $\mathbf{C}(-) = \mathbf{P}(\mathbf{P}(-)^*)^*$.

<u>Remarks</u>: (a) Recall - [Ver;II.1.4.] dualized - that choosing projective resolutions just means "categorically" to give a left adjoint to the embedding $K^{-,b}(P(S)) \longrightarrow D^b(S)$. From this point of view, $\mathbf{C}$ becomes a functor from $D^b(S)$ into $K^{+,b}(P(S))$, the homotopy-category of projective co-resolutions. Composing each of these two functors with the respective inclusion $K^{+/-,b}(P(S)) \longrightarrow K^{(b)}(P(S))$, the "norm" can hence be thought of as a natural transformation of functors from $D^b(S)$ into $K^{(b)}(P(S))$. Similarly, (5.6.3) constitutes a functor into the category of (distinguished) triangles of functors, if one d/cares to introduce such a thing.
(b) As usual, any choice of a dualizing module ω_S will define a "norm" and then "complete resolutions" and so on, but all these objects will be isomorphic. Hence we will in general not distinguish these constructions and rather allow ourselves to choose the "suitable" dualizing module depending on the concrete situation.

To analyze further the distinguished triangle (5.6.3), we introduce the following two invariants of an object X in $D^b(S)$:

(5.6.5.) $m_X = \max(i : H^i(X) \neq 0)$ and

$m_X^* = \max(i : H^i(\mathrm{RHom}_S(X,S)) \neq 0)$.

(In terms of the t-structures mentioned in (4.8.), these numbers measure "how far" the given object X is from the respective "hearts" of these structures : X is in the essential image of $MCM(S)$ in $D^b(S)$ iff $m_X = m_X^* = 0$.)

Now choosing projective resolutions "minimally" - so that $\mathbf{P}^i(X) = 0$ for $i > m_X$ - it follows that the morphisms in (5.6.3.) can be assumed to satisfy :

- $\mathbf{r}(X)$ is the identity in each degree not larger than $-m_X^*$,
- $\mathbf{i}(X)$ is the identity in each degree not smaller than m_X ,

and hence

- the Norm-map $\mathbf{N}(X)$ is <u>non-zero</u> only in the "twilight-zone" of those degrees i for which $-m_X^* \leq i \leq m_X$.

Finally, having chosen complete resolutions for objects in $D^b(S)$, we define accordingly their <u>MCM-approximations</u>, extending (5.4.1.), as

(5.6.6.) $\mathbf{M}(X) = \Omega_0\mathbf{CR}(X)$ for any X in $D^b(S)$.

<u>Remark</u>: (c) If compared to the situation for actual modules, what is missing in this more general context is an analogue of the functor $\mathbf{H}$, which would hence associate to any complex in $D^b(S)$ its "perfect part" in $D^b_{perf}(S)$. But, even if such an analogue exists, there will be the same shortcoming as in (5.4.) : $D^b(S)$ can in general not be obtained by "glueing together" $\underline{D^b(S)}$ and $D^b_{perf}(S)$ as simple examples already show - see also section 7 .
Remark that <u>if</u> there exists a stabilized higher K-theory as asked for in (4.9.), an obstruction for the glueing will ge given by any non-zero connecting homomorphism $\underline{K}_i(S) \longrightarrow K_{i-1}(S)$.

To resume the situation, we have seen :

<u>Theorem 5.6.7.</u>: For any ring S which is strongly Gorenstein, one may extend (4.4.1.) to a diagram of exact functors between triangulated categories, commutative up to isomorphisms of such functors :

$$\begin{tikzcd}[column sep=large]
 & D^b(S) \arrow[dl, "\mathbf{CR}"'] \arrow[d, "\mathbf{M}"] \arrow[dr, "\text{projection}"] & \\
\underline{APC(S)} \arrow[r, "\Omega_0", "\simeq"'] \arrow[rr, bend right, "\simeq", "\sigma_{\leqq}"'] & \underline{MCM(S)} \arrow[r, "\iota_S", "\simeq"'] & \underline{D^b(S)}
\end{tikzcd}$$

In particular, the functors **CR** (resp. **M**) define quasi-inverses of $\sigma_{\leqq}$ (resp. ι_S) and extend naturally the functors defined earlier, (4.5.1.) resp. (5.4.1.) .

6. - *The Tate-Cohomology*

6.1. The essential use of the triangulated structures on any of the equivalent categories $\underline{APC(S)}$, $\underline{MCM(S)}$ or $\underline{D^b(S)}$ is that they define one and the same cohomology theory for S-modules (or, more generally, for complexes in $D^b(S)$) :

Definition 6.1.1.: Let M , N be two (complexes of) modules (with bounded, finitely generated cohomology) over a ring S which is strongly Gorenstein. Then the i-th Tate-cohomology group of M with values in N (over S) is defined to be

$$\underline{\mathrm{Ext}}_S^i(M,N) =_{\mathrm{def}} \mathrm{Hom}_{\underline{D^b(S)}}(M,T^iN) \qquad ,$$

for any integer i .

A first more concrete description of these groups for actual S-modules can be obtained from Theorem (4.4.1.) as follows :

Lemma 6.1.2.: Let M and N be in mod-S . Then, for any i ,

(i) $\underline{\mathrm{Ext}}_S^i(M,N) = \underline{\mathrm{Hom}}_S(\Omega_S^i\mathbf{M}(M),N)$ - with notations as in section 2 -,

and also

(ii) $\underline{\mathrm{Ext}}_S^i(M,N) = H^i\mathrm{Hom}_S^{\cdot}(\mathbf{CR}(M),N)$.

<u>Proof</u>: By the defining properties of MCM-approximations, $\mathbf{M}(-)$ yields an inverse of ι_S , and by (4.4.1.(3)) the loop-space functor Ω_S represents an inverse of the translation functor on $\underline{MCM(S)}$, so that

$$\mathrm{Hom}_{\underline{D^b(S)}}(M,T^iN) = \mathrm{Hom}_{\underline{D^b(S)}}(T^{-i}M,N) = \mathrm{Hom}_{\underline{MCM(S)}}(\Omega_S^i\mathbf{M}(M),\mathbf{M}(N)) \ .$$

But, on the other hand, (5.4.), (the restriction of) $\mathbf{M}$ on $\underline{\text{mod-}S}$ is a right adjoint of the embedding of $\underline{MCM(S)}$ into $\underline{\text{mod-}S}$, so that

$$\mathrm{Hom}_{\underline{MCM(S)}}(\Omega_S^i\mathbf{M}(M),\mathbf{M}(N)) = \mathrm{Hom}_{\underline{\text{mod-}S}}(\Omega_S^i\mathbf{M}(M),N) \qquad ,$$

whence (i) .
To verify (ii), remark that $\mathbf{CR}(-)$ establishes an inverse of $\sigma_{\leqq}$, by (5.6.7.), and that by (4.4.1.(v)) one may replace $\sigma_{\leqq}$ by $\sigma_{\leqq k}$ for some sufficiently large k - depending on i -, so that the cohomology group in question can be calculated as :

$$\mathrm{Hom}_{\underline{D^b(S)}}(M,T^iN) = \mathrm{Hom}_{K(S)}(\sigma_{\leqq k}\mathbf{CR}(M),T^iN)$$

which last group of homotopy classes of morphisms of complexes equals by definition $H^i\mathrm{Hom}_S^{\cdot}(\sigma_{\leqq k}\mathbf{CR}(M),N)$ - see [ALG X;p.82] for example.
Now for k large enough (again compared to i), "$\sigma_{\leqq k}$" can obviously be dropped as the "complex" N is bounded above.

6.2. Before we give a more "classical" description of the newly introduced cohomology theory, we use the general machinery - in particular (4.4.1.) and (5.6.) - to investigate how the "ordinary" Ext's compare with the "<u>stabilized</u>" <u>Ext's</u> .
This will be done by calculating the Tate-cohomology as homotopy-groups in <u>APC(S)</u> , considered as a full triangulated subcategory of $K(P(S))$, the homotopy category of all complexes of finitely generated projective S-modules.
For this purpose and the readers convenience, let us recall the following facts on homotopy-groups of such complexes :

<u>Lemma 6.2.1.</u>: Let S be <u>any</u> ring, X and Y two complexes of finitely generated <u>projective</u> (right) S-modules. Denote - as before - by $\mathrm{Hom}_S^{\cdot}(X,Y)$ the complex of morphisms of the underlying graded S-modules, (which, following [ALG X.81], should be $\mathrm{Homgr}_S(X,Y)$).
(i) $\mathrm{Hom}_S^{\cdot}(X,Y)$ <u>is acyclic if</u>
(a) X is bounded above and Y is acyclic, <u>or</u>
(b) The injective dimension of S as a right module over itself is finite, X is acyclic and Y is bounded below.
(ii) The natural morphism of complexes $Y \otimes_S \mathrm{Hom}_S^{\cdot}(X,S) \longrightarrow \mathrm{Hom}_S^{\cdot}(X,Y)$ - as defined in [ALG X.99;(16)] say - <u>is an isomorphism</u> if X is bounded below and Y is bounded above.

<u>Proof</u>: (a) of (i) is well-known - dualize [Ha;I.4.4.] e.g. - and (ii) is obvious from the definitions of the complexes involved. Statement (b) in (i) can be seen most easily by extending the argument in (v) of the proof of (4.4.1.).

This Lemma yields the following informations on homotopy classes of morphisms between the various functors **CR** , **P** or **C** as introduced in (5.6.) :

<u>Lemma 6.2.2.</u>: Let S be a ring which is strongly Gorenstein, X and Y two complexes in $D^b(S)$. Then
(i) The complex of abelian groups $\mathrm{Hom}_S^{\cdot}(\mathbf{CR}(X),\mathbf{C}(Y))$ is acyclic and the restriction-morphism $\mathbf{r}(Y) : \mathbf{CR}(Y) \longrightarrow \mathbf{P}(Y)$ of (5.6.3) induces a quasi-isomorphism

$$\mathrm{Hom}_S^{\cdot}(\mathbf{CR}(X),\mathbf{r}(Y)) : \mathrm{Hom}_S^{\cdot}(\mathbf{CR}(X),\mathbf{CR}(Y)) \xrightarrow{\simeq} \mathrm{Hom}_S^{\cdot}(\mathbf{CR}(X),\mathbf{P}(Y)) .$$

(ii) The complex of abelian groups $\mathrm{Hom}_S^{\cdot}(\mathbf{P}(X),\mathbf{CR}(Y))$ is acyclic and the Norm-map of Y , $\mathbf{N}(Y)$, induces a quasi-isomorphism

$$\mathrm{Hom}_S^{\cdot}(\mathbf{P}(X),\mathbf{N}(Y)) : \mathrm{Hom}_S^{\cdot}(\mathbf{P}(X),\mathbf{P}(Y)) \xrightarrow{\simeq} \mathrm{Hom}_S^{\cdot}(\mathbf{P}(X),\mathbf{C}(Y)) .$$

Proof: The first statement in (i) is case (b), the first statement in (ii) is case (a) of the foregoing Lemma.
The remaining assertions follow then by applying $\mathrm{Hom}^{\cdot}_S(\mathbf{CR}(X),-)$ resp. $\mathrm{Hom}^{\cdot}_S(\mathbf{P}(X),-)$ to the distinguished triangle (5.6.3.) for Y.

To state the main result on Tate-cohomology we use - as in (5.6.) - the

Notations 6.2.3.: If X is any complex of finitely generated projective S-modules (any object of $D^b(S)$), we denote X^* the complex of finitely generated projective S^{op}-modules $\mathrm{Hom}^{\cdot}_S(X,S)$, (resp. the object $\mathrm{RHom}_S(X,S)$ in $D^b(S^{op})$).

Accordingly we denote for two objects X and Y in $D^b(S)$ by

$$\mathrm{Tor}_i^S(Y,X^*) = H^{-i}(Y \overset{\mathbb{L}}{\otimes}_S \mathrm{RHom}_S(X,S))$$

the corresponding hyper-Tor group, which is certainly defined as soon as S is of finite injective dimension as a right module over itself.

Remark that by the very definition, these hyper-Tor groups can be calculated as $\mathrm{Tor}_i^S(Y,X^*) = H_i(\mathbf{P}(Y) \otimes_S \mathbf{P}(\mathrm{Hom}^{\cdot}_S(\mathbf{P}(X),S)))$, hence using (5.6.4.) and (6.2.1.(ii)) one obtains for any two complexes X, Y in $D^b(S)$ with S strongly Gorenstein :

$$\text{(6.2.4.)} \qquad \mathrm{Tor}_i^S(Y,X^*) = H^{-i}\mathrm{Hom}^{\cdot}_S(\mathbf{C}(X),\mathbf{P}(Y)) \quad .$$

With these preparations - and the notations as in (5.6.) - we get :

Theorem 6.2.5.: Let S be a ring which is strongly Gorenstein.

(1) For any two complexes X and Y in $D^b(S)$, the complex of abelian groups $\mathrm{Hom}^{\cdot}_S(\mathbf{CR}(X),\mathbf{CR}(Y))$ is quasi-isomorphic to the mapping cone over $\mathbf{N}(X,Y) = \mathrm{Hom}^{\cdot}_S(\mathbf{N}(X),\mathbf{N}(Y))$.
More precisely, there is a morphism of complexes $\mathbf{d}(X,Y)$, so that the following diagram is commutative :

$$\begin{tikzcd}
\mathrm{Hom}^{\cdot}_S(\mathbf{CR}(X)[1],\mathbf{CR}(Y))[1] = \mathrm{Hom}^{\cdot}_S(\mathbf{CR}(X),\mathbf{CR}(Y)) \arrow[r, "\mathbf{d}(X,Y)"] \arrow[d, "(1)", "\mathrm{Hom}^{\cdot}_S(\mathbf{i}(X),\mathbf{r}(Y))[1]"'] \arrow[dr, phantom, "(1)"] & C(\mathbf{N}(X,Y)) \arrow[dl, "d"'] \\
\mathrm{Hom}^{\cdot}_S(\mathbf{C}(X),\mathbf{P}(Y)) \arrow[r, "\mathbf{N}(X,Y)"] \arrow[ur, phantom, "(+)"] & \mathrm{Hom}^{\cdot}_S(\mathbf{P}(X),\mathbf{C}(Y)) \arrow[u, "j"']
\end{tikzcd} ,$$

where $\mathbf{d}(X,Y)$ is a quasi-isomorphism, the triangle marked (+)

is the "typical" distinguished triangle associated to the mapping cone over $\mathbf{N}(X,Y)$.

(2) The above diagram is <u>self-dual</u> in the sense that "transposition" with respect to $(\ )^* = \mathrm{Hom}^{\cdot}_{(op)}(-,S)$ yields isomorphisms of distinguished triangles in the (full) derived category of abelian groups :

$$\begin{array}{ccccccc}
\mathrm{Hom}^{\cdot}_{S^{op}}(\mathbf{C}(X),\mathbf{P}(Y)) & \xrightarrow{\mathbf{N}} & \mathrm{Hom}^{\cdot}_{S^{op}}(\mathbf{P}(X),\mathbf{C}(Y)) & \xrightarrow{\mathbf{d}^{-1}\mathbf{j}} & \mathrm{Hom}^{\cdot}_{S^{op}}(\mathbf{CR}(X),\mathbf{CR}(Y)) & \xrightarrow{(1)} \\
\simeq\updownarrow(\ )^* & & \simeq\updownarrow(\ )^* & & \simeq\updownarrow(\ )^*[-1] & \\
\mathrm{Hom}^{\cdot}_{S}(\mathbf{C}(Y^*),\mathbf{P}(X^*)) & \xrightarrow{\mathbf{N}} & \mathrm{Hom}^{\cdot}_{S}(\mathbf{P}(Y^*),\mathbf{C}(X^*)) & \xrightarrow{\mathbf{d}^{-1}\mathbf{j}} & \mathrm{Hom}^{\cdot}_{S}(\mathbf{CR}(Y^*),\mathbf{CR}(X^*)) & \xrightarrow{(1)}
\end{array}$$

for any two objects X and Y in $D^b(S^{op})$. ($\mathbf{d}^{-1}$ represents of course the inverse of $\mathbf{d}$ in the derived category.)

(3) Evaluating (1) and (2) by taking cohomology, one obtains a long - even unlimited - exact sequence of abelian groups

$$\cdots\longrightarrow \underline{\mathrm{Ext}}^{i-1}_S(X,Y) \longrightarrow \mathrm{Tor}^S_{-i}(Y,X^*) \xrightarrow{H^i\mathbf{N}(X,Y)} \mathrm{Ext}^i_S(X,Y) \xrightarrow{c^i} \underline{\mathrm{Ext}}^i_S(X,Y) \longrightarrow\cdots$$

(where c^i is the map obtained by applying the projection functor

$$\mathrm{Ext}^i_S(X,Y) = \mathrm{Hom}_{D^b(S)}(X,T^iY) \longrightarrow \mathrm{Hom}_{\underline{D^b(S)}}(X,T^iY) = \underline{\mathrm{Ext}}^i_S(X,Y) \quad),$$

as well as isomorphisms

$$\underline{\mathrm{Ext}}^i_S(X,Y) \cong \underline{\mathrm{Ext}}^i_{S^{op}}(Y^*,X^*)$$

for all integers i and all objects X , Y in $D^b(S)$.

(4) All statements as well as diagrams in (1) - (3) are functorial in X and Y .

<u>Proof</u>: (1) Remark first that the <u>existence</u> of the claimed morphism $\mathbf{d}(X,Y)$ follows readily from the fact that

$$\mathbf{N}(X,Y).\mathrm{Hom}^{\cdot}_S(\mathbf{i}(X),\mathbf{r}(Y)) = \mathrm{Hom}^{\cdot}_S(\mathbf{i}(X)\mathbf{N}(X),\mathbf{N}(Y)\mathbf{r}(Y)) = 0$$

by definition of the morphisms involved, (5.6.3.).

That $\mathbf{d}(X,Y)$ is necessarily a <u>quasi-isomorphism</u> follows from the so-called "diagram of 9 squares" - see [BBD;1.1.11.]- obtained by

applying $\mathrm{Hom}^{\cdot}_{S}(-,-)$ to the distinguished triangle (5.6.3.) for X in its contravariant argument and to the corresponding triangle for Y in its covariant argument.
(In the following display of this diagram, we abbreviate $\mathrm{Hom}^{\cdot}_{S}(-,-)$ as $\cdot(-,-)$ and suppress the arguments X and Y.

The rows and columns in "solid" arrows represent distinguished triangles, the right-most column (bottom row) are the "anti-distinguished" triangles obtained (up to a canonical isomorphism) by translating the left-most column (the top row).

The square marked (-) is anti-commutative.)

$$
\begin{tikzcd}
\cdot(\mathbf{CR},\mathbf{CR}) \arrow[r, "\simeq"] & \cdot(\mathbf{CR},\mathbf{P}) \arrow[r] & \underline{\cdot(\mathbf{CR},\mathbf{C})} \arrow[r] & \cdot(\mathbf{CR},\mathbf{CR})[1] \\
\underline{\cdot(\mathbf{P},\mathbf{CR})} \arrow[r] \arrow[u] & \cdot(\mathbf{P},\mathbf{P}) \arrow[r, "\simeq"] \arrow[u] \arrow[dr, phantom, "(\#)"] & \cdot(\mathbf{P},\mathbf{C}) \arrow[r] \arrow[u] & \underline{\cdot(\mathbf{P},\mathbf{CR})[1]} \arrow[u, dashed] \\
\cdot(\mathbf{C},\mathbf{CR}) \arrow[r] \arrow[u] & \cdot(\mathbf{C},\mathbf{P}) \arrow[r] \arrow[u] \arrow[ur, dotted, "\mathbf{N}(X,Y)"'] & \cdot(\mathbf{C},\mathbf{C}) \arrow[r] \arrow[u, "\simeq"'] \arrow[dr, phantom, "(-)"] & \cdot(\mathbf{C},\mathbf{CR})[1] \arrow[u, dashed] \\
\cdot(\mathbf{CR}[1],\mathbf{CR}) \arrow[r, dashed, "\simeq"] \arrow[u, "\simeq"] \arrow[ur, dotted, "\cdot(\mathbf{i},\mathbf{r})"] & \cdot(\mathbf{CR}[1],\mathbf{P}) \arrow[r, dashed] \arrow[u] & \underline{\cdot(\mathbf{CR}[1],\mathbf{C})} \arrow[r, dashed] \arrow[u] & \cdot(\mathbf{CR}[1],\mathbf{CR})[1] \arrow[u, dashed, "\simeq"']
\end{tikzcd}
$$

(6.2.6.)

In this diagram, by (6.2.2.), the underlined complexes are acyclic and then the morphisms marked "≃" are quasi-isomorphisms.
The factorization - marked by (#) above - of $\mathbf{N}(X,Y) = \mathrm{Hom}^{\cdot}_{S}(\mathbf{N}(X),\mathbf{N}(Y))$ over the quasi-isomorphism $\mathrm{Hom}^{\cdot}_{S}(\mathbf{P}(X),\mathbf{N}(Y))$ implies the existence of a morphism from $\mathrm{Hom}^{\cdot}_{S}(\mathbf{CR}(X),\mathbf{P}(Y))$ into the mapping cone over $\mathbf{N}(X,Y)$ which is necessarily a quasi-isomorphism.
Now the claimed morphism $\mathbf{d}(X,Y)$ can be obtained by composing this with the quasi-isomorphism $\mathrm{Hom}^{\cdot}_{S}(\mathbf{CR}(X),\mathbf{CR}(Y)) \xrightarrow{\simeq} \mathrm{Hom}^{\cdot}_{S}(\mathbf{CR}(X),\mathbf{P}(Y))$, the horizontal arrow in the upper left corner, whence $\mathbf{d}(X,Y)$ itself is also a quasi-isomorphism.
Remark that this proof exhibits isomorphisms in the derived category (of abelian groups) between the distinguished triangles represented respectively by the second column or the third row and the triangle

given by ($\mathbf{N}(X,Y)$, $\mathbf{d}^{-1}j$, $\cdot(\mathbf{i},\mathbf{r})[1]$) .
(Strictly speaking, this last triple does not represent a distinguished triangle, unless the morphism $\mathrm{Hom}_S^{\cdot}(\mathbf{i}(X),\mathbf{r}(Y))[1]$, whose source is by definition $\mathrm{Hom}_S^{\cdot}(\mathbf{CR}(X)[1],\mathbf{CR}(Y))[1]$, is composed with the canonical - [SGA 4;XVII.1.1.5.] - isomorphism from $\mathrm{Hom}_S^{\cdot}(\mathbf{CR}(X),\mathbf{CR}(Y))$ into its source, as indicated in the diagram of (1).)
Taking into account the foregoing remark, it is clear from the definitions that "transposition" corresponds to "reflecting" the above diagram on the "diagonal".
For example,

$$\mathbf{C}(X)^* = \mathbf{P}(\mathbf{P}(X)^*)^{**} \cong \mathbf{P}(X^*)$$

and

$$\mathbf{P}(Y)^* \cong \mathbf{P}(Y^{**})^* = \mathbf{P}(\mathbf{P}(Y^*)^*)^* = \mathbf{C}(Y^*)$$

by (6.2.3.), (5.6.4.), and the duality on complete resolutions has been observed already in (4.6.1.).
That these dualities are in fact compatible, so that the given diagram in (2) commutes, is left to the reader.
Concerning (3), recall that always $\mathrm{Ext}_S^i(X,Y) = H^i\mathrm{Hom}_S^{\cdot}(\mathbf{P}(X),\mathbf{P}(Y))$, so that (6.2.4.) and passing to cohomology in (6.2.2.(ii)) and (6.2.6.) yields the identifications needed. (See also Remark (b) below.)
The functoriality asserted in (4) is evident.

qed

<u>Remarks</u>: (a) Obviously, the Theorem contains an understatement : As usual, one may replace throughout its formulation - or proof - the term "abelian group" by "module over the centre of S ".
(b) The proof of the Theorem - in particular (6.2.6.) - reveals the following possibilities to obtain in practice the natural transformation $c^{\cdot} : \mathrm{Ext}_S^{\cdot}(-,-) \longrightarrow \underline{\mathrm{Ext}}_S^{\cdot}(-,-)$: Identify these groups with

- $H^{\cdot}$ of $\mathrm{Hom}_S^{\cdot}(\mathbf{P},\mathbf{P})$, $\mathrm{Hom}_S^{\cdot}(\mathbf{P},\mathbf{C})$ or $\mathrm{Hom}_S^{\cdot}(\mathbf{C},\mathbf{C})$ for $\mathrm{Ext}_S^{\cdot}$ - this corresponds to the "central square" of (6.2.6.) -,

and accordingly with

- $H^{\cdot}$ of $\mathrm{Hom}_S^{\cdot}(\mathbf{CR},\mathbf{P})$, $\mathrm{Hom}_S^{\cdot}(\mathbf{CR},\mathbf{CR})$ or $\mathrm{Hom}_S^{\cdot}(\mathbf{C},\mathbf{CR}[1])$ for $\underline{\mathrm{Ext}}_S^{\cdot}$,

so that $c^{\cdot}$ becomes represented - in the corresponding order - by

- $H^{\cdot}$ of $\mathrm{Hom}_S^{\cdot}(\mathbf{r},\mathbf{P})$, $\mathbf{d}^{-1}j$ or $\mathrm{Hom}_S^{\cdot}(\mathbf{C},\mathbf{i})$.

Furthermore, in the first of these incarnations, one may drop the "P" in the covariant argument and take the complex itself instead - using once again (4.4.1.(v)) and (6.2.1.(a)).

The first of these descriptions is closest to a "classical" treatment, whence we make it explicit - it has been used implicitly already, e.g. in (4.4.1.(v)), (6.1.2.) :

- Assume given two (complexes of) S-modules X , Y (with bounded and finitely generated cohomology).
- Choose a projective resolution $\mathbf{P}(X)$ and construct from it a complete resolution $\mathbf{CR}(X)$ as in (5.6.) or (4.5.).
- The homotopy-classes of morphisms of complexes of degree i from $\mathbf{P}(X)$,(resp. $\mathbf{CR}(X)$), into Y represent the i-th (stabilized) cohomology group of X with values in Y .
- Representing a homotopy-class by an actual morphism of complexes from $\mathbf{P}(X)$ to Y , composing it with the restriction morphism $\mathbf{r}(X)$ from $\mathbf{CR}(X)$ onto $\mathbf{P}(X)$ and passing again to the homotopy-class describes the image of an element from $\mathrm{Ext}^i_S(X,Y)$ under c^i in $\underline{\mathrm{Ext}}^i_S(X,Y)$.

The above Theorem is then just the "triangulated" version of saying that this procedure is independent of the choices involved - and furthermore identifies the "mapping cone" over these natural maps c^i .

(6.3.) As observed in (5.6.), for a given object X in $D^b(S)$ the "Norm-map" $\mathbf{N}(X)$ can be chosen to be non-trivial in only finitely many degrees. This implies that the exact sequence in (6.2.5.(3)) is quite "degenerate".

More precisely, in addition to the invariants m_X , m^*_X introduced in (5.6.5.), consider also

(6.3.1.) $$i_X = \inf(i : H^i(X) \neq 0) \quad \text{and}$$
$$i^*_X = \inf(i : H^i(\mathrm{RHom}_S(X,S)) \neq 0) .$$

In terms of these invariants, one gets immediately

<u>Lemma 6.3.2.</u>: With notations as above, for any given complexes X and Y in $D^b(S)$, one has :

(i) $\mathrm{Tor}^S_{-i}(Y,X^*) = 0$ for all $i > m^*_X + m_Y$, and

(ii) $\mathrm{Ext}^i_S(X,Y) = 0$ for all $i < \max(i_Y - m_X , i^*_X - m^*_Y)$.

<u>Proof</u>: By (6.2.4.), $\mathrm{Tor}^S_{-i}(Y,X^*) = H^i(\mathbf{P}(Y) \otimes_S \mathbf{P}(\mathbf{P}(X)^*))$, and the complex $\mathbf{P}(Y) \otimes_S \mathbf{P}(\mathbf{P}(X)^*))$ can be assumed to be zero in degrees larger

than $m_Y + m_X^*$ by definition of these numbers, (5.6.5.), whence (i). For (ii), remark that by duality it is enough to have the claimed vanishing behaviour for all $i < i_Y - m_X$, which is obvious.

In view of this Lemma, we set

(6.3.3.) $$\underline{m} = \underline{m}(X,Y) = \max(\ i_Y - m_X\ ,\ i_X^* - m_Y^*\ ) - 1 \quad \text{and}$$
$$\bar{m} = \bar{m}(X,Y) = m_X^* + m_Y \qquad ,$$

to obtain

<u>Corollary 6.3.4.</u>: Using the just defined integers, depending on X and Y in $D^b(S)$, the long exact sequence of (6.2.5.(3)) breaks up according to

(i) The natural homomorphism $c^i(X,Y) : \mathrm{Ext}_S^i(X,Y) \longrightarrow \underline{\mathrm{Ext}}_S^i(X,Y)$ is
- an <u>isomorphism</u> for $i > \bar{m}$ and
- a <u>surjection</u> for $i = \bar{m}$.

(ii) The connecting homomorphism $\underline{\mathrm{Ext}}_S^i(X,Y) \longrightarrow \mathrm{Tor}^S_{-i-1}(Y,X^*)$ is
- an <u>isomorphism</u> for $i < \underline{m}$ and
- an <u>injection</u> for $i = \underline{m}$.

(iii) It remains an exact sequence

$$0 \longrightarrow \underline{\mathrm{Ext}}_S^{\underline{m}}(X,Y) \longrightarrow \mathrm{Tor}^S_{-\underline{m}-1}(Y,X^*) \to \cdots \to \mathrm{Ext}_S^{\bar{m}}(X,Y) \longrightarrow \underline{\mathrm{Ext}}_S^{\bar{m}}(X,Y) \longrightarrow 0.$$

<u>Remark</u>: The converging spectral sequence
$$E_2^{i,j} = \mathrm{Ext}_S^i(H^{-j}(X),S) \Longrightarrow H^{i+j}(\mathrm{RHom}_S(X,S))$$
for a complex X in $D^b(S)$ shows that always
$$-m_X \leqq i_X^* \leqq m_X^* \leqq \operatorname{vdim} S - i_X \qquad ,$$
which implies readily that for any two complexes X and Y as above necessarily $\underline{m} < \bar{m}$, so that the exact sequence in (iii) above is in any case "non-trivial".(Easy examples show furthermore that in general the bounds given in (6.3.2.) cannot be improved, so that in this sense the above Corollary is "best possible".)

<u>Example 6.3.5.</u>: Let M, N be two S-modules, S strongly Gorenstein. Then
$$\underline{m} = \underline{m}(M,N) = \max(\ 0\ ,\ i_M^* - m_N^*\ ) - 1 \geqq -1 \qquad ,$$
$$0 \leqq \bar{m} = \bar{m}(M,N) = m_M^* \leqq \operatorname{vdim} S \qquad .$$

So one gets the universal bounds - improving (6.1.3.) -

$$\underline{\mathrm{Ext}}^i_S(M,N) = \mathrm{Ext}^i_S(M,N) \qquad \text{for all} \quad i > \operatorname{vdim} S \geq m^*_M \ ,$$

$$\underline{\mathrm{Ext}}^i_S(M,N) = \mathrm{Tor}^S_{-i-1}(N, \mathrm{RHom}_S(M,S)) \qquad \text{for all} \quad i < \ -1 \ \leq \underline{m} \ .$$

If for any S-module N we call - in accordance with the local commutative case -

m^*_N the "<u>co-depth</u>" of N, and recall that by definition

i^*_N is the "<u>grade</u>" of N, [A-B;2.21.],

the above can be rephrased as

<u>For S-modules M, N, the Tate-cohomology of M with values in N yields "new" (stable) groups</u> $\underline{\mathrm{Ext}}^i_S(M,N)$ <u>only in the range</u>

$$-1 \leq \max(0, \operatorname{grade}(M) - \operatorname{codepth}(N)) - 1 \leq i \leq \operatorname{codepth}(M) \leq \operatorname{vdim} S \ .$$

6.4. It follows from the example - and could be expected in view of (4.4.1.) and the definition - that Tate-cohomology takes a particularly simple form on maximal Cohen-Macaulay modules.

<u>Corollary 6.4.1.</u>: Let M be a maximal Cohen-Macaulay module over S and denote $M^* = \mathrm{Hom}_S(M,S)$ its S-dual. Then

(i) $\underline{\mathrm{Ext}}^i_S(M,-) = \mathrm{Ext}^i_S(M,-)$ for all $i > 0$,

(ii) $\underline{\mathrm{Ext}}^i_S(M,-) = \mathrm{Tor}^S_{-i-1}(-,M^*)$ for all $i < -1$,

(iii) There is an exact sequence of functors

$$0 \longrightarrow \underline{\mathrm{Ext}}^{-1}_S(M,-) \longrightarrow - \otimes_S M^* \xrightarrow{\mathbf{n}(M,-)} \mathrm{Hom}_S(M,-) \longrightarrow \underline{\mathrm{Ext}}^0_S(M,-) \longrightarrow 0 \ ,$$

where for any (right) S-module the morphism $\mathbf{n}(M,N)$ is given by

$$(\mathbf{n}(M,N)(n \otimes \varphi))(m) = n.\varphi(m)$$

for any $n \in N$, $\varphi \in M^*$ and $m \in M$, and

(iv) $\underline{\mathrm{Ext}}^0_S(M,N) = \underline{\mathrm{Hom}}_S(M,N)$, the S-linear homomorphisms from M to N modulo those which factor over projective modules.

<u>Proof</u>: The first two statements are obtained from the above example and the foregoing Corollary as by definition M is MCM iff

$$\operatorname{codepth}(M) = 0 = \operatorname{grade}(M) \ .$$

(iv) is just (6.1.2.(i)) reformulated for $i = 0$.
Instead of deducing (iii) directly from the above results, we make the

Remark: In more generality, M.Auslander - [Aus 1;Ch.3,Prop.1 bis], see also [A-B;2.8.] - introduced and studied the exact sequence of functors

$$0 \longrightarrow \mathrm{Tor}_2^T(-,D(M)) \longrightarrow -\otimes_T M^* \xrightarrow{n(M,-)} \mathrm{Hom}_T(M,-) \longrightarrow \mathrm{Tor}_1^T(-,D(M)) \longrightarrow 0$$

on mod-T , where

- T is an arbitrary ring,
- M is a (right) T-module which admits a presentation

$$Q \xrightarrow{f} P \longrightarrow M \longrightarrow 0$$

  by finitely generated projective T-modules P and Q , and
- D(M) is the left T-module obtained by dualizing the presentation with respect to T :

$$0 \longrightarrow M^* \longrightarrow P^* \xrightarrow{f^*} Q^* \longrightarrow D(M) \longrightarrow 0 .$$

  (D(M) is well-defined in the category of left T-modules modulo projectives.)

If now T = S is strongly Gorenstein, M a maximal Cohen-Macaulay S-module, then

- $D(M) = \mathrm{Ker}(f)^* = (\Omega_S^2 M)^*$ and $\Omega_S^2 M$ is again MCM, (4.2.2.(ii)), so that
- $\mathrm{Tor}_i^S(-,D(M)) = \mathrm{Tor}_i^S(-,(\Omega_S^2 M)^*) = \underline{\mathrm{Ext}}_S^{-i-1}(\Omega_S^2 M,-)$ for all $i > 0$

  by assertion (ii) above. But, in view of (6.1.2.(i)) together with the fact that Ω_S on $\underline{\mathrm{MCM}(S)}$ represents the inverse of the translation functor, this shows
- $\mathrm{Tor}_i^S(-,D(M)) = \underline{\mathrm{Ext}}_S^{-i+1}(M,-)$ for all $i > 0$,

whence the exact sequence (6.4.1.(iii)) becomes a special case of the exact sequence above.

One may take (6.4.1.) as the starting point for a "synthetic" definition of Tate-cohomology by "dimension shifts" :
For any ring S which is strongly Gorenstein, there is a unique family of cohomological bi-functors $\underline{\mathrm{Ext}}_S^{\cdot}(-,-)$, characterized by

(i) (δ-functor) In classical terminology, $\underline{\mathrm{Ext}}_S^{\cdot}(-,-)$ is a δ-functor in both arguments, contravariant in the first, covariant in the second.

(ii) (effaceable and co-effaceable) For any module U of finite projective dimension and for any integer i , one has

$$\underline{\mathrm{Ext}}^i_S(U,-) = 0 = \underline{\mathrm{Ext}}^i_S(-,U) \quad ,$$

and for any finitely generated S-module N there exist modules U and V of finite projective dimension, such that N is a quotient of U - obvious -, and a submodule of V - by (5.1.4.) .

(iii) ("<u>initial condition</u>") For a maximal Cohen-Macaulay S-module in the contravariant argument, these cohomological functors are determined by (6.4.1.).

It is clear, that these three "axioms" determine the Tate-cohomology groups completely and in a unique way for all modules in mod-S .

6.5. As a final observation in this direction, we will use the above description to "narrow the gap" in the "unstable range" , (6.3.5.), between -1 and vdim S .

For this, remark first that the very "historical" reason for the introduction of the loop-space functor Ω_S on <u>mod-S</u> was the isomorphism of functors $\mathrm{Ext}^i_S(\Omega^j_S-,-) \xrightarrow{\simeq} \mathrm{Ext}^{i+j}_S(-,-)$ for all $i > 0$, $j \geqq 0$ - cf. [ALG X.128;Cor.4].

On the other hand, by definition of the groups involved, there are natural surjections $\mathrm{Ext}^i_S(-,-) \longrightarrow \underline{\mathrm{Hom}}_S(\Omega^i_S-,-)$ for all $i \geqq 0$.

These surjections yield precisely the image of the natural transformations $c^i : \mathrm{Ext}^i_S(-,-) \longrightarrow \underline{\mathrm{Ext}}^i_S(-,-)$ in case of a ring S which is strongly Gorenstein. More specifically :

<u>Proposition 6.5.1.</u>: For any two modules in mod-S one has natural isomorphisms

$$\underset{\substack{n \\ n+i \geqq 0}}{\underrightarrow{\lim}} \underline{\mathrm{Hom}}_S(\Omega^{i+n}_S M, \Omega^n_S N) \xrightarrow{\simeq} \underline{\mathrm{Ext}}^i_S(M,N)$$

for all integers i.

The direct limit on the left is essentially constant in the sense that all transition-maps are isomorphisms as soon as $i+n$ is at least equal to vdim S .

In particular, this limit "collapses" if $i \geqq$ vdim S , but defines a <u>finite ascending filtration</u> of $\underline{\mathrm{Ext}}^i_S(M,N)$ by abelian subgroups

$$F^i_k = \mathrm{Im}(\underline{\mathrm{Hom}}_S(\Omega^{i+k}_S M, \Omega^k_S N) \longrightarrow \underline{\mathrm{Ext}}^i_S(M,N)) \quad , \ k \geqq \max(0,-i) \ .$$

For $i \geqq 0$, F^i_k equals the image of $\mathrm{Ext}^i_S(\Omega^k_S M, \Omega^k_S N)$ in $\underline{\mathrm{Ext}}^i_S(M,N)$.

<u>The proof</u> can be obtained directly from the "axiomatic" treatment above, or, more conceptually, by applying the functor of MCM-approximations : As $\mathbf{M}(-)$ is a right adjoint of the embedding $\underline{\mathrm{MCM}(S)} \longrightarrow \underline{\mathrm{mod}\text{-}S}$, it commutes necessarily with the loop-space functor. But as Ω_S on $\underline{\mathrm{MCM}(S)}$ represents the inverse of the translation functor, one gets

$$\underline{\mathrm{Hom}}_S(\Omega_S^{i+n}M,\Omega_S^nN) \longrightarrow \mathrm{Hom}_{\underline{\mathrm{MCM}(S)}}(\mathbf{M}(\Omega_S^{i+n}M),\mathbf{M}(\Omega_S^nN)) \quad \text{by applying } \mathbf{M} \text{ and}$$

$$\begin{aligned}\mathrm{Hom}_{\underline{\mathrm{MCM}(S)}}(\mathbf{M}(\Omega_S^{i+n}M),\mathbf{M}(\Omega_S^nN)) &= \mathrm{Hom}_{\underline{\mathrm{MCM}(S)}}(\Omega_S^{i+n}\mathbf{M}(M),\Omega_S^n\mathbf{M}(N))\\ &= \mathrm{Hom}_{\underline{\mathrm{MCM}(S)}}(T^n\mathbf{M}(M),T^{i+n}\mathbf{M}(N))\\ &= \mathrm{Hom}_{\underline{\mathrm{MCM}(S)}}(\mathbf{M}(M),T^i\mathbf{M}(N))\\ &= \underline{\mathrm{Ext}}_S^i(M,N)\end{aligned}$$

using the definition (6.1.1.) and the equivalence ι_S of (4.4.1.(2)). Applying Ω_S and passing to the limit establishes then the result in view of the fact that one has an isomorphism of functors $\mathbf{M}\Omega_S^j = \Omega_S^j$, as soon as $j \geq \mathrm{vdim}\ S$, cf. (4.2.2.(iv)).

This proposition shows in which sense Tate-cohomology "stabilizes" the ordinary Ext's and provides the link to the "stable module theory" of M.Auslander-M.Bridger,[A-B] .

Furthermore, it reveals a forth equivalent description of $\underline{\mathrm{MCM}(S)}$, the category of all maximal Cohen-Macaulay modules modulo projectives over a ring which is strongly Gorenstein :

<u>It can be obtained - up to natural equivalence - from $\underline{\mathrm{mod}\text{-}S}$, the stable module category, by "inverting" the loop-space functor Ω_S .</u>

To formalize this statement, remark first the following general fact :

<u>Lemma 6.5.2.</u>: Assume given a pair $(\underline{A},e)$, where $\underline{A}$ is an additive category, $e : \underline{A} \longrightarrow \underline{A}$ an additive endo-functor of $\underline{A}$.
Then there exists an additive functor $j : (\underline{A},e) \longrightarrow (\underline{A}^\#,t)$ into an additive category $\underline{A}^\#$ together with an (additive) <u>auto-equivalence</u> $t : \underline{A}^\# \longrightarrow \underline{A}^\#$ such that :

(i) j transforms e into t^{-1} , i.e.: $t^{-1}j$ is isomorphic to $j.e$,

(ii) Any additive functor $F : (\underline{A},e) \longrightarrow (\underline{B},s)$ into an additive category $\underline{B}$ with auto-equivalence s , which transforms e into s^{-1} , factors uniquely over j by an additive functor $F^\# : (\underline{A}^\#,t) \longrightarrow (\underline{B},s)$ which transforms t into s , that is : $F = F^\# j$ and $F^\# t$ is isomorphic to $sF^\#$.

The functor j is unique up to "equivalences of categories with auto-equivalences under $(\underline{A},e)$".
(In other words, - beware of the universe -, there is a left adjoint of the forgetful functor from $(\underline{Ad},aut)$, the "category of additive categories with auto-equivalences" to $(\underline{Ad},end)$, the "category of additive categories with endo-functors".)

Proof: The objects of $\underline{A}^{\#}$ are given by all pairs (X,n), where X is an object in $\underline{A}$, n an integer.
The morphisms are given by

$$\mathrm{Hom}_{\underline{A}^{\#}}((X,n),(Y,m)) = \varinjlim_{i\geq -m,-n} \mathrm{Hom}_{\underline{A}}(e^{i+m}X, e^{i+n}Y) \quad ,$$

the auto-equivalence t by $t(X,n) = (X,n+1)$ and the "identity" on the direct limits representing the morphisms.
The functor j sends an object X from $\underline{A}$ to $(X,0)$ and is given on morphisms by the natural map of $\mathrm{Hom}_{\underline{A}}(X,Y)$ into the direct limit $\varinjlim_{i\geq 0} \mathrm{Hom}_{\underline{A}}(e^iX,e^iY)$.

If $F : (\underline{A},e) \longrightarrow (\underline{B},s)$ is now a functor as in (ii), set

$$F^{\#}(X,n) = s^nF(X) \quad .$$

It is left to the reader to convince himself that these data solve the "universal mapping problem" claimed - cf. also [BBD;1.1.5.] .

Coming back to the category of S-modules, it was stated in (2.2.2.) that the composition of the natural embedding of mod-S into $D^b(S)$ with the projection from $D^b(S)$ onto $\underline{D^b(S)}$ factors over the functor ι_S, which is defined on $\underline{\text{mod-S}}$ and transforms the loop-space functor into the inverse T^{-1} of the translation functor on $\underline{D^b(S)}$.
Hence $\iota_S : \underline{\text{mod-S}} \longrightarrow \underline{D^b(S)}$ factors over the category $(\underline{\text{mod-S}}^{\#},\Sigma_S)$, constructed from $\underline{\text{mod-S}}$ by "inverting Ω_S" in the sense above, - so that Σ_S represents the "suspension functor".
The induced functor $\iota_S^{\#}$ transforms then the "suspension functor" into the translation functor and Proposition (6.5.1.) is equivalent to :

Theorem 6.5.3.: $\iota_S^{\#} : (\underline{\text{mod-S}}^{\#},\Sigma_S) \longrightarrow (\underline{D^b(S)},T)$ is an equivalence of "graded" categories, [Ver;1.1.0.].

In particular, $\underline{\text{mod-S}}^{\#}$ inherits a triangulated structure via this equivalence and the reader is asked to describe the distinguished triangles using - and extending - (4.7.).

7. - *Multiplicative structure, Duality and Support*

So far, we have dealt with the definition and properties of Tate-cohomology for a fixed ring S which is strongly Gorenstein.
The main aspect here will be the functorial behaviour of this cohomology theory, culminating with a general duality theorem and its applications, which encompass the "classical" such theorem for Tate-cohomology of integral group rings as well as the analogue of Grothendieck-Serre-duality for finite morphisms (of commutative rings).

But we start with a short discussion of

7.1. Yoneda- or extension-products

The very definition of Tate-cohomology as groups of morphisms in the "graded" category $\underline{D^b(S)}$ implies that composition of such morphisms yields bilinear pairings (the "Yoneda-products") for any objects X , Y and Z in $D^b(S)$ and all integers i and j :

$$(7.1.1.)\qquad \underline{\mathrm{Ext}}^i_S(Z,Y) \times \underline{\mathrm{Ext}}^j_S(X,Z) \longrightarrow \underline{\mathrm{Ext}}^{i+j}_S(X,Y) \qquad ,$$

functorial in its arguments and associative in the usual sense.

As the natural transformation $c^\cdot : \mathrm{Ext}^\cdot_S(-,-) \longrightarrow \underline{\mathrm{Ext}}^\cdot_S(-,-)$ is induced by the projection functor from $D^b(S)$ onto $\underline{D^b(S)}$ by (6.2.5.(3)), it is necessarily compatible with these products.

Considering the behaviour of these products with respect to duality, one has as usual - see [SGA 4;XVII.1.1.] - that the following diagram is commutative "up to the sign $(-1)^{ij}$ ", (where $-^*$ represents $\mathrm{RHom}_S(-,S)$ as in (6.2.3.), the horizontal arrows are given by the respective products, the vertical isomorphisms by "transposition" as in (6.2.5.), - followed on the left hand side by the exchange of factors) :

$$(7.1.2.)\qquad \begin{array}{ccc} \underline{\mathrm{Ext}}^i_S(Z,Y) \times \underline{\mathrm{Ext}}^j_S(X,Z) & \longrightarrow & \underline{\mathrm{Ext}}^{i+j}_S(X,Y) \\ \simeq\updownarrow & (-1)^{ij} & \updownarrow\simeq \\ \underline{\mathrm{Ext}}^j_{S^{op}}(Z^*,X^*) \times \underline{\mathrm{Ext}}^i_{S^{op}}(Y^*,Z^*) & \longrightarrow & \underline{\mathrm{Ext}}^{i+j}_{S^{op}}(Y^*,X^*) \end{array} \quad .$$

In other words, for any two given classes f in $\underline{\mathrm{Ext}}^i_S(Z,Y)$, g in $\underline{\mathrm{Ext}}^j_S(X,Z)$, one has

$$(f\circ g)^* = (-1)^{ij}(g^*\circ f^*) \quad \text{in} \quad \underline{\mathrm{Ext}}^{i+j}_{S^{op}}(Y^*,X^*) \quad .$$

Recall furthermore, [ALG X.128], that on the (hyper-)Tor-groups $\mathrm{Tor}^S_{\cdot}(Y,X^*)$, - cf. (6.2.3.-4.) - , there are natural <u>left</u> actions of both $\mathrm{Ext}^{\cdot}_S(Y,Z)$ and $\mathrm{Ext}^{\cdot}_S(X^*,W^*)$ for all objects Z , W in $D^b(S)$.

These actions <u>anti-commute</u>, or, equivalently, if we define a <u>right</u> action of $\mathrm{Ext}^{\cdot}_S(W,X)$ on $\mathrm{Tor}^S_{\cdot}(Y,X^*)$:

$$\mathrm{Tor}^S_i(Y,X^*) \times \mathrm{Ext}^j_S(W,X) \longrightarrow \mathrm{Tor}^S_{i-j}(Y,W^*)$$

by

$$x\circ f =_{def} (-1)^{ij}(f^*\circ x) \quad ,$$

for all integers i and j , this newly defined action of $\mathrm{Ext}^{\cdot}_S(W,X)$ on $\mathrm{Tor}^S_{\cdot}(Y,X^*)$ becomes compatible with the original left action of $\mathrm{Ext}^{\cdot}_S(Y,Z)$.

Putting all these different actions together, we get :

<u>Proposition 7.1.3.</u>: Let W , X , Y , Z be objects in $D^b(S)$. Then the morphisms in the exact sequence (6.2.5.(3)) for X and Y are

- <u>right</u>-linear with respect to $\mathrm{Ext}^{\cdot}_S(W,X)$ and
- <u>left</u>-linear with respect to $\mathrm{Ext}^{\cdot}_S(Y,Z)$.

More precisely, given classes f in $\mathrm{Ext}^i_S(Y,Z)$ and g in $\mathrm{Ext}^j_S(W,X)$, right-multiplication with g (resp. $c^j(g)$) and left-multiplication with f (resp. $c^i(f)$) yields a homomorphism of exact sequences :

$$\begin{tikzcd}
\cdots \arrow[r] & \underline{\mathrm{Ext}}^{k-1}_S(X,Y) \arrow[r] \arrow[d, "c^i(f)\circ"', "\circ c^j(g)"] & \mathrm{Tor}^S_{-k}(Y,X^*) \arrow[r] \arrow[d, "f\circ"', "\circ g"] & \mathrm{Ext}^k_S(X,Y) \arrow[r, "c^k"] \arrow[d, "f\circ"', "\circ g"] & \cdots \\
\cdots \arrow[r] & \underline{\mathrm{Ext}}^{i+j+k-1}_S(W,Z) \arrow[r] & \mathrm{Tor}^S_{-k-i-j}(Z,W^*) \arrow[r] & \mathrm{Ext}^{k+i+j}_S(W,Z) \arrow[r, "c^{k+i+j}"] & \cdots
\end{tikzcd}$$

Taking into account the already observed degeneracy of these exact sequences, (6.3.4.), one can further analyze the structure of the Norm-map $\mathbf{N}(X,Y)$, (6.2.5.(1)), :

Example 7.1.4.: Let x be an element of $\mathrm{Ext}_S^k(X,Y)$. If there is a pair (Z,f) with Z an object in $D^b(S)$, and f an element in $\mathrm{Ext}_S^i(Y,Z)$ such that

$$k+i > \bar{m}(X,Z)\ , \quad \text{(with notations as in (6.3.3.))},$$

$$f \circ x \neq 0 \quad \text{in} \quad \mathrm{Ext}_S^{k+i}(X,Z)\ ,$$

then necessarily :

$$c^k(x) \neq 0 \quad \text{in} \quad \underline{\mathrm{Ext}}_S^k(X,Y)\ ,$$

that is : x is not in the image of $H^\cdot(\mathbf{N}(X,Y))$, as

$$c^i(f) \circ c^k(x) = c^{i+k}(fx) = fx$$

by the first assumption and (6.3.4.(i)).

Hence, in a sense which is left to be made precise by the reader, the (cohomological) Norm-map $H^\cdot(\mathbf{N}(X,Y)) : \mathrm{Tor}^S_{-\cdot}(Y,X^*) \longrightarrow \mathrm{Ext}_S^\cdot(X,Y)$ takes its values only in the subgroup of those classes of $\mathrm{Ext}_S^\cdot(X,Y)$ which get annihilated by the Yonedaproduct with any class of sufficiently high degree. (If one considers in analogy to the geometric situation the projection of $D^b(S)$ onto $\underline{D^b(S)}$ as a "localization functor", - a point of view which is supported by (6.5.3.) - , one may say that $H^\cdot(\mathbf{N}(X,Y))$ takes its values in the "zeroth local cohomology" of sections with support in $D^b_{perf}(S)$, off which one localizes.)

It seems rather suggestive to believe, that these classes constitute in fact the image of the cohomological Norm-map.

Similarly, its kernel contains all elements which are "eventually decomposable" in the appropriate (obvious) sense.

To conclude this subsection, specialize (7.7.1.) to the case where $X = Y = Z$. This yields for any object X in $D^b(S)$ a "stabilized Yoneda-Ext-algebra" $\underline{\mathrm{Ext}}_S^\cdot(X,X)$, which is a $\mathbb{Z}$-graded ring, canonically isomorphic to the <u>graded opposite</u> ring of $\underline{\mathrm{Ext}}_{S^{op}}^\cdot(X^*,X^*)$.

If X is in fact (isomorphic to) a single S-module M, this algebra comes further equipped with an ascending filtration by subgroups, - see (6.5.1.) - :

$$(7.1.5.) \qquad F_i^\cdot(M) = \mathrm{Im}(\mathrm{Ext}_S^\cdot(\Omega_S^i M, \Omega_S^i M) \longrightarrow \underline{\mathrm{Ext}}_S^\cdot(M,M))\ ,$$

and this filtration is "good" in the sense that

- $F_i^r(M).F_j^s(M)$ is contained in $F_{i+j}^{r+s}(M)$ for all i,j,r,s and
- one has equality for sufficiently large i,j.

7.2. Operations of a perfect complex

Assume that S and T are rings which are both strongly Gorenstein. Consider a complex L of S-T-bimodules, perfect as both a complex of left S-modules or right T-modules.

Then the derived tensor-product $- \otimes_S^{\mathbb{L}} L$ provides an exact functor from $D^b(S)$ into $D^b(T)$ which carries perfect complexes into such.
Hence it passes trivially to the respective quotients and induces an exact functor between the stabilized categories

$$(7.2.1.) \qquad - \underline{\otimes}_S^{\mathbb{L}} L : \underline{D^b(S)} \longrightarrow \underline{D^b(T)} \quad .$$

It admits a right adjoint iff $RHom_T(L,T)$ is still perfect as a complex of right S-modules. In this case, the adjoint is the exact functor

$$(7.2.2.) \qquad R\underline{Hom}_T(L,-) : \underline{D^b(T)} \longrightarrow \underline{D^b(S)} \quad ,$$

which is induced from $RHom_T(L,-)$ by passing to the quotient-categories.
These functors are then most easily described on $\underline{APC(-)}$:
If A is an acyclic complex of finitely generated projective (right) S-modules - hence an object of $\underline{APC(S)}$ - , the total complex associated to $A \otimes_S L$, considered as a complex of right T-modules, is necessarily an object of $\underline{APC(T)}$ and represents naturally $A \otimes_S^{\mathbb{L}} L$.

In the same way, one has a natural identification - cf. (6.2.1.(ii)) -

$$R\underline{Hom}_T(L,B) = B \otimes_T Hom_T(L,T)$$

for any complex B in $\underline{APC(T)}$.

Rewriting this in terms of complete resolutions, it says that there are functorial isomorphisms

$$(7.2.3.) \qquad \mathbf{CR}_S(X) \otimes_S^{\mathbb{L}} L = \mathbf{CR}_T(X \otimes_S^{\mathbb{L}} L) \quad ,$$

$$\mathbf{CR}_S(R\underline{Hom}_T(L,Y)) = Hom_T^{\cdot}(L,\mathbf{CR}_T(Y))$$

for X in $D^b(S)$, Y in $D^b(T)$.

The reader should observe that the notation "$R\underline{Hom}_T(L,-)$" is some-what misleading : There are no cohomology groups naturally attached to $\underline{\mathrm{RHom}}_T(L,-)$, in particular $\underline{\mathrm{Ext}}_T^{\cdot}(L,-)$,(which groups vanish anyway as L is perfect over T), does not represent its cohomology.

(As an example, remark that for $S = T = L$, the functor $\underline{\mathrm{RHom}}_S(S,-)$ is naturally isomorphic to the identity functor on $\underline{D^b(S)}$.)

Instead, consider the cohomology groups

$$E^{\cdot} = \underline{\mathrm{Ext}}_T^{\cdot}(X \overset{\mathbb{L}}{\otimes}_S L, Y) = \underline{\mathrm{Ext}}_S^{\cdot}(X, \underline{\mathrm{RHom}}_T(L,Y)) \quad .$$

They are obtained as the common abutment of the two spectral sequences

$$'E_2^{i,j} = \underline{\mathrm{Ext}}_T^i(\mathrm{Tor}_j^S(X,L),Y) \Longrightarrow E^{i+j}$$

and

$$''E_2^{i,j} = \underline{\mathrm{Ext}}_S^i(X,\mathrm{Ext}_T^j(L,Y)) \Longrightarrow E^{i+j} \quad ,$$

which depend functorially on X in $D^b(S)$ and Y in $D^b(T)$.

Furthermore, with respect to the natural transformations $c^{\cdot}$, these spectral sequences "stabilize" (and extend) the corresponding ones for the "ordinary" Ext's .

7.3. Change of rings

Let $f : T \longrightarrow S$ in the following be a ring homomorphism between two rings which are both strongly Gorenstein.

7.3.1. Then certainly $f^* = - \overset{\mathbb{L}}{\otimes}_T S$ transforms perfect complexes over T into such over S . But, almost by definition, it induces a functor from $D^b(T)$ into $D^b(S)$ - and not just $D^-(S)$ - only if S considered as a left T-module (by restriction of scalars along f) is of finite flat dimension. In this case, f^* passes trivially to the quotients and the induced exact functor will be denoted $\underline{f^*} : \underline{D^b(T)} \longrightarrow \underline{D^b(S)}$.

7.3.2. The functor f_* , which "restricts the scalars", will transform $D^b(S)$ into $D^b(T)$ iff f_*S , the underlying (right) T-module of S , is finitely generated over T . As S generates $D^b_{perf}(S)$, (1.2.1.), f_* will preserve perfectness iff furthermore f_*S is of finite projective dimension over T as a right module.

If in particular S is <u>perfect</u>, (that is: finitely generated and of finite projective dimension), on both sides over T - with respect to f - , there is a naturally induced pair of adjoint functors $(\underline{f}^*, \underline{f}_*)$ between $\underline{D^b(T)}$ and $\underline{D^b(S)}$.

This is of course also a special case of (7.2.).

7.3.3. The right adjoint $f^!$ of f_* , obtained as the derived functor of $\mathrm{Hom}_T(f_*S,-)$, transforms modules of finite injective dimension into (complexes isomorphic to) such modules, but does not necessarily preserve finite generation, so that in general $D^b(T)$ is not carried into $D^b(S)$.

But, if $f^!T = \mathrm{RHom}_T(f_*S,T)$ is <u>perfect</u> - as a complex of right modules over S - , one may apply (7.2.) to S considered as a complex of S-T-bimodules, which yields that $f^!$ induces an exact functor

$$\underline{f}^! : \underline{D^b(T)} \longrightarrow \underline{D^b(S)} ,$$

so that the pair $(\underline{f}^!, \underline{f}_*)$ is adjoint again.

7.3.4. Summarizing, in the most favourable case where

- f_*S is perfect on both sides over T and
- $\mathrm{RHom}_T(f_*S,T)$ is perfect as a complex of right S-modules,

there is a trivially induced triple of adjoint functors $(\underline{f}^*, \underline{f}_*, \underline{f}^!)$ between $\underline{D^b(S)}$ and $\underline{D^b(T)}$.

It is left to the reader to specialize (7.2.) accordingly to obtain the corresponding "stabilized" change-of-rings spectral sequences.

Even under the foregoing strong hypotheses on f , there is still a shortcoming: Without further restrictions - like S and T both being commutative - , we ignore the behaviour of the deduced triple of adjoint functors with respect to the dualities induced by dualizing modules for S or T , (4.6.1.).

Hence it seems rather doubtful that the class of "bi-perfect" ring homomorphisms constitutes already the appropriate class of morphisms to turn (strongly) Gorenstein rings into a suitable category.

Furthermore, the following suggests that the above class of ring homomorphisms might be too small :

7.4. An extension of Tate-cohomology

To put the analogue of "Grothendieck-Serre-duality" for Tate-cohomology into its proper general context, it will be convenient first to extend the definition of the stabilized Ext-groups, allowing arbitrary modules as covariant argument.

This also constitutes the "right" generalization of the classical case, in which the contravariant argument is the fixed augmentation module, whereas the covariant argument remains unrestricted.

The main reason to treat Tate-cohomology here up to now as an essentially "symmetric" theory was provided by its behaviour with respect to the "trivial duality" of (6.2.5.(2)), based on (4.6.1.), which seemingly places contra- and co-variant argument on the same footing.

But, examining (6.2.5.(3)) or (6.4.) a little bit closer, it becomes evident that in fact the contravariant argument is "special" :

Fixing, say, a maximal Cohen-Macaulay S-module M and considering (6.4.1.(i)-(iii)) as a definition, one may interpret the family of functors $(\underline{\mathrm{Ext}}_S^i(M,-))_i$ to yield a (covariant) cohomological δ-functor on all S-modules - and not just on the finitely generated ones.

To formulate a more precise statement, we will use - and extend - the description given in Remark (b) of (6.2.5.) .

Following [Ver;II.1.1.] , let us recall that for an arbitrary ring S and $* = -$ or $* = b$,

$$D^*(\mathrm{Mod}\text{-}S) = K^{-,*}(\mathrm{Mod}\text{-}S)/K^{-,\emptyset}(\mathrm{Mod}\text{-}S)$$

denotes the derived category which is obtained from $K^{-,-/b}(\mathrm{Mod}\text{-}S)$, the homotopy-category of all bounded-above complexes of "arbitrary" right S-modules (with bounded cohomology) , by factoring out its thick subcategory $K^{-,\emptyset}(\mathrm{Mod}\text{-}S)$ of all acyclic complexes.

It is over such a category that the covariant argument will range.

Assuming again that S is a ring which is strongly Gorenstein, we have

Definition - Proposition 7.4.1.: For any complexes Y in $D^-(\mathrm{Mod}\text{-}S)$ and X in $D^b(S)$ set

$$R\underline{\mathrm{Hom}}_S^{\cdot}(X,Y) = \mathrm{Hom}_S^{\cdot}(\mathbf{CR}(X),Y) \quad .$$

(i) This defines a bi-functor into the (full) derived category of abelian groups,

$$R\underline{\mathrm{Hom}}_S^{\cdot}(-,-) : D^-(\mathrm{Mod}\text{-}S) \times \underline{D^b(S)}^{op} \longrightarrow D(\mathrm{Ab}) \quad ,$$

which is exact in both arguments.

(ii) The corresponding cohomology groups are still denoted

$$\underline{\mathrm{Ext}}^i_S(X,Y) =_{\mathrm{def}} H^i\mathrm{Hom}^{\cdot}_S(\mathbf{CR}(X),Y))$$

and they coincide naturally with the Tate-cohomology groups as defined in (6.1.1.) if Y has in fact bounded and finitely generated cohomology only.

(iii) For Y in $D^b(\text{Mod-}S)$ there is a convergent spectral sequence

$$E_2^{i,j} = \underline{\mathrm{Ext}}^i_S(\mathbf{M}(X),H^j(Y)) \Longrightarrow \underline{\mathrm{Ext}}^{i+j}_S(X,Y) \quad ,$$

whose E_2-terms can be calculated using (6.4.1.) .

Proof: As stated, the functor $R\underline{\mathrm{Hom}}^{\cdot}_S(-,-)$ is a priori defined on $K^{-,-}(\text{Mod-}S) \times \underline{APC(S)}^{op}$ and is obviously exact in each argument.

Given that $\mathbf{CR}(-)$ induces an equivalence between $\underline{D^b(S)}$ and $\underline{APC(S)}$ by (5.6.) and (4.4.1.(1.)) , it remains only to be seen that as soon as Y is an acyclic, bounded above complex of (arbitrary right) S-modules, the complex of groups $\mathrm{Hom}^{\cdot}_S(\mathbf{CR}(X),Y))$ is also acyclic.

But this is well-known (and easily established) for any , not necessarily acyclic, complex of projectives in the contravariant argument, hence holds a fortiori in the special case under consideration.

That this definition indeed extends (6.1.1.), follows from Remark (b) of (6.2.5.) .

The last assertion - which we will not use - is left as an exercise.

Remark: (a) This definition is certainly not symmetric in its arguments in the sense that even if Y is in $D^b(\text{Mod-}S)$, one cannot derive the covariant argument first by replacing it with an injective resolution :

If I is a bounded-below complex of injective modules, X an arbitrary acyclic complex (for example a complete resolution), then the complex $\mathrm{Hom}^{\cdot}_S(X,I)$ is acyclic ! (In other terms, identifying $D^b(\text{Mod-}S)$ with $K^{+,b}(\text{Mod-}S)/K^{+,\emptyset}(\text{Mod-}S)$, the functor $R\underline{\mathrm{Hom}}^{\cdot}_S(-,-)$ is not obtainable by passage to the quotient from a functor defined on $K^{+,b}(\text{Mod-}S)$ in its covariant argument.)

(b) One may still extend the definition of the Yoneda-products to this situation : The "original" stabilized $\underline{\mathrm{Ext}}$'s act from the right, the "ordinary" Ext's (on $D^-(\text{Mod-}S)$) act from the left. This is obvious.

A first reason to extend the definition of the Tate-cohomology groups is that they behave (slightly) better with respect to ring homomorphisms :

Proposition 7.4.2.: Let $f : T \longrightarrow S$ be a homomorphism of rings which are both strongly Gorenstein.

(i) If S is of finite flat dimension as a left T-module, there is a natural isomorphism of (bi-)functors on $D^-(\text{Mod-}S) \times \underline{D^b(T)}^{op}$,

$$R\underline{\mathrm{Hom}}^{\cdot}_S(\underline{f}^*-,-) = R\underline{\mathrm{Hom}}^{\cdot}_T(-,f_*-) \quad .$$

(ii) If S is finitely generated and of finite projective dimension as a right T-module, there is a natural isomorphism of (bi-)functors on $D^-(\text{Mod-}T) \times \underline{D^b(S)}^{op}$,

$$R\underline{\mathrm{Hom}}^{\cdot}_T(\underline{f}_*-,-) = R\underline{\mathrm{Hom}}^{\cdot}_S(-,f^!-) \quad .$$

The proof follows immediately from (7.3.) . The hypotheses just guarantee that the underlined functors are defined.

7.5. The Duality-Theorem

Using the now extended definition of the Tate-cohomology groups, we can formulate the rather broad version of a

Duality-Theorem 7.5.1.: Assume the following conditions to hold :

- S is a ring which is strongly Gorenstein, T is an arbitrary ring,
- M is a complex in $D^b(S^{op})$,
- N is a bounded-above complex of S-T-bimodules with only finitely many non-zero cohomology modules. (Hence it represents an object in $D^b(\text{Mod-}S^{op} \otimes T)$.)
- W is a complex of right T-modules, quasi-isomorphic to a finite*) complex of right injective T-modules.

Then there are two spectral sequences converging to the same limit $\mathbf{E}^{\cdot}$:

$${}'E_2^{i,j} = \mathrm{Ext}^i_T(\underline{\mathrm{Ext}}^{-j}_{S^{op}}(M,N),W) \quad ===> \mathbf{E}^{i+j}$$

and

$${}''E_2^{i,j} = \underline{\mathrm{Ext}}^{i-1}_S(M^*,\mathrm{Ext}^j_T(N,W)) \quad ===> \mathbf{E}^{i+j} \quad .$$

(Here again M^* is short for $R\mathrm{Hom}_{S^{op}}(M,S^{op})$, - cf. (6.2.3.) - .)

Proof: Let $W \longrightarrow I$ be an injective resolution of W over T . By assumption, I can - and will be - chosen to be a finite complex. The common abutment of the two spectral sequences will be given by the cohomology of the total complex associated to the double complex

$$C^{\cdot\cdot} = \mathrm{Hom}^{\cdot}_T(\mathrm{Hom}^{\cdot}_{S^{op}}(\mathbf{CR}(M),N),I) \quad .$$

Remark that the terms of the double complex $"C^{\cdot\cdot}$ are given by

$$"C^{i,j} = \mathrm{Hom}_S(\mathbf{CR}^{-i+1}(M^*), \prod_k \mathrm{Hom}_T(N^k, I^{k+j})) .$$

As $\mathrm{Hom}_T^{\cdot}(N,I)$ is bounded (above and below) by the assumptions on N and I, the natural filtration on $\mathbf{CR}(M^*)[1]$ induces a regular filtration on $"C^{\cdot\cdot}$, and the E_2-terms of the associated convergent spectral sequence are precisely the groups $"E_2^{i,j}$ of the theorem as follows from (7.4.1.(ii)), taking into account that the components of $\mathbf{CR}(M^*)$ are finitely generated projective S-modules.

This finishes the proof of (7.5.1.) .

Let us immediately record the simplest case in which both spectral sequences of (7.5.1.) are degenerate, (and which generalizes for example the corresponding well-known statement for integral group rings - see [C-E;XII.6.5.]) :

Corollary 7.5.2.: Maintaining the assumptions and notations of the foregoing theorem, assume furthermore that $W = I$ is actually already an injective T-module.

Then each of the two spectral sequences in (7.5.1.) collapses, the edge homomorphisms $"E_2^{j,0} \longrightarrow \mathbf{E}^j$ and $\mathbf{E}^j \longrightarrow {}'E_2^{0,j}$ are defined and are isomorphisms of abelian groups for any integer j .

More suggestively, let us denote by $\mathbf{D}(-)$ either contravariant functor on $D(\text{Mod-}T)$ or $D(\text{Mod-}S^{op}\otimes T)$ deduced from $\mathrm{Hom}_T(-,I)$.

Then the composition of the aforementioned edge-homomorphisms yields an isomorphism of abelian groups for any integer j :

(7.5.3.) $$\underline{\mathrm{Ext}}_S^{j-1}(M^*, \mathbf{D}(N)) \xrightarrow{\sim} \mathbf{D}(\underline{\mathrm{Ext}}_{S^{op}}^{-j}(M,N)) ,$$

and these isomorphisms are natural in M and N .

Before giving more sophisticated applications of the Duality Theorem, we want to analyze a little bit closer its proof :

All isomorphisms occurring above in the identification of the total complexes associated to $C^{\cdot\cdot}$ respectively "$C^{\cdot\cdot}$" are completely natural - except for the "interchange-isomorphism" which presupposes the choice of a "commutation-factor" and introduces signs.

These signs appear, if one studies the behaviour of this duality with respect to Yoneda-products.

In the following discussion, assume N and W in (7.5.1.) to be fixed and consider the variation of the spectral sequences in M.

If L is another object in $D^b(S^{op})$, the Yoneda-product from the <u>right</u> with $\underline{\mathrm{Ext}}_S^k(L^*,M^*)$ maps $"E_2^{i,j}(M) = \underline{\mathrm{Ext}}_S^{i-1}(M^*,\mathrm{Ext}_T^j(N,W))$ into $"E_2^{i+k,j}(L) = \underline{\mathrm{Ext}}_S^{i+k-1}(L^*,\mathrm{Ext}_T^j(N,W))$, as well as $E^{\cdot}(M)$ into $E^{\cdot+k}(L)$, - consider these groups as the cohomology of the total complexes associated to "$C^{\cdot\cdot}(M)$" or "$C^{\cdot\cdot}(L)$" respectively.

This action is natural in the sense that it commutes with the respective differentials of the (doubly primed) spectral sequences and is compatible with the induced filtrations on the limit terms. In other words, any given element of $\underline{\mathrm{Ext}}_S^k(L^*,M^*)$ defines in a natural way a morphism of spectral sequences of bi-degree $(k,0)$.

Dually, $\underline{\mathrm{Ext}}_{S^{op}}^k(M,L)$ acts from the right on $\underline{\mathrm{Ext}}_{S^{op}}^{\cdot}(L,N)$, and applying $\mathrm{Ext}_T^i(-,W)$ to it, it furnishes a natural <u>left</u> action of this group which maps $'E_2^{i,j}(M) = \mathrm{Ext}_T^i(\underline{\mathrm{Ext}}_{S^{op}}^{-j}(M,N),W)$ into $'E_2^{i,j+k}(L) =$
$= \mathrm{Ext}_T^i(\underline{\mathrm{Ext}}_{S^{op}}^{-j-k}(L,N),W)$.

Chasing through the identifications in the proof above and taking into account (7.1.2.), the connection between these two actions can be expressed as follows :

<u>Corollary 7.5.2.</u>: With the foregoing notations, if α is a class in $\underline{\mathrm{Ext}}_{S^{op}}^k(M,L)$, and α^* its transpose in $\underline{\mathrm{Ext}}_S^k(L^*,M^*)$, there is the following diagram of morphisms of spectral sequences, commutative in an obvious sense :

$$\begin{tikzcd}
\mathrm{Ext}_T^i(\underline{\mathrm{Ext}}_{S^{op}}^{-j}(M,N),W) \arrow[r, Rightarrow] \arrow[d, "(-1)^{jk}\mathrm{Ext}_T^i(-\circ\alpha,W)"'] & E^{i+j}(M) \arrow[d, "-\circ\alpha^*"] & \underline{\mathrm{Ext}}_S^{i-1}(M^*,\mathrm{Ext}_T^j(N,W)) \arrow[l, Rightarrow] \arrow[d, "-\circ\alpha^*"] \\
\mathrm{Ext}_T^i(\underline{\mathrm{Ext}}_{S^{op}}^{-j-k}(L,N),W) \arrow[r, Rightarrow] & E^{i+j+k}(L) & \underline{\mathrm{Ext}}_S^{i+k-1}(L^*,\mathrm{Ext}_T^j(N,W)) \arrow[l, Rightarrow]
\end{tikzcd}$$

More precisely, its terms are given by

$$C^{i,j} = \mathrm{Hom}^i_T(\mathrm{Hom}^{-j}_{S^{op}}(\mathbf{CR}(M),N),I) = \mathrm{Hom}_T(\prod_k \mathrm{Hom}_{S^{op}}(\mathbf{CR}^k(M),N^{k-j}),I^i)$$

Now, as I is supposed to be finite, both spectral sequences associated to this double complex are biregular, [EGA III.0.11.3.3.(ii)], and the $'E_2$-terms of the spectral sequence associated to the first filtration of this double complex are the groups $'E_2^{i,j}$ of the theorem as I is assumed to be an injective resolution of W.

Here we did not need that N has bounded cohomology. This condition only intervenes in dealing with the second spectral sequence.

For this, remark first that the cohomology of (the total complex associated to) $C^{\cdot,\cdot}$ does not change if one replaces N by a quasi-isomorphic complex in $K^{-,-}(\mathrm{Mod}\text{-}S^{op}\otimes T)$: that $\mathrm{Hom}^{\cdot}_S(\mathbf{CR}(M),-)$ preserves such quasi-isomorphisms was explained above, (7.4.1.), and for $\mathrm{Hom}^{\cdot}_T(-,I)$ it is obvious. Hence we may and will assume that N is indeed a finite complex of S-T-bimodules.

Then, in the foregoing explicit description of $C^{i,j}$ the product may be replaced by a (finite) direct sum so that the natural morphism of double complexes

$$N \otimes_{S^{op}} \mathrm{Hom}^{\cdot}_{S^{op}}(\mathbf{CR}(M),S^{op}) \longrightarrow \mathrm{Hom}^{\cdot}_{S^{op}}(\mathbf{CR}(M),N)$$

is in fact an isomorphism, compatible with the right T-module structures on either side.

Composing this with the "interchange-isomorphism" of double complexes - [ALG X.71] - :

$$\mathrm{Hom}^{\cdot}_{S^{op}}(\mathbf{CR}(M),S^{op}) \otimes_S N \longrightarrow N \otimes_{S^{op}} \mathrm{Hom}^{\cdot}_{S^{op}}(\mathbf{CR}(M),S^{op}) \quad ,$$

and passing to the associated total complexes, one identifies the double complex $C^{\cdot,\cdot}$ with the double complex

$$'C^{\cdot,\cdot} = \mathrm{Hom}^{\cdot}_T(\mathbf{CR}(M)^* \otimes_S N,I) \quad .$$

Here, as before, $-^*$ denotes the dual - this time with respect to S^{op} .

Now apply finally to $'C^{\cdot,\cdot}$ term by term the adjunction-isomorphism which induces an isomorphism of the total complexes associated :

$$\mathrm{Hom}^{\cdot}_T(\mathbf{CR}(M)^* \otimes_S N,I) = \mathrm{Hom}^{\cdot}_S(\mathbf{CR}(M)^*,\mathrm{Hom}^{\cdot}_T(N,I))$$

and observe that $\mathbf{CR}(M)^*$ is canonically isomorphic to $\mathbf{CR}(M^*)[1]$ by (4.6.1.) .

Putting all this together, it follows that the total complexes associated to $C^{\cdot,\cdot}$ or

$$''C^{\cdot,\cdot} = \mathrm{Hom}^{\cdot}_S(\mathbf{CR}(M^*)[1],\mathrm{Hom}^{\cdot}_T(N,I))$$

are naturally isomorphic.

7.6. To prepare for another application of the Duality Theorem, assume - still with the notations and hypotheses of (7.5.1.) - that N is in fact a complex of S-bimodules, whose <u>right</u> T-module structure is given by restricting the scalars on the right with respect to a homomorphism of rings $f : T \longrightarrow S$.

Recall that $\mathrm{Hom}_T(S,-)$, considered as the right adjoint of the forgetful functor f_* , carries injective T-modules into injective modules over S . In particular, for a complex W of T-modules as in (7.5.1.), the complex $f^! W$ is quasi-isomorphic to a finite complex of injective S-modules.

Hence, using the adjunction $(f_* , f^!)$, the spectral sequences in (7.5.1.) become

$$(7.6.1.)\qquad \begin{aligned} {}'E_2^{i,j} &= \mathrm{Ext}_S^i(\underline{\mathrm{Ext}}_{S^{op}}^{-j}(M,N), f^! W) \Longrightarrow E^{i+j} \\ {}''E_2^{i,j} &= \underline{\mathrm{Ext}}_S^{i-1}(M^*, \mathrm{Ext}_S^j(N, f^! W)) \Longrightarrow E^{i+j} \end{aligned} ,$$

(which, equivalently, are the spectral sequences of (7.5.1.) for $T = S$ and $f^! W$ instead of W).

Now we will be interested in the case where W can be chosen to be a dualizing module for T ,(and T strongly Gorenstein), such that $f^! W$ is still at least a "dualizing complex" for S - with the purpose that then for N a MCM S-module on the right, the doubly primed spectral sequence above degenerates.

We ignore the general situation in which this can be done, but let us exhibit the following*) particular one :

<u>Definition 7.6.2.</u>: In analogy to the commutative case, let us call a ring homomorphism $f : T \longrightarrow S$ a (central) <u>duality morphism of finite type</u> iff

(i) S is strongly Gorenstein and T is a <u>commutative Gorenstein ring</u> (of finite Krull dimension),

(ii) The morphism f turns S into a <u>central</u> T-algebra, such that the underlying T-module $f_* S$ is <u>finitely generated</u> and of <u>finite projective dimension</u> over T , say $d = \mathrm{projdim}_T f_* S$,

(iii) $f_*(-)[-d]$ <u>detects maximal Cohen-Macaulay S-modules</u> in the sense that M in mod-S is MCM iff

$$\mathrm{Ext}_T^i(f_* M[-d], T) = \mathrm{Ext}_T^{i+d}(f_* M, T) = 0 \quad \text{for} \quad i \neq 0 \quad .$$

Keeping closely to the usual proof in the commutative case, we will establish, as claimed, that for such duality morphisms

<u>$f^!$ transforms dualizing T-modules into dualizing S-modules</u>

in the following more precise sense :

<u>Proposition 7.6.3.</u>: Assume that $f : T \longrightarrow S$ satisfies (7.6.2.) . Then the following holds :

(i) The cohomology of $f^!T$ is concentrated in degree d and

$$\omega_{S/T} =_{def} H^d(f^!T)$$

is a dualizing module for S .

(ii) In particular, $\omega_{S/T} = f^!T[d]$ in $D^b(S)$ and $f^!T$ is a perfect complex of S-modules.

(iii) One has $d = \dim T - \operatorname{vdim} S$.

<u>Proof</u>: First, take $M = S$ in condition (iii), use that S is MCM and that

$$\operatorname{Ext}^i_T(f_*S[-d],T) = \operatorname{Ext}^{i+d}_S(S,f^!T) = H^{i+d}(f^!T) \quad .$$

This shows that the cohomology of $f^!T$ is indeed concentrated in degree d , or, equivalently, that one has $\omega_{S/T} = H^d(f^!T) \cong f^!T[d]$ in D(Mod-S).

Next, observe that $f^!T$ is in a natural way (represented by) a complex of <u>S-bimodules</u> (with bounded cohomology) and that forgetting the hence still existing <u>left</u> S-module structure on $\operatorname{RHom}_S(N,f^!T)$ for any given N in D(Mod-S) yields a natural isomorphism of complexes of T-modules

$$f_*\operatorname{RHom}_S(N,f^!T) = \operatorname{RHom}_T(f_*N,T)$$

in D(Mod-T) .

(For obvious reasons, we refrain from distinguishing "left" and "right" forgetful functor.)

This rather pedantic remark shows in particular that $\omega_{S/T}$ is in a natural way a S-bimodule and it remains to be seen that it is indeed (projective on both sides) and <u>invertible</u>. For this, we prove :

(a) The (right) S-module $\omega_{S/T}$ is <u>finitely generated</u> as already the T-module

$$f_*\omega_{S/T} = H^d(f_*f^!T) = H^d(\operatorname{RHom}_T(f_*S,T))$$

is finitely generated by condition (ii) of (7.6.2.) .

(b) $\omega_{S/T}$ is of <u>finite projective dimension</u> as a (right) S-module :
For any S-module M , one has by the above already that

$$\mathrm{Ext}^i_S(M,\omega_{S/T}) = \mathrm{Ext}^i_S(M,f^!T[d]) = \mathrm{Ext}^i_T(f_*M[-d],T) \quad .$$

But by condition (iii) of (7.6.2.) , these groups vanish for $i \neq 0$ if(f) M is MCM and (5.1.1.(ii)) applies.

(c) $\omega_{S/T}$ is <u>MCM over S</u> :
This is essentially the (local) duality theorem for <u>commutative</u> Gorenstein rings. Namely, by the foregoing and (4.6.1.), one has

$$\begin{aligned} \mathrm{RHom}_T(f_*\omega_{S/T}[-d],T) &= \mathrm{RHom}_T(f_*f^!T,T) \\ &= \mathrm{RHom}_T(\mathrm{RHom}_T(f_*S,T),T) \\ &= f_*S[0] \quad , \end{aligned}$$

whence $\omega_{S/T}$ satisfies the condition in (7.6.2.(iii)) .

Now apply (5.1.1.(iv)) to conclude from (a) - (c) that $\omega_{S/T}$ is <u>finitely generated projective</u> as a right S-module (and by symmetry then also as a left S-module). This already proves (ii) .

To get (i) completely, it remains still to verify that $\omega_{S/T}$ is an <u>invertible</u> S-bimodule. But in view of the above and by symmetry again, this reduces to the condition that the natural ring homomorphism

$$S \longrightarrow \mathrm{Hom}_S(\omega_{S/T},\omega_{S/T}) \quad ,$$

associating to an element of S the left multiplication with it, is an isomorphism of S-bimodules.

It is evidently enough, to prove instead that the underlying T-linear map is bijective. This is essentially the chain of identifications in (c) above, read backwards,and taking into account that

$$\begin{aligned} \mathrm{RHom}_T(f_*f^!T,T) &= f_*\mathrm{RHom}_S(f^!T,f^!T) \\ &= f_*\mathrm{Hom}_S(\omega_{S/T},\omega_{S/T})[0] \end{aligned}$$

in $D(\text{Mod-}T)$, $\omega_{S/T} = f^!T[d]$ being projective over S .
This finishes the proof of (i) .

Finally, we show that (i) ==> (iii) :
As $\omega_{S/T}$ is finitely generated projective over S , one has necessarily

$$\mathrm{injdim}_S\omega_{S/T} = \mathrm{vdim}\ S \quad .$$

As T is assumed to be Gorenstein, one has $\mathrm{injdim}_T T = \dim T$. Then, if I denotes an injective resolution of T as a module over itself, the complex $\mathrm{Hom}_T(S,I)$ represents $f^!T$ in $D^b(\text{Mod-}T)$.

Choosing I of minimal length, this means in view of (i) that $\omega_{S/T} = H^d(f^!T)$ admits an injective resolution of length at most $\dim T - d$, whence

$$\operatorname{vdim} S = \operatorname{injdim}_S \omega_{S/T} \leqq \dim T - d \quad .$$

On the other hand,

$$\operatorname{Ext}^i_T(f_*(-),T) = \operatorname{Ext}^{i-d}_S(-,f^!T[d]) = \operatorname{Ext}^{i-d}_S(-,\omega_{S/T}) \quad ,$$

so that necessarily

$$\operatorname{injdim}_S \omega_{S/T} \geqq \operatorname{injdim}_T T - d = \dim T - d \quad ,$$

which finishes the proof of the proposition.

Remarks: (a) The notation $\omega_{S/T}$ for the "relative dualizing module" follows the general (mis-)usage in the commutative case, ω_f would be more accurate.

(b) The proposition implies that $f_*\omega_{S/T}$ is a T-module of finite projective dimension, equal to d . More precisely, choosing a projective resolution $\mathbf{P} \longrightarrow f_*S$ of f_*S over T of (minimal) length d , the object $f_*f^!T[d]$ in $D(\text{Mod-}T)$ is represented by $\operatorname{Hom}^{\cdot}_T(\mathbf{P},T)[d]$, which is then a projective resolution of $f_*\omega_{S/T}$.

This yields the existence of a "projection formula" (see also [Kl]): If N is any complex of T-modules in $D^-(\text{Mod-}T)$, then

$$f_*f^!(N) = N \overset{\mathbb{L}}{\otimes}_T f_*f^!T = (N \overset{\mathbb{L}}{\otimes}_T f_*\omega_{S/T})[-d] \quad \text{in} \quad D^-(\text{Mod-}T) \ ,$$

and this object is represented by the complex

$$f_*f^!(N) = \operatorname{Hom}^{\cdot}_T(\mathbf{P},N) = N \otimes_T \operatorname{Hom}^{\cdot}_T(\mathbf{P},T) \quad .$$

In particular, if ω_T is any dualizing module for T , then

$$\omega_S =_{\text{def}} f^!\omega_T[d]$$

represents a dualizing module for S , satisfying

$$f_*\omega_S = \omega_T \otimes_T f_*\omega_{S/T} \quad .$$

(c) As for examples of such duality morphisms, tighten up (7.6.2.) by imposing - in addition to (i) and (ii) - either of the following :

(iii_f) S is a (central, finite) <u>flat</u> T-algebra with respect to f , (in which case f_*S is in fact finitely generated projective over T , whence $d = 0$) ,

<u>or</u>

(iii_c) S is also commutative.

It is then well-known, that the conditions (i) and (ii) of (7.6.2.) together with either (iii_f) or (iii_c) imply the remaining condition.

Having sorted out the above class of "duality morphisms", we need furthermore to recall the following facts from local duality theory for commutative Gorenstein rings.

Let (R,m,k) be a commutative local Gorenstein ring of Krull dimension r, m denoting the unique maximal ideal of R, $k = R/m$ the residue class field.

If $i : k \longrightarrow E$ denotes an injective envelope of k over R, there exists an injective resolution of R as a module over itself which is of (minimal) length r and exhibits E as its last term.

This defines a canonical class γ in $\mathrm{Ext}_R^r(E,R)$ which has the following property :

If A is any R-module of <u>finite length</u>, the (Yoneda-)product (from the left) with γ provides an isomorphism

$$\gamma\circ - : \mathrm{Hom}_R(A,E) \xrightarrow{\sim} \mathrm{Ext}_R^r(A,R) \quad ,$$

and for all $i \neq r$ one has $\mathrm{Ext}_R^i(A,R) = 0$. In other words, the product with γ yields an isomorphism in $D(\text{Mod-}S)$ between the single R-module $\mathrm{Hom}_R(A,E)$ and the complex $\mathrm{RHom}_R(A,R)[r]$.

In particular, as $\mathrm{Hom}_R(k,E)$ is isomorphic to k itself, it follows by induction on the length of A that A, $\mathrm{Hom}_R(A,E)$ and $\mathrm{Ext}_R^r(A,R)$ are all R-modules of the same finite length.

Composing the inverse of the isomorphism above with the evaluation-map

$$ev_A : \mathrm{Hom}_R(A,E) \otimes_R A \longrightarrow E \quad ,$$

one obtains (equivalently) a non-degenerate R-bilinear pairing

$$\langle\ ,\ \rangle_A = ev_A \circ ([\gamma\circ -]^{-1} \otimes_R id_A) : \mathrm{Ext}_R^r(A,R) \otimes_R A \longrightarrow E \quad ,$$

(from which γ can be recovered as $\langle\gamma\lambda, a\rangle = \lambda(a)$ for all a in A, λ in $\mathrm{Hom}_R(A,E)$).

The foregoing takes on a slightly simpler form in case that R contains a copy of its residue class field, hence if it is in fact an augmented k-algebra. Then E can be chosen to be - and is always isomorphic to - the R-module $\mathrm{Hom}_k(R,k)$, so that $i : k \longrightarrow E$ equals the k-dual of the projection $R \longrightarrow k$.

It follows that $\mathrm{Hom}_R(A,E) \cong \mathrm{Hom}_k(A,k)$ as R-modules and that the bilinear pairing $\langle\ ,\ \rangle_A$ takes its values already in k.

These statements extend trivially to the affine case in the following sense.

Assume given again an arbitrary commutative Gorenstein ring T of finite Krull dimension t.

If S is some finite subset of $\operatorname{maxspec}(T)$, the set of all maximal ideals of T, the module $T/I(S)$ is semi-simple, namely by definition isomorphic to the product of all the residue class fields $k(m) = T/m$, m ranging over all the maximal ideals in S.

An injective envelope of this module, denoted E_S, is then the product of injective envelopes of the fields - and T-modules - $k(m)$.

If now A is a T-module of finite length, it is supported at finitely many maximal primes, and the T-modules $\operatorname{Hom}_T(A,E_S)$, obtained from finite subsets of $\operatorname{maxspec}(T)$ which contain these finitely many maximal ideals, are all isomorphic.

Accordingly, we will denote - ambiguously - any of these modules by A'.

The local theory then yields for any dualizing T-module ω_T the existence of a natural T-linear isomorphism

$$A' \xrightarrow{\sim} \operatorname{Ext}^t_T(A,\omega_T) \qquad , \tag{7.6.4.}$$

and the T-modules A, A' and $\operatorname{Ext}^t_T(A,\omega_T)$ are all of the same finite length.

7.7. Now we can finally formulate the application of the Duality Theorem (7.5.1.) which seems to be most useful.

For this, assume given :

- $f : T \longrightarrow S$ a duality morphism of finite type as in (7.6.2.), of "virtual codimension" $d = \operatorname{projdim}_T f_*S$,
- ω_T a dualizing T-module, $\omega_S = f^!\omega_T[d]$ the associated dualizing module over S as in Remark (b) above,
- M a complex of left S-modules in $D^b(S^{op})$ as in (7.5.1.),
- N a S-bimodule which is maximal Cohen-Macaulay as either left or right S-module, considered as a T-module by restriction of scalars along f.

Then, summarizing the foregoing discussion and substituting these data into (7.5.1.), we get :

<u>Proposition 7.7.1.</u>:

(i) Under the assumptions just made, the spectral sequence $"E^{i,j}$ for M, N and $W = \omega_T$ collapses at the E_2-level, leaving a single spectral sequence

$$E_2^{i,j} = \mathrm{Ext}_T^i(\underline{\mathrm{Ext}}_{S^{op}}^{-j}(M,N),\omega_T) \implies \underline{\mathrm{Ext}}_S^{i+j-d-1}(M^*,\mathrm{Hom}_S(N,\omega_S)) .$$

(ii) Applying to the E_2-terms of this spectral sequence the trivial duality induced by $\mathrm{Hom}_{S^{op}}(-,S^{op})$ - (4.6.1.) and (6.2.5.(2)) - it becomes

$$^*E_2^{i,j} = \mathrm{Ext}_T^i(\underline{\mathrm{Ext}}_S^{-j}(N^*,M^*),\omega_T) \implies \underline{\mathrm{Ext}}_S^{i+j-d-1}(M^*,\mathrm{Hom}_S(N,\omega_S)) .$$

<u>Proof</u>: (i) Just observe that under the assumptions made,

$\mathrm{Ext}_T^i(f_*N,\omega_T) = \mathrm{Ext}_S^i(N,f^!\omega_T)$ by definition of $(f_*, f^!)$

$= \mathrm{Ext}_S^{i-d}(N,\omega_S)$ by definition of ω_S and (7.6.3.), Remark (7.6.(b)),

$= 0$ for $i \neq d$, N being MCM as a right S-module .

(ii) As N is by assumption also MCM as a <u>left</u> S-module, (6.2.5.(3)) applies. (Remark that N finitely generated would be enough - at the expense of interpreting N^* again as $\mathrm{RHom}_{S^{op}}(N,S^{op})$.)

In view of the local duality theory for commutative rings which we just recalled, the spectral sequences of the proposition degenerate if the Tate-cohomology groups involved are of finite length.

For convenience (only), we will replace in the following M by M^* so that M will henceforth represent a complex of <u>right</u> S-modules in $D^b(S)$.

Furthermore, if N is any S-bimodule, let us denote :

$^*N = \mathrm{Hom}_{S^{op}}(N,S^{op})$, the <u>left</u> S-dual of N ,

and

$N^* = \mathrm{Hom}_S(N,S)$, the <u>right</u> S-dual of N .

Both these modules are naturally S-bimodules again, but only their underlying right module structure will be used in the sequel.

Also, the following terminology will shorten formulations :

<u>Definition 7.7.2.</u>: Let $f : T \longrightarrow S$ be a homomorphism of rings satisfying (7.6.2.) .

(i) Two complexes of S-modules, Y in $D^-(\text{Mod-}S)$ and X in $D^b(S)$, are (essentially) <u>stably transversal</u> (with respect to f), iff for any integer j , the T-module $\underline{\mathrm{Ext}}_S^j(X,Y)$ is of finite length.

(ii) The ring homomorphism f will be called <u>stably regular outside</u> S for a subset S of Spec(T) iff for all complexes X and Y as above, the groups $\underline{\mathrm{Ext}}^{\cdot}_{S}(X,Y)$ are supported on S.

Accordingly, f is <u>stably regular</u> at a prime $\mathbf{p}$ in Spec(T) iff the localizations $\underline{\mathrm{Ext}}^{\cdot}_{S}(X,Y)_{\mathbf{p}}$ vanish for all X and Y.

<u>Definition 7.7.3.</u>: Let S be a ring which is strongly Gorenstein. Then S has only <u>isolated singularities</u> if there exists a homomorphism $f : T \longrightarrow S$ which satisfies (7.6.2.) and is stably regular outside a finite set of maximal ideals of T.

In this case, E_f will represent any injective envelope over T of $T/I(S)$, S some finite subset of maxspec(T) off which f is stably regular.

<u>Remark</u>: The terminology should be justified by the following elementary observation (cf. also [Aus 2]) :

If $f : T \longrightarrow S$ is a homomorphism as in (7.6.2.), let $\mathbf{p}$ be a prime ideal in T. Then $S_{\mathbf{p}} = S \otimes_T T_{\mathbf{p}}$ is a flat S-module (on both sides) and localization in $\mathbf{p}$ defines trivially an exact functor from $D(S)$ into $D(S_{\mathbf{p}})$. It is left as an exercise to establish that with S also $S_{\mathbf{p}}$ is strongly Gorenstein and that Tate-cohomology localizes :

$$\underline{\mathrm{Ext}}^{\cdot}_{S}(X,Y) \otimes_T T_{\mathbf{p}} = \underline{\mathrm{Ext}}^{\cdot}_{S_{\mathbf{p}}}(X_{\mathbf{p}},Y_{\mathbf{p}}) .$$

In particular, f is stably regular at $\mathbf{p}$ if $S_{\mathbf{p}}$ is a ring of finite global dimension.

If S is furthermore commutative, also the converse holds : Let $\mathbf{q}$ be a prime in S lying over $\mathbf{p}$ in T. If $S_{\mathbf{q}}$ is not regular, consider the cyclic S-module $S/\mathbf{q}$ whose localization at $\mathbf{q}$ is the residue class field $k(\mathbf{q})$ of $S_{\mathbf{q}}$. Localizing subsequently in $\mathbf{p}$ and $\mathbf{q}$ yields maps

$$\underline{\mathrm{Hom}}_{S}(S/\mathbf{q},S/\mathbf{q}) \longrightarrow \underline{\mathrm{Hom}}_{S}(S/\mathbf{q},S/\mathbf{q}) \otimes_T T_{\mathbf{p}} \longrightarrow \underline{\mathrm{Hom}}_{S_{\mathbf{q}}}(k(\mathbf{q}),k(\mathbf{q}))$$

under which the identity on $S/\mathbf{q}$ is mapped onto the identity on $k(\mathbf{q})$.

As $k(\mathbf{q})$ is not of finite projective dimension, it represents not a zero-object in $\underline{\text{mod-}S_{\mathbf{q}}}$ by (2.1.). Consequently, the localization of $\underline{\mathrm{Hom}}_{S}(S/\mathbf{q},S/\mathbf{q})$ in $\mathbf{p}$ is not zero and hence f is not stably regular at this prime of T.

It seems worth-while to resume this as a separate

Lemma 7.7.4.: Let S be a commutative Gorenstein ring of finite Krull dimension. Then for a prime $\mathbf{p}$ in $\mathrm{Spec}(S)$ the groups $\underline{\mathrm{Ext}}_S^{\cdot}(X,Y)_{\mathbf{p}}$ are zero for all (complexes of) S-modules Y (in $D^-(\text{Mod-}S)$) and X (in $D^b(S)$) if and only if $S_{\mathbf{p}}$ is a regular local ring.

Coming back to the promised application of the Duality Theorem, let us recall that $t = \dim T$ and $s = \mathrm{vdim}\, S = t - d$ by (7.6.3.(iii)). Then we have :

Theorem 7.7.5. (The Duality Theorem for isolated singularities) : Assume again given a homomorphism of rings $f : T \longrightarrow S$ which satisfies (7.6.2.), ω_T and ω_S dualizing modules over T and S respectively as in (7.7.1.), M a complex of S-modules in $D^b(S)$ and N a S-bimodule which is MCM on both sides.

(i) If M and the underlying right S-module of N are stably transversal with respect to f, for any integer j, there exists a natural isomorphism of T-modules

$$\gamma^j_{M,N} : \underline{\mathrm{Ext}}_S^{-j}({}^*N,M)' \xrightarrow{\sim} \underline{\mathrm{Ext}}_S^{j+s-1}(M,\omega_S \otimes_S N^*) .$$

(ii) Let L be a second complex in $D^b(S)$, also stably transversal to the underlying right S-module of N. If then α is a class in $\underline{\mathrm{Ext}}_S^k(M,L)$, one has a commutative diagram of T-linear maps

$$\begin{array}{ccc} \underline{\mathrm{Ext}}_S^{-j}({}^*N,L)' & \xrightarrow[\sim]{\gamma^j_{L,N}} & \underline{\mathrm{Ext}}_S^{j+s-1}(L,\omega_S \otimes_S N^*) \\ {\scriptstyle (\alpha\circ -)'}\downarrow & & \downarrow{\scriptstyle (-\circ\alpha)} \\ \underline{\mathrm{Ext}}_S^{-k-j}({}^*N,M)' & \xrightarrow[\sim]{\gamma^{k+j}_{M,N}} & \underline{\mathrm{Ext}}_S^{k+j+s-1}(M,\omega_S \otimes_S N^*) \end{array} ,$$

where the vertical maps are given by the Yoneda-products with α.

(iii) Assume that f is stably regular outside a finite set of maximal ideals of T - so that in particular S has only isolated singularities. Then the inverse of $\gamma^j_{M,N}$ yields a non-degenerate T-bilinear pairing

$$_M\langle\ ,\ \rangle^j_N : \underline{\mathrm{Ext}}_S^{j+s-1}(M,\omega_S \otimes_S N^*) \otimes_T \underline{\mathrm{Ext}}_S^{-j}({}^*N,M) \longrightarrow E_f ,$$

associative in the sense that for any class α in $\underline{\mathrm{Ext}}_S^k(M,L)$ as above, x in $\underline{\mathrm{Ext}}_S^{j+s-1}(L,\omega_S \otimes_S N^*)$ and y in $\underline{\mathrm{Ext}}_S^{-k-j}({}^*N,M)$:

$$_M\langle x\circ\alpha, y\rangle^{k+j}_N = {}_L\langle x,\alpha\circ y\rangle^j_N .$$

(iv) Restricting (iii) to the special case $j = 0$ and $M = {}^*N$, it follows for any S-bimodule N, which is MCM on both sides, the existence of a T-linear <u>trace-map</u> :

$$\tau_N = {}_{*N}\langle\ ,id_{*N}\rangle_N^0 : \underline{Ext}_S^{s-1}({}^*N,\omega_S \otimes_S N^*) \longrightarrow E_f \quad ,$$

such that in general the pairing in (iii) can be obtained by composing this trace with the Yoneda-product :

$$_M\langle x,y\rangle_N^j = \tau_N(x \circ y)$$

for all x in $\underline{Ext}_S^{j+s-1}(M,\omega_S \otimes_S N^*)$ and y in $\underline{Ext}_S^{-j}({}^*N,M)$.

<u>The proof</u> consists just of a reformulation of (7.7.1.) in these special circumstances.

Consider the spectral sequence ${}^*E_2^{i,j} \Longrightarrow E^{i+j}$ of (7.7.1.(ii)) for the arguments M^* (instead of M), N and ω_T, ω_S as stated. Then ${}^*E_2^{i,j} = Ext_T^i(\underline{Ext}_S^{-j}({}^*N,M),\omega_T)$, as $M = M^{**}$ in $D^b(S)$.

M being stably transversal to ${}^*N$ with respect to f by assumption, the local duality theory shows that ${}^*E_2^{i,j} = 0$ for $i \neq t$ and that $\underline{Ext}_S^{-j}({}^*N,M)'$ is naturally isomorphic to ${}^*E_2^{t,j}$.

Consequently, the spectral sequence degenerates at the E_2-level, the edge homomorphisms $e_2^{t,j} : {}^*E_2^{t,j} \longrightarrow E^{t+j}$ are defined and yield isomorphisms of T-modules

$$\underline{Ext}_S^{-j}({}^*N,M)' \xrightarrow[\sim]{} {}^*E_2^{t,j} \xrightarrow[\sim]{e_2^{t,j}} E^{t+j} \quad .$$

Furthermore, N being MCM as a right S-module, there is a natural isomorphism - of S-bimodules even - from $\omega_S \otimes_S N^*$ into $Hom_S(N,\omega_S)$, whose inverse induces an isomorphism of left S-modules

$$E^{t+j} = \underline{Ext}_S^{t+j-d-1}(M,Hom_S(N,\omega_S)) \xrightarrow{\sim} \underline{Ext}_S^{j+s-1}(M,\omega_S \otimes_S N^*)$$

in view of $s = t - d$. Composing these isomorphisms yields the desired maps $\gamma_{M,N}^j$ and proves (i) .

Assertion (ii) then follows from (7.5.2.) by applying the duality $Hom_{S^{op}}(-,S^{op})$ to the contravariant arguments of $E_2^{\cdot,\cdot}$, taking into account the behaviour of Yoneda-products with respect to this duality, (7.1.2.) .

Given (i) and (ii), (iii) and (iv) are well-known and easily established consequences.

Remark: The trace-maps exhibited in (iv) should be of paramount importance - and hard to come by.

This will be supported by (   ) which indicates furthermore that an "explicit" determination of these traces may require a "calculus of residues" as in algebraic geometry.

7.8. Noether Different and Support of Tate-Cohomology

Before treating concrete examples, we will give another result on the "universal" support of Tate-cohomology, supplementing in the commutative case Lemma (7.7.4.). Informally stated, the result is :

Assume that $f : T \longrightarrow S$ is a ring homomorphism from a commutative ring of finite Krull dimension and of finite global dimension into the centre of a ring S which is strongly Gorenstein such that f_*S , the underlying T-module of S , is finitely generated projective. Then :

The Noether different of f annihilates all the Tate-cohomology groups $\hat{\mathrm{Ext}}_S(-,-)$ over S .

To make this statement precise, we first need to define the Noether different in the general non-commutative setting, naturally extending the usual notion in the commutative case as introduced by E. Noether and investigated more closely by R. Berger, [Be]. Such extensions have been studied by several authors, see [Fos] for further references.

To begin with, assume given an arbitrary commutative ring A and a ring homomorphism $g : A \longrightarrow B$, B an associative ring with unit as usual, such that B becomes a (central) A-algebra with respect to g .

Then, recall that $B^e_A = B \otimes_A B^{op}$ is the enveloping algebra of g - or, less precisely of B over A .

Deviating from our general convention, we will only deal with left modules over enveloping algebras.

In particular, B becomes a left B^e_A-module in the usual way by

$$\lambda : B^e_A \otimes_A B \longrightarrow B \quad , \quad \lambda(b'\otimes b''\otimes b) = b'bb'' \quad .$$

The deduced map

$$\mu : B^e_A \longrightarrow B \qquad , \quad \mu(b'\otimes b'') = \lambda(b'\otimes b''\otimes 1) = b'b''$$

is then a homomorphism of left B^e_A-modules, also called the natural left augmentation of B^e_A .

The kernel of μ is denoted $I_{B/A}$ and accordingly called the (left) <u>augmentation ideal</u> of B_A^e.

It represents naturally the two-sided A-linear derivations on B, [ALG III.132], and is generated as a left B_A^e-ideal - or even as a left B-module - by the images of the universal derivation $d_{B/A} : B \longrightarrow I_{B/A}$ of B over A, given by $d_{B/A}(b) = b\otimes 1 - 1\otimes b$.

It follows that the <u>right annihilator</u> $\mathrm{ann}(I_{B/A})_{B_A^e}$ <u>is mapped by</u> μ <u>into the centre</u> $Z(B)$ <u>of B</u> :

An element w in B_A^e annihilates the left ideal $I_{B/A}$ from the right if and only if $d_{B/A}(b).w = 0$ for all b in B. Hence

$$0 = \mu(d_{B/A}(b)w) = b.\mu(w) - \mu(w).b$$

for all b as required.

<u>Definition 7.8.1.</u>[†)]: The (classical) <u>Noether different</u> $\vartheta_N^0(g)$ of the A-algebra $g : A \longrightarrow B$ is that ideal of the centre $Z(B)$ of B which is given by

$$\vartheta_N^0(g) = \mu(\mathrm{ann}(I_{B/A})_{B_A^e})$$

For our purposes, it is more convenient to interpret the Noether different in homological terms :

- $Z(B)$ may be identified with $\mathrm{Hom}_{B_A^e}(B,B)$, the ring of B-bimodule endomorphisms of B, by associating to such an endomorphism Θ the element $\Theta(1)$.
- $\mathrm{ann}(I_{B/A})_{B_A^e}$ can be identified with $\mathrm{Hom}_{B_A^e}(B,B_A^e)$ by associating to a B-bimodule homomorphism Θ from B into B_A^e again the element $\Theta(1)$. Conversely, an element w in the right annihilator of $I_{B/A}$ defines the left B_A^e-homomorphism $\Theta_w(b) = (b\otimes 1)w = (1\otimes b)w$.
- $Z(B) = \mathrm{Hom}_{B_A^e}(B,B)$ acts naturally from the right on $\mathrm{Hom}_{B_A^e}(B,-)$ by the "Yoneda-product" so that the induced map $\mathrm{Hom}_{B^e}(B,\mu)$ is naturally $Z(B)$-linear on the right.
  Furthermore, evaluation in the unit of B identifies this map with the restriction of μ to $\mathrm{ann}(I_{B/A})_{B_A^e}$:

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†) This definition is E. Noether's original one in the commutative case, extended by M. Auslander- O. Goldman to the non-commutative case. The following homological description seems due to D.G. Higman; see once again [Be] and [Fos].

$$\begin{array}{ccccc}
\mathrm{Hom}_{B_A^e}(B,\mu) & : & \mathrm{Hom}_{B_A^e}(B,B_A^e) & \longrightarrow & \mathrm{Hom}_{B_A^e}(B,B) \\
 & & \downarrow\simeq & & \downarrow\simeq \\
\mu|\,\mathrm{ann}(I_{B/A})_{B_A^e} & : & \mathrm{ann}(I_{B/A})_{B_A^e} & \longrightarrow & Z(B)
\end{array}$$

From this description of the Noether different one has immediately

<u>Proposition 7.8.2.</u>: Let $g : A \longrightarrow B$ be an A-algebra as above. Then $\vartheta_N^0(g) = Z(B)$ if and only if B is projective as a left module over the enveloping algebra B_A^e.

If B is commutative and g_*B, the underlying A-module of B, is projective, then $\vartheta_N^0(g) = B$ holds if and only if $g : A \longrightarrow B$ is an <u>étale</u> A-algebra.

<u>Proof</u>: By the above, $\vartheta_N^0(g) = Z(B)$ iff $\mathrm{Hom}_{B_A^e}(B,\mu)$ is surjective. This happens iff the short exact sequence of left B_A^e-modules

$$0 \longrightarrow I_{B/A} \longrightarrow B_A^e \xrightarrow{\mu} B \longrightarrow 0$$

splits, hence if and only if B is B_A^e-projective as a left module.

The characterization of a (commutative) finite and ètale algebra by its Noether different then just restates [EGA IV;18.3.1.(ii)] .

The significance of Noether differents for stable homological algebra - hence in particular for Tate-cohomology - is due to the following, elementary observation which "derives" in a straightforward manner the treatment of bimodules as given by H. Bass in [Ba;II.2] :

Let X be a bounded above complex of B-bimodules whose underlying complex of A-bimodules is "symmetric" in the sense that its left and right module structures over the commutative ring A are the "same".

Then X corresponds bi-uniquely to a complex of left B_A^e-modules and such complexes represent hence "all" the objects of the derived category $D^-(B_A^e\text{-Mod})$ as defined in (7.4.).

On the other hand, taking tensor-products with respect to the underlying <u>left</u> B-module structure of X and remembering the <u>right</u> B-module structure of X afterwards, defines an <u>exact</u> functor

$$\mathbf{h}(-,X) = - \overset{\mathbb{L}}{\otimes}_B X : D^-(\text{Mod-}B) \longrightarrow D^-(\text{Mod-}B)$$

This functor commutes canonically with translation on $D^-(\text{Mod-}B)$ by [ALG X.61;Rem.(2)] and a quasi-isomorphism of complexes of B-bimodules yields a natural equivalence of functors.

Leaving the pain of defining properly the "category of additive and exact endo-functors on $D^-(\text{Mod-}B)$" to the reader, $\mathbf{h}(-,-)$ can hence be interpreted as a functor from $D^-(B^e_A\text{-Mod})$ into that category.

Look at the following examples :

- If $X = B$, considered as a (complex of) left B^e_A-module(s) , $\mathbf{h}(-,B)$ is isomorphic to the identity-functor on $D^-(\text{Mod-}B)$.
- If $X = B^e_A$ considered as a (complex of) left B^e_A-module(s), the associativity formula for the (derived) tensor-product
$$- \overset{\mathbb{L}}{\otimes}_B (B \overset{\mathbb{L}}{\otimes}_A B^{op}) \cong - \overset{\mathbb{L}}{\otimes}_A B^{op}$$
shows that
$$\mathbf{h}(-,B^e_A) \cong g^*g_* : D^-(\text{Mod-}B) \longrightarrow D^-(\text{Mod-}A) \longrightarrow D^-(\text{Mod-}B) \quad ,$$
as soon as the natural "augmentation" onto the 0^{th}-homology :
$$B \overset{\mathbb{L}}{\otimes}_A B^{op} \longrightarrow H^0(B \overset{\mathbb{L}}{\otimes}_A B^{op}) = B \otimes_A B^{op} = B^e_A$$
is a quasi-isomorphism, which is equivalent to
$$H^{-i}(B \overset{\mathbb{L}}{\otimes}_A B^{op}) = \mathrm{Tor}^A_i(B,B) = 0 \quad \text{for all} \quad i \neq 0 \quad .$$
(Remark that $B = B^{op}$ as A-modules, $g(A)$ being central in B .)
- The morphism μ of left B^e_A-modules yields a morphism of functors $\mathbf{h}(-,\mu)$, and if $\mathrm{Tor}^A_i(B,B) = 0$ for $i \neq 0$, it becomes identified with the adjunction co-unit $g^*g_* \xrightarrow{\varepsilon} \mathrm{id}$.
In case that B is projective as an A-module, one may forget about deriving tensor-products and the identification just states the obvious :
$$\begin{aligned} x(b'b'') = (1_X\otimes\mu)(x\otimes_B b'\otimes_A b'') &= (1_X\otimes\mu)(xb'\otimes_B 1\otimes_A b'') \\ &= \varepsilon(X)(xb'\otimes_A b'') = (xb')b'' \end{aligned}$$
for all elements x in X and b',b'' in B .

- Finally, if z is an element of the centre of B , considered as a B-bimodule endomorphism of B as above, $\mathbf{h}(-,z)$ yields a natural transformation of the identity-functor on $D^-(\text{Mod-}B)$ - and, generalizing [Ba;II.2.1.Prop.], one may prove that all such transformations are obtained this way.

Consequently, if z is already an element of the Noether different $\vartheta_N^0(g)$, obtained as $\mu \circ w$ for some $w : B \longrightarrow B_A^e$, we may interpret these data as natural transformations of functors :

$$\begin{tikzcd} h(-,z) : h(-,B) \arrow[r, "h(-{,}w)"] \arrow[d, "\simeq"] & h(-,B_A^e) \arrow[r, "h(-{,}\mu)"] \arrow[d, "\simeq"] & h(-,B) \arrow[d, "\simeq"] \\ \rho(z) : id_{D^-(\mathrm{Mod}\text{-}B)} \arrow[r] & g^*g_* \arrow[r, "\varepsilon"] & id_{D^-(\mathrm{Mod}\text{-}B)} \end{tikzcd},$$

where $\rho(z)$ denotes "right multiplication with z ", regarded as a natural transformation of the identity-functor on $D^-(\mathrm{Mod}\text{-}B)$.

Hence, <u>if</u> :

- B is noetherian and g_*B, the underlying A-module, is finitely generated projective (which ensures in particular $\mathrm{Tor}_i^A(B,B) = 0$ for $i \neq 0$),
- A is a ring of finite global dimension, and
- X is a complex in $D^b(B)$, - see (0.2.) for notation - ,

<u>then</u> :

- multiplication with an element z in the Noether different $\vartheta_N^0(g)$ from the right on X factors through a <u>perfect</u> complex, namely g^*g_*X .

Specializing to the case of Gorenstein rings, this yields the result announced :

<u>Theorem 7.8.3.</u>: Let $f : T \longrightarrow S$ be a homomorphism of rings where

- S is strongly Gorenstein and f_*S is finitely generated projective,
- T is a commutative ring of finite Krull dimension such that all localizations $T_{\mathbf{p}}$, $\mathbf{p}$ a prime in $\mathrm{Spec}(T)$, are regular local rings,
- and, as usual, $f(T)$ is contained in the centre $\mathbf{Z}(S)$ of S .

Then the natural action of $\mathbf{Z}(S)$ on $\underline{\mathrm{Ext}}_S^{\cdot}(X,Y)$ annihilates the Noether different $\vartheta_N^0(f)$ for all Y in $D^-(\mathrm{Mod}\text{-}S)$ and X in $D^b(S)$.

<u>For the proof</u>, simply remark that the hypotheses on T guarantee that its global dimension is finite.

<u>Remark</u>: Classically, in the commutative case, one rather studies the <u>Kähler or Dedekind different</u>, denoted here $\vartheta_K(-)$ or $\vartheta_D(-)$ resp. .

The relations for a commutative ring S and a morphism $f : T \longrightarrow S$ as in the Theorem are the following :

(For the definitions and properties of $\vartheta_{K/D}(f)$ the reader may consult [Be] or [H-K] and the literature cited there.)

(i) If $I_{S/T}$ is finitely generated as a left S^e_T-ideal, the Noether different $\vartheta^0_N(f)$ contains the Kähler different $\vartheta_K(f)$, [Be;II. Satz 4].

(ii) If $I_{S/T}/(I_{S/T})^2$ is finitely generated as a S-module, say by m elements, then the Kähler different contains the m-th power of the Noether different, [Be;II.Satz 3].

(iii) If f is generically unramified in the sense that for K the field of fractions of T the extension $f \otimes_T K : K \longrightarrow S \otimes_T K$ is separable, then Noether and Dedekind different of f coincide, [Be;III.Satz 7].

(iv) If f is a complete intersection morphism in the sense that in addition to f being finite and flat all (geometric) fibres are - necessarily artinian - complete intersection algebras, then all three differents are the same; E. Kunz, [Ku].

The properties (i) - (iii) do not depend on S being Gorenstein, the morphism f being finite and flat - which implies S Cohen-Macaulay - is enough. (In (iv), the Gorenstein property is implied by the other assumptions.) A special property of Gorenstein rings is nevertheless :

(v) The Dedekind different is a principal ideal in S, [H-K;7.25].

Does (v) still hold for the Noether different of a not necessarily commutative but still strongly Gorenstein ring S ?

Another remark concerns the tensor-product used to define the enveloping algebra. It should depend on the category of algebras one is working with and not just on the ring theoretic properties.

For example, treating analytic commutative local algebras over some field, hence quotients of rings of convergent or formal power series, the tensor-product should be completed to obtain the "correct" analytic different.

Similarly, in the graded commutative case, the tensor-product could be replaced by the graded tensor-product (with respect to the given commutation factor), see [ALG III.47], to obtain a graded version of the different.

All this shows that Noether differents are rather a general homological tool not necessarily bound to the theory of Gorenstein rings.

We conclude this section with three, fairly well-known examples and an application to (commutative) complete intersections.

The notations will be as before, $g : A \longrightarrow B$ denoting an A-algebra with enveloping algebra B_A^e .

Example 7.8.4.: (Group algebras)

Let A be an arbitrary commutative ring and G a finite group. Set $B = A[G]$, the group algebra of G over A , and let $g : A \longrightarrow A[G]$ be the canonical map.

Then $g_* B$ is freely generated over A by the elements of G and the ideal $I_{B/A}$ in B_A^e is generated by the elements

$$d_{B/A}(h) = h\otimes 1 - 1\otimes h \quad , \quad h \text{ any element in } G .$$

It follows that the element

$$w = \sum_{g \in G} g\otimes g^{-1}$$

annihilates $I_{B/A}$ from the right :

$$d_{B/A}(h)w = \sum_{g \in G} hg\otimes g^{-1} - g\otimes g^{-1}h = 0$$

and hence, as

$$\mu(w) = \sum_{g \in G} 1 = (G:1) \qquad ,$$

the order of G is an element of the Noether different. (Compare for example [C-E;XII.2.2.-2.5.] .)

Example 7.8.5.: (Simple extensions)

Let A again be arbitrary commutative, x a variable and $f(x)$ a polynomial with coefficients in A , $B = A[x]/(f(x))$ the simple extension defined by $f(x)$.

Then B_A^e is isomorphic to $A[x,y]/(f(x),f(y))$, y a second variable, and $\mu: B_A^e \longrightarrow B$ becomes the substitution " $x = y$ " .

Accordingly, $x - y$ generates $I_{B/A}$ and one may write

$$f(x) - f(y) = (x - y)g(x,y) \quad \text{for some } g(x,y) \text{ in } A[x,y] .$$

Evaluating this equation in the quotient ring B_A^e , it follows that the class of the "difference quotient" $g(x,y)$ in B_A^e annihilates the ideal $I_{B/A}$, whence

$$\mu(g(x,y)) = g(x,x) = f'(x)$$

shows that the derivative $f'(x)$ belongs to the Noether different of B over A . (In fact, it generates the Noether different,[Be;II.1.Kor.3] .)

Example 7.8.6.: (Exterior Algebras)

Let A be a commutative ring, V a free A-module of rank n. Consider $B = \Lambda_A V$, the exterior algebra of V over A and the natural map $g : A \longrightarrow B$.

If $v_1, \ldots, v_n$ constitutes an A-basis of V, the ideal $I_{B/A}$ in B_A^e is generated by the elements

$$d_{B/A}(v_i) = v_i \otimes 1 - 1 \otimes v_i \quad , \quad i = 1, \ldots, n \quad .$$

Then the product

$$w = (v_1 \otimes 1 + 1 \otimes v_1) \ldots (v_n \otimes 1 + 1 \otimes v_n)$$

is certainly in the right annihilator of $I_{B/A}$ and its image under μ,

$$\mu(w) = 2^n v_1 \ldots v_n \quad ,$$

is easily seen to generate the Noether different of g.

If one takes the graded enveloping algebra of B over A instead, which is $B \otimes_A B$ with the multiplication

$$(x_1 \otimes y_1)(x_2 \otimes y_2) = (-1)^{\deg y_1 . \deg x_2} (x_1 x_2 \otimes y_1 y_2)$$

for homogeneous elements x_2, y_1, then one verifies that the corresponding graded Noether different is the zero ideal.

In both examples (7.8.5.&6.), the information obtained is not always optimum :

Take A a field of characteristic two, $B = A[x]/(x^2)$ considered either as a simple extension or the exterior algebra of a free module of rank one. Then B is strongly Gorenstein and Theorem (7.8.3.) applies to the canonical map $g : A \longrightarrow B$. The various Noether differents mentioned above will all be zero but the element x in B annihilates all Tate-cohomology groups over B as follows from (   ).

Apart from this rather trivial example, it follows already from the general theory above that Noether differents are not always sufficient to describe precisely the "universal" support of Tate-cohomology, even for commutative (Gorenstein) algebras S over a field K which allow "many" finite flat morphisms from regular rings T into it :

By (7.8.2.), the Noether different yields the non-étale locus of such a finite flat map, hence the union of all such Noether differents will describe (maximally) the non-smooth locus of S over K whereas the Tate-cohomology groups are supported only on the non-regular locus of S by (7.7.4.) . (For more examples along these lines, see [BEH] .)

Extending (7.8.5.) or directly using Remark (iv) above, we get the following result on the Tate-cohomology of complete intersections in the equi-characteristic case :

Let k be a field, and denote by $P = k\langle x_1,\ldots,x_n\rangle$ either

- the localization of the polynomial ring over k in n variables at the maximal ideal $\mathfrak{m}_P = (x_1,\ldots,x_n)$ or
- the ring of formal (or convergent) power series over a (complete non-trivially valued) field k in n variables.

Assume given a sequence $\underline{f} = (f_1,\ldots,f_m)$ of elements in the unique maximal ideal $\mathfrak{m} = (x_1,\ldots,x_n)$ and let $J(\underline{f})$ denote the corresponding <u>Jacobian ideal</u> generated by all maximal minors of the Jacobian matrix of $\underline{f}$ with respect to the chosen coordinate-functions $(x_1,\ldots,x_n)$, whose entries are hence given by the partial derivatives $\partial f_j/\partial x_i$.

With these notations we have :

<u>Corollary 7.8.7.</u>: If $\underline{f} = (f_1,\ldots,f_m)$ constitutes a <u>regular sequence</u> in $\underline{m} = (x_1,\ldots,x_n)$, the quotient ring $R = P/\underline{f}P$ is a complete intersection ring , hence Gorenstein, of dimension $n - m$.

The Jacobian ideal $J(\underline{f})$ then annihilates all Tate-cohomology groups over R which become consequently modules over the <u>Jacobian ring</u> of R , $\bar{R} = R/J(\underline{f})R$ in a natural way.

<u>For the proof</u> it needs only to be remarked that a composition

$$k\langle x_{i_1},\ldots,x_{i_{n-m}}\rangle \xrightarrow{\text{incl.}} P \xrightarrow{\text{proj.}} R$$

yields a finite flat homomorphism of rings if the corresponding minor

$$\det(\partial f_j/\partial x_i)^{j=1,\ldots,m}_{i\neq i_1,\ldots,i_{n-m}}$$

of the Jacobian matrix is not zero. In this case, the associated ("analytic" !) Kähler different - which equals here the (analytic) Noether different in accordance with Remark (iv) above - is generated by that minor, see [Ku; Sätze 1 & 2] .

8. - *First Examples*

First, let us consider those cases where S is of small virtual dimension.

8.1. Quasi-Frobenius Rings

Let S be an artinian ring which is injective as a right module over itself. Then it is also injective on the left, [C-R;p.135], hence S is strongly Gorenstein of virtual dimension zero.

Such rings are also called quasi-Frobenius - which motivated Heller's definition of a Frobenius category, (4.8.) .

For S quasi-Frobenius, obviously all finitely generated S-modules are maximal Cohen-Macaulay, whence $\underline{D^b(S)} = \underline{MCM(S)} = \underline{mod\text{-}S}$.

Furthermore, complete resolutions are simply obtained by connecting a projective with an injective resolution of the given module.

Let rad(S) denote the radical of S , soc(S) its socle.

Proposition 8.1.1.: (cf. [C-R;6.28ff], for example)
Let S be a ring which is quasi-Frobenius. Then :

(i) soc(S) is a two-sided ideal of S , canonically isomorphic to the S-dual of S/rad(S) .

(ii) A finitely generated right S-module M contains a projective (or injective) direct summand if and only if $M.soc(S) \neq 0$.

This shows, by (2.1.), that the isomorphism classes of objects in $\underline{MCM(S)}$ are in bijective correspondence with the isomorphism classes of finitely generated right $\bar{S} = S/soc(S)$ -modules.

Furthermore, supplementing (7.7.4.) and (7.8.3.) in this case, it follows from the proposition :

Corollary 8.1.2.: For any two (complexes of) S-modules M and N (in $D^b(S)$) over a quasi-Frobenius ring S , the Tate-cohomology groups $\underline{Ext}_S^{\cdot}(M,N)$ are annihilated by all central elements in the socle of S .

In case of quasi-Frobenius rings, the "canonical octahedron" of a module, (5.3.1.), degenerates, the functor **M** becomes the identity and **H** may be chosen to associate to a module its injective envelope.

Hence, in case of quasi-Frobenius rings, the theory developed here reduces essentially to the representation theory of artinian rings which are self-injective.

Before treating more specific (graded) examples of such rings, where Tate-cohomology admits a rather concrete geometric interpretation, we want to mention some more classes of rings which are strongly Gorenstein.

For example

8.2. Integral Representations of Finite Groups

Let G be a finite group, $S = \mathbb{Z}[G]$ its integral group ring. Then S is strongly Gorenstein in the terminology adopted here and its injective dimension equals one. The natural ring homomorphism from $\mathbb{Z}$ into S satisfies the conditions (7.6.2.). It follows that a S-module M - or, equivalently, an integral representation of G - is maximal Cohen-Macaulay over S if and only if the underlying $\mathbb{Z}$-module is MCM over the integers which means nothing but M being $\mathbb{Z}$-free of finite rank in view of the structure theory of modules over principal ideal domains for example.

Hence, one indeed regains the classical theory of complete resolutions as initiated by J. Tate.

The (original) Tate-cohomology of G with values in an integral representation M, usually denoted $\hat{H}^{\cdot}(G,M)$, is accordingly nothing but $\underline{\mathrm{Ext}}^{\cdot}_{\mathbb{Z}[G]}(\mathbb{Z},M)$, $\mathbb{Z}$ considered as a trivial G-module.

Classically, most of the results on Tate-cohomology are deduced using the cocommutative Hopf-algebra structure on $\mathbb{Z}[G]$. To point out the connections, let us make the following remarks :

Assume given two integral representations M and N of G, with M being MCM. Then, using the co-multiplication and inversion on the group ring, $\mathrm{Hom}_{\mathbb{Z}}(M,N)$ becomes in a natural way a (right) $\mathbb{Z}[G]$-module again and one easily gets that

(8.2.1.) $$\hat{H}^{i}(G,\mathrm{Hom}_{\mathbb{Z}}(M,N)) = \underline{\mathrm{Ext}}^{i}_{\mathbb{Z}[G]}(M,N)$$

for all integers i and functorially in both M and N.

Furthermore, the duality theory for Tate-cohomology in this case is usually developed using the cup-products induced from the Hopf-algebra structure. But, by the various uniqueness theorems for homological products, (or here also simply by [ALG X.201,Exerc. 10(c)]), these products are known to be the "same" as the Yoneda-products up to a sign (remember that we deal exclusively with <u>right</u> modules !). It follows then easily

- for example by invoking the various uniqueness theorems once again - that the duality theorems obtained here reduce to the classical statements.

An immediate generalization of integral representation theory which is still covered is obtained by replacing $\mathbb{Z}$ with any Dedekind ring.

The resulting group ring is again strongly Gorenstein of virtual dimension one and maximal Cohen-Macaulay modules over it correspond to lattices.

Finally, as far as group representations are concerned, let us mention the following two cases which display features rather similar to those described here but which are not covered. May be, these constitute a good testing ground for a more general theory of - not yet defined - "Gorenstein categories" :

(8.2.2.) Pro-finite Poincaré Groups

Considering the results on such groups - see [Ser 2] for example - , and in particular the existing duality theory for them, (loc.cit.; app.), it seems reasonable to expect that large parts of the foregoing theory can be extended to cover these groups.

Even more promising, consider

(8.2.3.) Groups of finite virtual cohomological dimension

For these groups, F.T. Farrell introduced a cohomology theory generalizing naturally the Tate-cohomology for finite groups. If the group is furthermore a "virtual duality group" - see [Br;VIII & X] for details - , all the necessary ingredients like complete resolutions and the duality theory are available and developed in complete analogy to the classical case. This indicates that our finiteness assumptions - S noetherian and MCMs being finitely generated - should be largely redundant.

Coming back to the classical case of finite groups some questions even for the theory developed here impose themselves. For example, what are the analogues for strongly Gorenstein rings of more sophisticated results like the theorem of Tate-Nakayama in class field theory and what are the geometric interpretations in case of isolated Gorenstein singularities.

For a simple example along these lines, consider the generalization of "Herbrand quotients" to hypersurface singularities and their suspected geometric meaning outlined in chapter below.

8.3. Higher dimensional Gorenstein Rings

Here we give two criteria which enable one to recognize strongly Gorenstein rings of arbitrarily large virtual dimension.

The first one shows that strongly Gorenstein rings may be obtained by "deforming" those of small virtual dimension :

Proposition 8.3.1.: Assume given a commutative regular local ring T with residue class field k . If then $f : T \longrightarrow S$ is a finite and flat T-algebra - see (7.6.) - such that the ring $S_0 = S \otimes_T k$, the "geometric fibre" of f , is quasi-Frobenius, then S is strongly Gorenstein and its virtual dimension equals $\dim T$.

Furthermore, f satisfies the conditions (7.6.2.) and consequently a finitely generated S-module is maximal Cohen-Macaulay if and only if the underlying T-module is free of finite rank.

The proof, which uses the well-known behaviour of injective dimension with respect to quotients by central non-zero-divisors, is left to the reader.

It follows by the usual arguments that being strongly Gorenstein is a generic property of finite flat maps :

Corollary 8.3.2.: Let T be a commutative ring of finite Krull dimension all of whose localizations (in maximal ideals) are regular local.

If then $f : T \longrightarrow S$ is a finite flat T-algebra, $\mathfrak{m}$ a maximal ideal of T such that the corresponding fibre $S(\mathfrak{m}) = S \otimes_T T/\mathfrak{m}$ is a quasi-Frobenius ring, then there exists a Zariski-open neighbourhood U of $\mathfrak{m}$ in $\mathrm{Spec}(T)$ such that $S_U = S \otimes_T T_U$ is strongly Gorenstein.

In particular, S itself is strongly Gorenstein if and only if all geometric fibres over maximal ideals in T are quasi-Frobenius.

A simple example where the foregoing applies is given by group rings $T[G]$, with G a finite group, T a commutative ring satisfying the hypotheses of the Corollary. Remark that then (7.8.4.) implies the well-known fact that $T[G]$ is of finite global dimension as soon as $(G:1)$, the order of G , is invertible in T .

Similarly, coming back to the example (7.8.6.), if V is a finitely generated projective T-module, the exterior algebra $\Lambda_T^{\cdot}(V)$ of V over T is strongly Gorenstein. If $V \neq 0$ it is not of finite global dimension. See also (8. .) for more details on this example.

The second criterion has been mentioned already in (4.1. Remark). It is due to J.-E. Roos, [Roo] :

<u>Proposition 8.3.3.</u>: Let S be an arbitrary ring endowed with an ascending and exhaustive filtration $(F_i)_{i \geq 0}$ by subgroups such that

- $F_i . F_j$ is contained in F_{i+j} for all i, j and
- the associated graded ring $gr(S)$ is strongly Gorenstein.

Then S itself is strongly Gorenstein and its virtual dimension is bounded above by the virtual dimension of $gr(S)$.

Furthermore, if M is any S-module allowing a filtration compatible with the given filtration on S and such that the associated $gr(S)$-module $gr(M)$ is maximal Cohen-Macaulay, then M is already MCM over S.

<u>The proof</u> follows essentially from the existence of a converging spectral sequence for any two filtered S-modules M and N for which the associated $gr(S)$-module $gr(M)$ is of finite type, starting with the E_1-terms $Ext^{\cdot}_{gr(S)}(gr(M), gr(N))$ and converging towards $Ext^{\cdot}_{S}(M,N)$, (see [Roo] or [Bjö;Ch.2,§3] for details).

This result applies in particular to primitive quotients of enveloping algebras of finite dimensional Lie-algebras over a field in view of the Poincaré-Birkhoff-Witt theorem, cf. [Roo] again.

Unfortunately - for our purposes ! -, "most" of these rings seem to be already of finite global dimension, for example, this holds for the enveloping algebras themselves.

The first example of such a ring which is not of finite global dimension has been given by J.T. Stafford, [Sta;Prop.3.5.] :

$U(sl_2\mathbb{C})/(C+1)$, where $C = H^2 + 2H + 4FE$ is the Casimir element, is of virtual dimension one, [Roo;Cor.2], but of infinite global dimension.

This example raises some interesting questions :
As $C + 1 = (H + 1)^2 + 4FE$, the associated graded ring is the homogeneous coordinate ring of the quadric $z^2 + xy$, where z is the class of H (or $H + 1$), x the class of $2E$ and y the class of $2F$ for example. Hence the associated graded ring has only an isolated singularity of type A_1 at the origin and it is well-known that this ring is of finite MCM-representation type. More precisely, see [BEH], for this ring the category of (ungraded) MCMs modulo projectives is equivalent to the category of finite dimensional $\mathbb{C}$-vectorspaces.

Furthermore, it is not hard to see that the module of infinite projective dimension over the given primitive quotient of $U(sl_2\mathbb{C})$ which was exhibited by J.T. Stafford in (loc.cit.) has as its associated graded module over the quadric precisely the unique indecomposable and non-free MCM over this A_1-singularity.

Hence, is it true that this module is the unique indecomposable and non-free MCM over this primitive quotient ?

Furthermore, given the correspondance between simple surface singularities over the complex numbers and semi-simple Lie-algebras as established by E. Briekorn, - see [Slo] - , and the fact that these are the only normal two-dimensional Gorenstein singularities which are of finite representation type, - see [BGS] - , it would be interesting to know :

- Are there always primitive quotients of the enveloping algebras of the corresponding Lie-algebras whose category of MCMs modulo projectives is equivalent to the corresponding category over the associated simple singularity ?
- Given the "explicit" knowledge of <u>MCM</u> for a simple singularity, see [Knö] , does an affirmative answer to the foregoing question yield a method to interpret these categories directly in terms of representations of the corresponding semi-simple Lie-algebra, adding another facet to the so-called "McKay-correspondance" ?

Apart from this possible connection with simple singularities :

- Is it true in general that primitive quotients of enveloping algebras of (semi-simple) finite dimensional Lie-algebras over a field are always of finite MCM-representation type in the sense that only finitely many distinct isomorphism classes of indecomposable MCMs exist for such rings ? (This could then be interpreted as saying that such rings are never "too far" from being of finite global dimension.)

Much more and natural examples of strongly Gorenstein rings to which (8.3.3.) applies and which are "almost never" of finite global dimension are obtained from <u>graded</u> or <u>Super-Lie-algebras</u>.

Before treating these examples, we will shortly comment on the necessary - and rather obvious - modifications of the theory in

8.4. <u>The graded Case</u>

As already mentioned several times and indicated in (4.8.), most of the foregoing results on MCMs over rings which are strongly Gorenstein should hold in a much broader context. A simple extension, for which all proofs remain literally the same, is furnished by considering graded rings and modules.

Hence assume, in addition to the general hypotheses of being noetherian and of finite injective dimension on both sides, that $S. = \bigoplus_{\delta \in \Delta} S_\delta$ is a Δ-graded ring for some commutative group of degrees $(\Delta,+)$.

Then replace the various categories defined so far for ungraded rings correspondingly by Mod-S. , $(\underline{\text{mod-S.}})$, $(\underline{D^*(S.)})$, ... , constructed out of the category of Δ-graded S.-modules with degree-preserving S.-linear maps as morphisms.

Accordingly, one defines graded complete resolutions, graded perfect complexes and so on, to obtain mutatis mutandis the "same" results for such graded strongly Gorenstein rings, results we will henceforward take for granted.

As usual, to compensate for the lack of sufficiently many morphisms, one introduces shift functors $-(\delta)$ for all elements δ of Δ , which may depend on the choice of a commutation factor ε on $\Delta \times \Delta$, (see [ALG III.47ff]) .

Given then a graded S.-module M. , the grading of the shifted module is given by $M.(\delta)_{\delta'} = M_{\delta+\delta'}$ and its (right) S.-module structure is modified according to

$$m(\delta).s = m.\varepsilon(\sigma,\delta).s \quad \text{in} \quad M.(\delta)_{\mu+\sigma} = M_{\delta+\mu+\sigma}$$

for homogeneous elements m in $M_{\delta+\mu}$ and s in S_σ .

These shift functors do not alter morphisms and preserve obviously graded perfect complexes, complete resolutions and MCMs .

As they also commute with the translation functor on complexes, one can consequently define Δ-graded Tate-cohomology groups by

$$(8.4.1.) \qquad \underline{\mathrm{Ext}}^i_{S.}(X.,Y.)_\delta = H^i(\mathrm{Hom}^{\cdot}_{S.}(\mathbf{CR}(X.),Y.(\delta)))$$

for all integers i , degrees δ in Δ and complexes of graded modules Y. in $D^-(\text{Mod-S.})$, X. in $D^b(S.)$ as in (7.4.1.) .

The Yoneda-products are compatible with the shift functors which may be expressed equivalently by saying that $D^*(\text{Mod-S.})$, $(\underline{D^b(S.)})$ and its equivalent companions are in fact $\mathbb{Z} \times \Delta$-graded categories.

The "multiplicative" structure of these categories depends on the chosen commutation-factor :

The graded endomorphism ring of S. as a graded right module over itself, $\bigoplus_{\delta \in \Delta} \mathrm{Hom}_{S.}(S.,S.(\delta))$, is not S. again in general.

. One certainly has $\mathrm{Hom}_{S.}(S.,S.(\delta)) = S_\delta$ as abelian groups, the identification obtained as usual by evaluating a homomorphism in the unit of the ring $S.$,(which is tacitely assumed to be an element of S_0).

To an element s in S_δ corresponds then the homomorphism h_s , given by

$$h_s(s') = h_s(1).s' = \varepsilon(\delta',\delta)ss' \quad \text{for } s' \text{ in } S_{\delta'} \ ,$$

as h_s is right $S.$-linear and in view of the definition of the module structure on $S.(\delta)$.

It follows that in terms of this identification, the "Yoneda-product" on $S.$, considered as its own endomorphism ring, satisfies

$$s \circ s' = \varepsilon(\delta',\delta)ss' \quad \text{for } s \text{ in } S_\delta \ , \ s' \text{ in } S_{\delta'} \ .$$

Consequently, <u>the graded S.-dual</u> of a graded $S.$-module $M.$, given by $M^*_. = \bigoplus_{\delta\in\Delta} \mathrm{Hom}_{S.}(M.,S.(\delta))$ as a Δ-graded abelian group, becomes a graded <u>left</u> module over this endomorphism ring - or, equivalently, it carries the structure of a graded <u>right</u> module over the graded ε-opposite ring ${}_\varepsilon S^{op}_.$ of $S.$, whose multiplication, say "#" , is determined by

$$s'\#s =_{def} \varepsilon(\delta',\delta)ss' \quad \text{for } s \text{ in } S_\delta \ , \ s' \text{ in } S_{\delta'} \ .$$

Hence all statements concerning duality, like in (4.6.), (6.2.) or chapter 7 , have to be interpreted over this ε-opposite ring ${}_\varepsilon S^{op}_.$. The details are left to the reader, as an example we only mention the extension of (7.1.2.) to the graded case :

For any two classes f in $\underline{\mathrm{Ext}}^i_{S.}(Z.,Y.)_\delta$, g in $\underline{\mathrm{Ext}}^j_{S.}(X.,Z.)_{\delta'}$, one finds

$$\text{(8.4.2.)} \qquad (f\circ g)^* = (-1)^{ij}\varepsilon(\delta',\delta)g^*\circ f^* \quad \text{in } \underline{\mathrm{Ext}}^{i+j}_{{}_\varepsilon S^{op}_.}(Y^*_.,X^*_.)_{\delta+\delta'} \ .$$

If we take $X = Y = Z$, and extend the commutation factor ε on Δ to the commutation factor ε' on $\mathbb{Z}\times\Delta$ by

$$\varepsilon'((i,\delta),(j,\delta')) = (-1)^{ij}\varepsilon(\delta,\delta')$$

for all integers i , j and degrees δ , δ' in Δ , the $\mathbb{Z}\times\Delta$-graded "stabilized Yoneda-Ext-algebra" $\underline{\mathrm{Ext}}^{:}_{S.}(X.,X.).$ is seen to be isomorphic to the graded ε'-opposite algebra of $\underline{\mathrm{Ext}}^{:}_{{}_\varepsilon S^{op}_.}(X^*_.,X^*_.)$ . .

After these brief comments on the necessary modifications for graded rings which are strongly Gorenstein, we can treat the example of

8.5. Graded or Super-Lie-Algebras

Let $(\Delta,+)$ be a commutative group (of degrees), $|.| : \Delta \longrightarrow \mathbb{Z}/2\mathbb{Z}$ a group homomorphism, Δ^+ its kernel, also called the subgroup of even degrees, Δ^- its complement, consisting of all odd degrees.

Given a commutative ring K, a Δ-graded super-K-Lie-algebra consists of a Δ-graded K-module $L. = \bigoplus_{\delta\in\Delta} L_\delta$ endowed with bilinear pairings

$$[\ ,\ ] : L_\delta \times L_{\delta'} \longrightarrow L_{\delta+\delta'} \qquad \text{for all pairs of degrees } (\delta,\delta'),$$

and quadratic maps

$$q : L_\delta \longrightarrow L_{2\delta} \qquad \text{for all odd degrees } \delta \text{ in } \Delta^-,$$

satisfying the usual axioms (considered first by Milnor-Moore and amended by G. Sjödin; see [Av] for details - and replace $|a|$ there by $|\deg a|$ for any homogeneous element a in $L_{\deg a}$).

As always, examples of such Lie-algebras arise from Δ-graded, associative K-algebras A. if one defines the Lie-bracket as

$$[a,b] = ab - (-1)^{|\delta||\delta'|}ba \quad \text{for } a \text{ in } A_\delta \text{ and } b \text{ in } A_{\delta'},$$

and the quadratic maps as

$$q(a) = a^2 \qquad \text{if } a \text{ is a homogeneous element of odd degree.}$$

The left adjoint to this "forgetful functor" associates then in the usual way to a Δ-graded super-K-Lie-algebra $L.$ its universal enveloping algebra $U.(L)$, which is a Δ-graded, cocommutative K-Hopf-algebra with respect to the commutation factor $\varepsilon(\delta,\delta') = (-1)^{|\delta||\delta'|}$ on $\Delta\times\Delta$. It comes equipped with the natural "inversion" (or "antipodism"; [C-E]), which is the isomorphism of Δ-graded Hopf-algebras from $U.(L)$ onto its ε-opposite (Hopf-)algebra ${}_\varepsilon U.(L)^{op}$ induced by the (ordinary) opposite of the identity on $L.$. This enables one in particular to consider the graded dual of a (right) $U.(L)$-module as a (right) $U.(L)$-module again.

Finally, denoting by L^+ the (ordinary and Δ^+-graded) sub-Lie-algebra of $L.$ spanned by all homogeneous elements of even degree, and by L^-

the K-submodule spanned by those of odd degree, $L^{\cdot} = L^{+} \oplus L^{-}$ is the underlying $\mathbb{Z}/2\mathbb{Z}$-graded (or "super"-)K-Lie-algebra of $L.$, and, accordingly, $U^{\cdot}(L)$ the associated enveloping "super"-algebra.

If now $L.$ is finitely generated projective as a K-module, the canonical filtration by tensor-degree on $U(L)$ yields as the associated graded ring (or K-algebra) $\mathbf{S}.(L^{+}) \otimes_K \Lambda^{\cdot}(L^{-})$, the ($\mathbb{Z}\times\Delta$-graded) tensor-product of the symmetric (K-)algebra on L^{+} with the exterior (K-)algebra on L^{-} , by the graded version of the Poincaré-Birkhoff-Witt theorem.

Remark that this associated graded algebra is nothing but the enveloping algebra of the underlying abelian Lie-algebra of $L.$, as usual.

To apply (8.3.3.), we only need the following

Lemma 8.5.1.: Let K be any commutative ring with connected prime spectrum, V and W finitely generated projective K-modules of rank v and w respectively. Set $S = \mathbf{S}.(V) \otimes_K \Lambda^{\cdot}(W)$. Then

(i) For the canonical augmentation module K of S , one has

$$\mathrm{Ext}^i_S(K,S) = 0 \quad \text{for} \quad i \neq v ,$$

and

$$\mathrm{Ext}^v_S(K,S) = \det(W) \otimes_K \det(V')$$

as K-modules, where V' denotes the K-dual of V , $\det(.)$ the "determinant" - or highest non-vanishing exterior power - of a finitely generated projective K-module.

(ii) If K is furthermore Gorenstein of finite Krull dimension k , then S is strongly Gorenstein of virtual dimension $k + v$. It is of finite global dimension if and only if this holds for K and in addition $W = 0$.

Proof: (i) follows most easily from the change-of-rings spectral sequence

$$\mathrm{Ext}^i_{\Lambda^{\cdot}(W)}(K,\mathrm{Ext}^j_S(\Lambda^{\cdot}(W),S)) \Longrightarrow \mathrm{Ext}^{i+j}_S(K,S)$$

obtained from the obvious k-algebra homomorphisms $S \longrightarrow \Lambda^{\cdot}(W) \longrightarrow K$. The S-module $\Lambda^{\cdot}(W)$ is resolved by the Koszul-komplex over the S-linear map $V \otimes_K S \longrightarrow S$ which identifies the generating set V with the subset $\mathbf{S}_1(V) \otimes_K 1$ of S . Consequently, $\mathrm{Ext}^j_S(\Lambda^{\cdot}(W),S) = 0$ for $j \neq v$, and $\mathrm{Ext}^v_S(\Lambda^{\cdot}(W),S) = \Lambda^{\cdot}(W) \otimes_K (\det(V))'$. But $\mathrm{Ext}^i_{\Lambda^{\cdot}(W)}(K,\Lambda^{\cdot}(W)) = 0$ for $i \neq 0$ and $\mathrm{Hom}_{\Lambda^{\cdot}(W)}(K,\Lambda^{\cdot}(W)) = \det(W)$, which shows that the spectral sequence degenerates to yield (i).

For (ii), observe that $S.(V)$ is (locally) a polynomial ring over K which is Gorenstein if and only if K has this property. Furthermore, its Krull dimension is $\operatorname{rank} V + \dim K$. Now S itself is a finite flat $S.(V)$-algebra with respect to the natural inclusion of $S.(V)$ as the first factor of S. Hence one may conclude by (8.3.2.), taking into account that the formation of exterior powers commutes with any change of rings and that over a field the exterior algebra of a finite dimensional vectorspace is quasi-Frobenius and furthermore of infinite global dimension as soon as the vectorspace is non-zero.

In view of this Lemma, (8.3.3.) yields

<u>Proposition 8.5.2.</u>: Assume given a commutative Gorenstein ring K of finite Krull dimension k, $(\Delta,+)$ a group of degrees with a "parity homomorphism" $|.| : \Delta \longrightarrow \mathbb{Z}/2\mathbb{Z}$ as above, $L.$ a Δ-graded K-super-Lie-algebra such that the underlying K-modules of L^+ or L^- are both finitely generated projective and have a rank, say l^+ or l^- respectively. Then

(i) $U.(L)$ as a Δ-graded ring is strongly Gorenstein of injective dimension equal to $k + l^+$. Its global dimension is bounded below by the global dimension of K.

(ii) The (total or derived) graded dual of the natural augmentation module K of S as an object in $D^b({}_\varepsilon U.(L)^{op})$ satisfies

$$K^* =_{def} \mathrm{RHom}_{U.(L)}(K,U.(L)). = \det(L^-) \otimes_K \det(L^+)'[-l^+] \quad .$$

(iii) If K is regular, hence of finite global dimension, for $U.(L)$ to be of infinite global dimension, it is necessary that $l^- \neq 0$ and it is sufficient that L^- has non-zero intersection with the centre of $L.$.

<u>Proof</u>: Applying (8.3.3.) to the filtration by tensor-degree on $U.(L)$, the Lemma above yields a priori that $U.(L)$ is strongly Gorenstein of virtual dimension at most equal to $k + l^+$. That one has indeed equality follows from (ii) which in turn is a consequence of the (indicated) proof of (8.3.3.) and (8.5.1.(i)) : Taking $M = K$ and $N = U.(L)$, the associated graded module of M is K again and that of N is the associated graded ring of $U.(L)$. But then (8.5.1.(i)) implies that the spectral sequence in the proof of (8.3.3.) for these $U.(L)$-modules degenerates, leaving

$$\mathrm{Ext}^{l^+}_{U.(L)}(K,U.(L)) = \mathrm{Ext}^{l^+}_{gr(U.(L))}(K,gr(U.(L))) = \det(L^-) \otimes_K \det(L^+)'$$

as the only, possibly non-vanishing cohomology group of K^*, which is precisely the assertion in (ii). The last statement in (i) follows immediately from the fact that $U.(L)$ is a supplemented flat K-algebra : Whenever A, B are two K-modules, one has

$$\mathrm{Ext}^{\cdot}_K(A,B) = \mathrm{Ext}^{\cdot}_{U.(L)}(A \otimes_K U.(L),B) \quad ,$$

B considered a (right) $U.(L)$-module via the augmentation.

The first part of (iii) follows from the last statement of (8.5.1.) as the global dimension of $U.(L)$ is also bounded above by the global dimension of its associated graded ring, once again using J.-E. Roos' result as in (8.3.3.); cf. also [Bjö;Ch.2,Thm.3.7.].

The proof of the remaining assertion of (iii) is obtained as follows: Without loss of generality, one may assume that K is a field and that there exists a homogeneous element u in L^- which is non-zero and in the centre of $L.$. If then $\underline{L}.$ denotes the still Δ-graded K-super-Lie-algebra which is the quotient of $L.$ by the ideal $K.u$, the universal enveloping algebra $U.(L)$ is isomorphic to $U.(\underline{L}) \otimes_K \Lambda^{\cdot}(Ku)$, and the $U.(L)$-module $U.(\underline{L}.)$ is easily seen to be of infinite projective dimension, a projective resolution being obtained by tensoring a resolution of K over $\Lambda^{\cdot}(Ku)$ with $U.(\underline{L})$ over K. In other words, one uses the well-known relation : $\mathrm{Ext}^{\cdot}_{\Lambda^{\cdot}(Ku)}(K,K) = \mathrm{Ext}^{\cdot}_{U.(L)}(U.(\underline{L}),K)$, (see [C-E; XIII.4.4.] for example).

Now let us set by (slight) analogy with the case of integral group representations

$$(8.5.3.) \qquad \hat{H}^i(L./K,M.) = \underline{\mathrm{Ext}}^i_{U.(L)}(K,M.)$$

for any (complex of) right, graded $U.(L)$-module(s) $M.$ (in $D^-(\mathrm{Mod}\text{-}U.(L))$) and any integer i, and call these groups the <u>Tate-cohomology of $L.$ with values in $M.$</u>.

These groups are naturally Δ-graded K-modules and the Yoneda-product defines on the total Tate-cohomology $\hat{H}^{\cdot}(L./K,M)$ the structure of a right $\mathbb{Z}\times\Delta$-graded module over the <u>Tate-cohomology ring of $L.$</u>, $\hat{H}^{\cdot}_{L.} = \hat{H}^{\cdot}(L./K,K)$.

Recall furthermore, that $H^{\cdot}(L./K,M.) = \mathrm{Ext}^{\cdot}_{U.(L)}(K,M.)$ is the ordinary <u>cohomology of $L.$</u> with values in $M.$, and that the <u>homology of $L.$</u> with values in $M.$ is given by $H.(L./K,M.) = \mathrm{Tor}^{U.(L)}_{\cdot}(M.,K)$.

Finally, set $\mathbf{det}(V^+ \oplus V^-) = \det(V^+) \otimes_K \det(V^-)'$, the "<u>super-determinant</u>" - or "<u>Berezinian</u>" in "super-terminology", see [Lei] -, for any

$\mathbb{Z}/2\mathbb{Z}$-graded K-module, where V^+, the submodule of even elements, and V^-, the submodule of odd elements, are both finitely generated projective. Remark that the "super-determinant" is a projective K-module of rank one over K and that its inverse (in the Picard group of K) is represented by its K-dual or, as well, as both the "super-determinant" of the graded module $V^- \oplus V^+$, which is obtained from the original one by "parity-change", or of the graded K-dual $(V^+)' \oplus (V^-)'$.

Now, combining (8.5.2.(ii)) and (6.2.5.(3)), we get for any object $M.$ in $D^-(\text{Mod-}U.(L))$ a long exact sequence of Δ-graded K-modules

$$(8.5.4.)\qquad \ldots \longrightarrow \hat{H}^{i-1}(L./K,M.) \longrightarrow H_{-i+1^+}(L./K,M.) \otimes_K \mathbf{det}(L.)^{-1} \longrightarrow$$

$$\xrightarrow{\mathbf{N}^i} H^i(L./K,M.) \xrightarrow{c^i} \hat{H}^i(L./K,M.) \longrightarrow \ldots ,$$

where $\mathbf{N}^i$ is the morphism induced by the "Norm-map", c^i the natural transformation from ordinary to "stabilized" or Tate-cohomology.

In particular one has - as in (6.3.5.) - for any graded U.(L)-module

$$\hat{H}^i(L./K,M.) = H^i(L./K,M.) \qquad \text{for } i > 1^+$$

and

$$\hat{H}^i(L./K,M.) = H_{1^+-i}(L./K,M.) \otimes_K \mathbf{det}(L.)^{-1} \qquad \text{for } i < -1 .$$

Just remark that

$$\begin{aligned} \mathrm{Tor}_i^{U.(L)}(M.,K^*) &= H^{-i}(M. \otimes^{\mathbb{L}}_{U.(L)} \mathrm{RHom}_{U.(L)}(K,U.(L))) && \text{by definition,}\\ &= H^{-i}(M. \otimes^{\mathbb{L}}_{U.(L)} \mathbf{det}(L.)^{-1}[-1^+]) && \text{by (8.5.2.(ii)),}\\ &= H^{-i-1}(M. \otimes^{\mathbb{L}}_{U.(L)} K) \otimes_K \mathbf{det}(L.)^{-1} && \text{as } \mathbf{det}(L.)^{-1} \text{ is projective over } K,\\ &= \mathrm{Tor}^{U.(L)}_{i+1^+}(M.,K) \otimes_K \mathbf{det}(L.)^{-1} && \text{again by definition.} \end{aligned}$$

Next, we want to consider the duality theory of chapter 7 in this situation. To avoid (notational) complications arising from the structure of K, assume henceforth that K is (commutative) semi-simple - hence a product of fields.

Then, given any object $M.$ in $D^-(\text{Mod-}U.(L))$, let $M^\#$ denote the graded K-dual of $M.$, converted into a complex of right $U.(L)$-modules again by use of the inversion.

Now the graded version of Theorem (7.5.1.) or its Corollary (7.5.2.)

reads :

<u>Proposition 8.5.5.</u>: For any object $N.$ in $D^b(\text{Mod-}U.(L))$ there are natural isomorphisms for any integer i

$$\hat{H}^{i+1^{+}-1}(L./K,N^{\#}_{.}) \otimes_K \mathbf{det}(L.) \longrightarrow (\hat{H}^{-i}(L./K,N.))' \quad .$$

<u>Proof</u>: Apply (7.5.2.) with $T = K$, $S = {}_{\varepsilon}U.(L)^{op}$ and $W = K$ to get isomorphisms

$$\underline{\mathrm{Ext}}^{i-1}_{{}_{\varepsilon}U.(L)^{op}}(K^*,\mathrm{Hom}_K(N.,K)) \longrightarrow \mathrm{Hom}_K(\underline{\mathrm{Ext}}^{-i}_{U.(L)}(K,N.),K)$$

as K is injective as a module over itself. Using then once again the known structure of K^* ,(8.2.5.(ii)), as above, and converting the occurring graded ${}_{\varepsilon}U.(L)^{op}$-modules into $U.(L)$-modules by means of the inversion exhibits the terms in the form claimed.

Finally, let us examine a little bit closer $\hat{H}^{\cdot}_{L.}$, the Tate-cohomology ring of a graded super-Lie-algebra $L.$, over a <u>field</u>.

<u>Proposition 8.5.6.</u>: Let K be a field, $L.$ a Δ-graded super-Lie-algebra over K , $|.| : \Delta \longrightarrow \mathbb{Z}/2\mathbb{Z}$ a "parity homomorphism" as before.

(i) The $\mathbb{Z}\times\Delta$-graded Tate-cohomology ring $\hat{H}^{\cdot}_{L.}$ of $L.$ is ε'-commutative, where ε' is the commutation factor

$$\varepsilon'((i,\delta),(j,\delta')) = (-1)^{ij + |\delta||\delta'|}$$

for pairs (i,δ) and (j,δ') in $\mathbb{Z}\times\Delta$.
In particular, $\hat{H}^0_{L.} = \underline{\mathrm{Hom}}_{U.(L)}(K,K)$. is a Δ-graded, ε-commutative K-algebra.

(ii) $\hat{H}^{\cdot}_{L.}$ is a self-injective K-algebra. More precisely, there is a non-degenerate K-bilinear pairing, associative for the Yoneda-product on $\hat{H}^{\cdot}_{L.}$, and homogeneous for the total Δ-degree :

$$\hat{H}^{*+1^{+}-1}_{L.} \otimes_K \hat{H}^{-*}_{L.} \longrightarrow \mathbf{det}(L.)^{-1} \quad .$$

<u>Proof</u>: As seen in (8.4.2.), $\underline{\mathrm{Ext}}^{\cdot}_{U.(L)}(K,K) = \hat{H}^{\cdot}_{L.}$ is ε'-opposite to $\underline{\mathrm{Ext}}^{\cdot}_{{}_{\varepsilon}U.(L)^{op}}(K^*,K^*)$. Using the inversion on $U.(L)$ and the fact that $\mathrm{RHom}_{U.(L)}(K,K)$ and $\mathrm{RHom}_{U.(L)}(K^*,K^*)$ are canonically isomorphic, one

gets an ε'-anti-isomorphism of $\hat{H}^{\cdot}_{L.}$ onto itself. The point now is, that this map is in fact the identity, implying the claimed ε'-commutativity. The details are left to the reader. They can easily be filled in, as there is the so-called "standard-resolution" of K over $U.(L)$ - see below or [HFJ] - to which the foregoing functors and identifications can be applied directly.

Here, let us only remark that (i) "stabilizes" the classical result which states that the (ordinary) cohomology ring $H^{\cdot}_{L.}$ itself is ε'-commutative; see for example [B-R;§0] and the sources cited there.

The proof of (ii) is obvious from (8.5.5.) above :
Just take $N. = K$, - so that $N^{\#}_{.}$ equals K again - , to obtain isomorphisms of K-modules

$$\hat{H}^{i+l\ -1}_{L.} \otimes_K \mathbf{det}(L.) \xrightarrow{\sim} (\hat{H}^{-i}_{L.})'$$

for all i . Such an isomorphism corresponds biuniquely to a non-degenerate K-bilinear pairing as claimed. That these pairings are associative with respect to the (Yoneda-)product on the total Tate-cohomology ring, follows from (7.5.4.) - as in (7.7.5.(iii)) - in its graded version; cf. (8.4.2.).

<u>Remarks</u>: (a) Assume chosen Δ-homogeneous K-bases $\{y_1,\dots,y_{l^-}\}$ of L^- and $\{x_1,\dots,x_{l^+}\}$ of L^+ respectively. Then the one-dimensional vectorspace $\mathbf{det}(L.)$ is generated formally by the "super volume-element"

$$\mathbf{vol}(L.) = \frac{x_1 \wedge x_2 \wedge \dots \wedge x_{l^+}}{y_1 \wedge y_2 \wedge \dots \wedge y_{l^-}} \quad .$$

In other words, $\mathbf{det}(L.)^{-1}$ is isomorphic as a Δ-graded K-vectorspace to $K(\mathbf{vol}(L.)^{-1}) = K(\sum_i \deg x_i - \sum_j \deg y_j)$, where $\deg(.)$ denotes the degree in Δ of a homogeneous element.

(b) The pairing in (ii) above "stabilizes" the usual pairing between ordinary cohomology and homology of $L.$: The following diagram is commutative, the horizontal arrows being isomorphisms of graded K-vectorspaces

$$\begin{array}{ccc}
\hat{H}^{i+l^+-1}_{L.} \otimes_K \mathbf{det}(L.) & \xrightarrow{\sim} & (\hat{H}^{-i}_{L.})' \\
\big\downarrow & & \big\downarrow {\scriptstyle (c^{-i})'} \\
H_{-i}(L./K,K) & \xrightarrow{\sim} & (H^{-i}_{L.})'
\end{array} \quad ,$$

the left vertical map being the connecting homomorphism of (8.5.4.) for
$M. = K$, tensored with the identity on $\mathbf{det}(L.)$.

The main source - in algebra and topology - of such graded super-Lie-algebras is provided by taking $\Delta = \mathbb{Z}$ and - see [Av],[Qu] or [HJF] for details - considering <u>either</u>

- the <u>homotopy-Lie-algebra</u> $\pi^*(R)$ of an augmented commutative K-algebra $R \longrightarrow K$,

<u>or</u>

- the <u>rational homotopy-Lie-algebra</u> $\pi_*(\Omega X) \otimes \mathbb{Q}$ of a simply connected topological space X with the Samelson product as bracket.

In the first case, the enveloping algebra is by definition $\mathrm{Ext}^{\cdot}_R(K,K)$, the usual Yoneda-Ext-algebra of R, and, if R happens to be a graded "Koszul-algebra", [Löf], R itself with its grading is the cohomology ring of its homotopy-Lie-algebra.

In the second case, the enveloping algebra is the rational homology of the loop-space ΩX of X. Here, it would be interesting to know, what the topological or homotopy-theoretic interpretation of MCMs and Tate-cohomology might be.

These examples show once again - similar to (8.2.2.&3.) - a short-coming of the theory here : Most super-Lie-algebras arising these ways are not finite dimensional but rather <u>locally finite</u> in the sense that each graded piece has this property. Given the results of [HJF], where it is shown that the injective dimension of the corresponding enveloping algebra can be bounded above by the L.S.-category of the topological space, at least those where the L.S.-category is finite should be enclosed into a more general theory.

We conclude with the simple

<u>Example 8.5.7.</u>: (Abelian super-Lie-algebras)

We return to the notations of (8.5.1.) : $L.$ is the abelian super-Lie-algebra over a field K with a v-dimensional vectorspace V in even degrees and a w-dimensional vectorspace W in odd degrees, the enveloping algebra $\mathbf{S}.(V) \otimes_K \Lambda^{\cdot}(W)$ denoted $S.$ again.

A complete resolution of K over $S.$ is obtained - as in (5.6.) - by connecting a projective resolution of K with the ${}_{\varepsilon}S.^{op}$-dual of a projective resolution of $K^* = (\det(W) \otimes_K \det(V'))[-v]$, (8.5.2.(ii)).

The natural projective resolution of K over $S.$ is well-known :

First, if $W = 0$, K is resolved projectively over $S.(V)$ by the ordinary Koszul-complex

$$\mathbf{P}_{S.(V)}(K) = (\Lambda^{\cdot}(dV) \otimes_K S.(V)\ ,\ d_V) \quad ,$$

where dV denotes an isomorphic copy of V, with the same Δ-grading but placed in complex-degree -1 (according to our general convention on complexes). In other words, dV is a $\mathbb{Z}\times\Delta$-graded K-vectorspace, concentrated in degrees $\{-1\}\times\Delta$. "Extending" the original Δ-grading on V to a $\mathbb{Z}\times\Delta$-grading with V concentrated in degrees $\{0\}\times\Delta$, the underlying $\mathbb{Z}\times\Delta$-graded K-vectorspace of the Koszul-complex becomes a ε'-commutative K-algebra with respect to the commutation factor ε' given in (8.5.6.).

Then, as is well-known, [ALG X.206;Exerc.(2)], d_V is the unique K-algebra-derivation of bi-degree $(1,0)$ extending the K-linear endomorphism of $dV \oplus V$ which maps dV isomorphically onto V and annihilates V.

As $RHom_{S.(V)}(K,S.(V)) = \det(V)^{-1}[-v]$, the algorithm of (5.6.) yields the natural projective co-resolution

$$\mathbf{C}_{S.(V)}(K) = (\det(V) \otimes_K \Lambda^{\cdot}(dV') \otimes_K S.(V)[v],\ d_V^{\#}) \quad ,$$

where $(dV)'$, the graded K-dual of dV, is accordingly concentrated in degrees $\{1\}\times\Delta$, and the differential $d_V^{\#}$ is the $S.(V)$-dual of d_V, shifted v times and multiplied with the identity on $\det(V)$.

Remark that as $\mathbb{Z}\times\Delta$-graded vectorspaces one has

$$\det(V)[v] \cong \det(dV)$$

and a natural isomorphism ([ALG X.149] for example),

$$\Lambda^{\cdot}(dV) \longrightarrow \det(dV) \otimes_K \Lambda^{\cdot}(dV')$$

as usual.

This isomorphism on the generating sets extends naturally to an isomorphism of complexes from $\mathbf{P}_{S.(V)}(K)$ onto $\mathbf{C}_{S.(V)}(K)$ and represents a natural "Norm-map" $\mathbf{N}_{S.(V)}(K)$ in this case.

Of course, the deduced complete resolution, (5.6.2.), is contractible in accordance with the fact that K is of finite projective dimension over the polynomial ring $S.(V)$.

These classical properties of the Koszul-complex have a (less known) "shifted" counterpart, first noted by D.Quillen in this context, see

[I11;I.4.3.] for a complete treatment:

Namely, if now $V = 0$, so that we start with an abelian super-Lie-algebra concentrated in odd degrees, the enveloping algebra is $\Lambda^{\cdot}(W)$, (which is in particular quasi-Frobenius), and the natural projective resolution of K over it is given by the "shifted" Koszul-complex

$$\mathbf{P}_{\Lambda^{\cdot}(W)}(K) = (\Gamma^{\cdot}(dW) \otimes_K \Lambda^{\cdot}(W) \ , \ d_W) \qquad ,$$

dW denoting a copy of W concentrated in degrees $\{-1\}\times\Delta$ as above, $\Gamma^{\cdot}(dW)$ the K-algebra of divided powers over dW. Remark that in this situation, $\Gamma^{\cdot}(W)$ is (by definition) isomorphic to the graded K-dual of $S.(W')$, - even as a graded Hopf-algebra.

Now the $\mathbb{Z}\times\Delta$-graded K-vectorspace underlying $\mathbf{P}_{\Lambda^{\cdot}(W)}(K)$ carries also a natural ε'-commutative K-algebra structure with respect to which d_W is the unique K-algebra derivation extending the obvious K-linear map as before.

As here $\mathrm{RHom}_{\Lambda^{\cdot}(W)}(K,\Lambda^{\cdot}(W)) = \det(W)[0]$, the algorithm of (5.6.) - see also (8.1.) - shows that the natural projective co-resolution of K is

$$\mathbf{C}_{\Lambda^{\cdot}(W)}(K) = (S.(dW') \otimes_K \det(W)^{-1} \otimes_K \Lambda^{\cdot}(W) \ , \ d_W^{\#}) \qquad ,$$

which is also a (minimal) injective resolution of K embedded by the map $K \longrightarrow \det(W)^{-1} \otimes_K \det(W)$ into $\det(W)^{-1} \otimes_K \Lambda^{\cdot}(W)$ as its socle.

In this case of course, the "Norm-map" $\mathbf{N}_{\Lambda^{\cdot}(W)}(K)$ is just the composition of the $\Lambda^{\cdot}(W)$-linear maps

$$\mathbf{P}_{\Lambda^{\cdot}(W)}(K) \xrightarrow{\text{augmentation}} K \xrightarrow{\sim} \det(W)^{-1} \otimes_K \det(W) \xrightarrow{\text{incl.}} \mathbf{C}_{\Lambda^{\cdot}(W)}(K) \quad .$$

Having hence clearified the situation in the "pure" case, where the super-Lie-algebra is concentrated in one parity and abelian, one obtains in general :

- For an abelian super-Lie-algebra $L^{\cdot} = V \oplus W$, the projective (co-)resolution is obtained as the tensor-product over K of the corresponding objects for V and W alone, considered as subalgebras of $L.$. The associated Norm-map $\mathbf{N}_{S.}(K)$ is the tensor-product of the corresponding Norm-maps $\mathbf{N}_{S.(V)}(K)$ and $\mathbf{N}_{\Lambda^{\cdot}(W)}(K)$.

It follows then easily :

-. The Tate-cohomology ring of $L. = V \oplus W$ is given by

(i) $\hat{H}^{\cdot}_{L.} = 0$ iff $W = 0$, as precisely in this case $S.$ is of finite global dimension.

(ii) If $W \neq 0$, the exact sequence (8.5.4.) for $M. = K$ breaks up into short exact sequences as the corresponding (cohomological) Norm-map $\mathbf{N}^i$ is zero. Accordingly, as $\mathbb{Z}\times\Delta$-graded K-vectorspaces, one has that Tate-cohomology is the direct sum of the ordinary cohomology and the shifted (by $[-v+1]$) and twisted (by $\mathbf{det}(L.)^{-1}$) ordinary homology of $L.$.

In terms of the explicit descriptions just mentioned, the multiplicative structure is as follows :

(iii) If $\dim_K W = 1$, say $W = K(\delta)$, the shifted vectorspace dW is generated by a $\mathbb{Z}\times\Delta$-homogeneous element of bi-degree $(-1,-\delta)$, say t^{-1}. Then, as a $\mathbb{Z}\times\Delta$-graded K-algebra, one finds

$$\hat{H}^{\cdot}_{L.} = \Lambda^{\cdot}(dV')[t,t^{-1}] ,$$

the algebra of "Laurent-polynomials" in t over the exterior K-algebra generated by dV'. In this identification, the pairing of (8.5.6.(ii)) is given by the "super-residue", which associates to a pair of such polynomials the coefficient of $\mathbf{vol}(dV').t^{-1}$ in their product.

(iv) If $\dim_K W > 1$, the Tate-cohomology ring is indeed the trivial algebra-extension of the ordinary cohomology by the (shifted and twisted) ordinary homology considered as a bimodule. Explicitely :

$$\hat{H}^{\cdot}_{L.} = \Lambda^{\cdot}(dV') \otimes_K (\mathbf{S}.(dW') \oplus \det(W)\otimes_K \Gamma^{-\cdot}(dW)[1]) ,$$

the graded tensor-product of the exterior K-algebra on dV' with the trivial ring-extension of the symmetric K-algebra $\mathbf{S}.(dW')$ by the indicated bimodule, where $\mathbf{S}.(dW')$ acts on its graded K-dual $\Gamma^{\cdot}(dW)$ by "contraction" or "inner products", as defined for example in [ALG III;§11,n^{o} 6]. The pairing (8.5.6.(ii)) is then again given by multiplication and then projection onto the super volume-element chosen. (Remark that $\mathbf{det}(L.)^{-1}$ occurs as the direct summand $\Lambda^{v}(dV') \otimes_K \det(W)$ in $\hat{H}^{v-1}_{L.}$.)

(v) The foregoing description of the pairings is in accordance with (7.7.5.) : Taking $T = \mathbf{S}.(V)$, $\omega_T = \Omega^{v}_{T/K} = \det(V) \otimes_K \mathbf{S}.(V)$, the module of (Kähler-)differential forms of maximal degree of T over K, as dualizing module and $f : T \longrightarrow \mathbf{S}.(V) \otimes_K \Lambda^{\cdot}(W)$ the obvious inclusion, then the associated dualizing module of $S.$ is given by Remark (b) in (7.6.) :

$$\begin{aligned}\omega_{S.} &= f^{!}\omega_T[0] = \mathrm{Hom}_{S.(V)}(S.(V)\otimes_K\Lambda^{\cdot}(W),\det(V)\otimes_K S.(V)) \\ &= (S.(V)\ \otimes_K\ \Lambda^{\cdot}(W))\ \otimes_K\ \det(W)^{-1}\ \otimes_K\ \det(V) \\ &= S.\ \otimes_K\ \mathbf{det}(L.)\end{aligned}$$

as a S.-bimodule, taking into account as before the natural identification of the graded K-dual of $\Lambda^{\cdot}(W)$ with $\Lambda^{\cdot}(W)\otimes_K\det(W)^{-1}$. Then the trace-map of (7.7.5.(iv)) for (the MCM-approximation of) K is the linear form

$$\tau_K\ :\ \underline{\mathrm{Ext}}^{v-1}_{S.}(*K,\omega_{S.}\otimes_{S.}K*) = \mathbf{det}(L.)\ \otimes_K\ \hat{H}^{v-1}_{L.} \longrightarrow K$$

- observe that $*K = K*$ as complexes of S.-bimodules and that $\omega_{S.}\otimes_{S.}\ K = \mathbf{det}(L.)$ by the above - , which exhibits the coefficient of $\mathbf{det}(L.)^{-1}$ as a direct summand of $\hat{H}^{v-1}_{L.}$.

In "super-geometric" terms, this linear form is nothing but the integration against the "fundamental cycle" of the underlying projective super-space of S. , as will become clear below.

This description of S. as a finite flat **S**.(V)-algebra yields the following representation of MCMs over S. :

The polynomial-algebra is concentrated in the even degrees Δ^+, so for any Δ-graded S.-module M. , the underlying **S**.(V)-module $f_*M.$ decomposes into a direct sum $f_*M. = M^+_. \oplus M^-_.$ of Δ^+-graded S.(V)-modules. By (8.3.1.) in its graded version, the original module M. is MCM over S. iff both summands $M^{+/-}$ are finitely generated free S.(V)-modules.

The missing piece of information to reconstruct M. from these free modules is the action of the 1-forms in $\Lambda^1(W) = W$ as $\Lambda^{\cdot}(W)$ is freely generated as an alternating K-algebra by them.

Hence, a MCM over S. can be represented as a collection

$$(M^+,M^-,\ A_i\ :\ M^+ \longrightarrow M^-\ ,\ B_i\ :\ M^- \longrightarrow M^+\ ;\ i = 1,\ldots,w)\quad ,$$

where $M^{+/-}$ are graded free S.(V)-modules of finite rank, A_i and B_i **S**.(V)-linear maps satisfying the relations

$$A_iB_i = 0\ ,\ B_iA_i = 0 \quad\text{and}\quad A_iB_j + A_jB_i = 0\ ,\ B_iA_j + B_jA_i = 0$$

for all $1 \leq i,j \leq w\ ;\ i \neq j$.

We finish this section with two more

<u>Remarks</u>: (c) The given multiplicative structure of $\hat{H}_{L.}^{\cdot}$ implies that $\hat{H}_{L.}^{0}$ is a <u>local</u> ring if $W \neq 0$. This proves that any MCM-approximation of K without free summands is indecomposable, (2.1.), as a S.-module as $\hat{H}_{L.}^{0}$ is its endomorphism ring in <u>MCM(S.)</u> . Furthermore, it has a well defined rank if constructed from the natural complete resolution, cf. (5.5.1.) . One may hence ask as in (5.5.3.) whether this is the minimum for a non-free MCM-approximation of a S.-module of finite length.

Also, it should be interesting to understand the representations of Gl(V)xGl(W) - or rather of the corresponding super-group - which occur in this indecomposable MCM-approximation of K , as this module determines essentially the Tate-cohomology by (6.1.2.) , (6.4.1.) .

(d) For an arbitrary K-super-Lie-algebra L. , extending naturally the Poincaré-Birkhoff-Witt theorem, one may obtain a "standard (complete) resolution" of K over U.(L) by deforming the projective (co-)resolution (and Norm-map) of the underlying abelian super-Lie-algebra, see for example [HFJ;Prop.1.13.] or [Qu ;App.B,6.7.] for the case of projective resolutions.

In other words, the underlying graded K-vectorspaces can be chosen to be the same, multiplication and differential become deformed.

It follows that, denoting $L.^{ab}$ the underlying abelian algebra, the Tate-cohomology of L. itself can be obtained as the cohomology of a differential, $\mathbb{Z}\times\Delta$-graded algebra $(H^{\cdot},D)$, D a homogeneous differential of bi-degree (1,0) , the underlying graded K-vectorspace of $H^{\cdot}$ being isomorphic to $\hat{H}_{L.ab}^{\cdot}$. In particular, the dimensions of the homogeneous components of $\hat{H}_{L.}^{\cdot}$ are bounded above by the corresponding dimensions in the abelian case, which can be easily determined from the explicit description above.

Although we ignore the meaning of MCMs for an arbitrary super-Lie-algebra, at least for certain of those, they admit a geometric interpretation. Here, we restrict ourselves again to the case of abelian such super-Lie-algebras, referring the reader to [BEH] for some more examples of the same kind.

We want to show, that over the category of abelian super-Lie-algebras - or, equivalently, graded K-vectorspaces - , the theory of MCMs and of coherent sheaves on the associated projective super-spaces are dual to each other. This extends naturally - and in a straightforward manner - the theory initiated by A.A. Beilinson, J. Bernstein, S.I. Gelfand and I.M. Gelfand. It relies on the connection between :

9. - *Maximal Cohen-Macaulay Modules and Geometry on Projective Superspaces*
(The theory of Beilinson - Bernstein - Gelfand - Gelfand)

Fix a field K and the group of degrees $\Delta = \mathbb{Z} \times \mathbb{Z}$ with "parity homomorphism" $|(a,b)| = b \bmod 2$ into $\mathbb{Z}/2\mathbb{Z}$. The "total degree" of (a,b) is given by $\int(a,b) = a+b$.

9.1. Linear Superspaces

Consider the abelian category mod-$K \times K$ and interpret it as follows : Its objects, which are by definition pairs (V,W) of finite dimensional K-vectorspaces, are thought of as Δ-graded K-vectorspaces concentrated in total degree 1, the even subspace V being placed in degree $(1,0)$, the odd subspace W in degree $(0,1)$. For short, we will also say that (V,W) endowed with this grading is a linear superspace.

The morphisms $F = (F^+,F^-) : (V_1,W_1) \longrightarrow (V_2,W_2)$ are accordingly just pairs of K-linear maps and can be interpreted as Δ-homogeneous morphisms of Δ-graded vectorspaces.

In particular, there are the following :

(i) A linear form λ on the even subspace V will be identified with the morphism $(\lambda,0) : (V,W) \longrightarrow (K,0)$ and, if $\lambda \neq 0$, its kernel is said to define a (K-rational) geometric point of (V,W).

(ii) An element w in the odd subspace W gives rise to a morphism $\langle w\rangle : (0,K) \longrightarrow (V,W)$, $\langle w\rangle(0,a) = (0,wa)$, and its image, if not zero, will represent a (K-rational) odd point of (V,W).

"Recall" that $\mathbf{P}(V) = \mathrm{Proj}_K \mathbf{S}.(V)$ is the scheme of the projective space of hyperplanes in V, so that "geometric points" of (V,W) correspond to the points of $\mathbf{P}(V)$, whereas the "odd points" of (V,W) are parametrized by $\mathbf{P}(W^*)$, $(-)^*$ denoting in this chapter the K-dual of a vectorspace.

(iii) A morphism F as above will be called (linearly) perfect iff its even component F^+ is surjective, whereas its odd component F^- is injective.

Two particular such morphisms are given by $i_V = (\mathrm{id}_V,0)$, the inclusion of the even subspace $V = (V,0)$ into (V,W), and by $p_W : (V,W) \longrightarrow (0,W) = W$, the projection onto the odd quotientspace.

Extending the notion of "super-determinant" for a $\mathbb{Z}/2\mathbb{Z}$-graded K-vectorspace, (8.5.), define for any morphism F as above

$$\mathbf{det}(F) = \mathbf{det}(V_2,W_2) \otimes_K \mathbf{det}(V_1,W_1)^{-1} = \mathbf{det}(\mathrm{Cok}\ F) \otimes_K \mathbf{det}(\mathrm{Ker}\ F)^{-1}$$

and call it accordingly the "super-determinant" of F .

If $v_i = \dim_K V_i$ and $w_i = \dim_K W_i$, then, as graded K-vectorspaces,

$$\mathbf{det}(F) = K(v_1 - v_2, w_2 - w_1) \quad ,$$

hence, if F is linearly perfect, $\mathbf{det}(F)$ is concentrated in negative degrees. We will also say that

$$\mathbf{ind}^+(F) = \mathrm{ind}(F^+) = \dim_K \mathrm{Ker}(F^+) - \dim_K \mathrm{Cok}(F^+) = v_1 - v_2$$

is the <u>even index</u> of F , whereas

$$\mathbf{ind}^-(F) = -\mathrm{ind}(F^-) = w_2 - w_1$$

represents the <u>odd index</u> of F .

On the category of linear superspaces just defined, we consider the following <u>duality functor</u> $\mathbf{D}$, given by

$$\mathbf{D}(V,W) = (W^*,V^*) \quad \text{and} \quad \mathbf{D}(F^+,F^-) = ((F^-)^*,(F^+)^*) \quad .$$

It is obviously a contravariant, exact functor satisfying $\mathbf{D}^2 = \mathrm{id}$, hence establishes indeed a "duality" . It interchanges "geometric" and "odd" points and preserves linearly perfect morphisms. One has also $\mathbf{ind}^{+/-}(F) = \mathbf{ind}^{-/+}(\mathbf{D}(F))$.

9.2. <u>The enveloping algebras and their modules</u>

In accordance with the theory of "super-Lie-algebras" mentioned in the foregoing chapter, we have a functor $\mathbf{U}.$ from the category mod-KxK , alias linear superspaces or abelian super-Lie-algebras, into the category of Δ-graded K-(Hopf-)algebras with inversion, given by

$$\mathbf{U}.(V,W) = \mathbf{S}.(V) \otimes_K \Lambda^{\cdot}(W) \quad \text{and} \quad \mathbf{U}.(F^+,F^-) = \mathbf{S}.(F^+) \otimes_K \Lambda^{\cdot}(F^-) \quad .$$

These (Hopf-)algebras are ε-commutative (and ε-cocommutative) with respect to the commutation factor

$$\varepsilon((a_1,b_1),(a_2,b_2)) = (-1)^{b_1 b_2} \quad :$$

Accordingly, we will not distinguish left and right $\mathbf{U}.$-modules, but rather assume that all modules are ε-symmetric bimodules, the left and right structure being obtained from one another by means of the inversion (or "graded commutativity").

By definition, $\mathbf{U}.(V,W)$ is both a supplemented $\mathbf{S}.(V)$-algebra as well as a supplemented $\Lambda^{\cdot}(W)$-algebra. In particular, $\mathbf{S}.(V)$ and $\Lambda^{\cdot}(W)$ will always be considered $\mathbf{U}.(V,W)$-modules with respect to the associated augmentations. Analogously, K just represents the canonical augmentation module of $\mathbf{U}.(V,W)$.

Remark that the algebra homomorphism $\mathbf{U}.(i_V) : \mathbf{S}.(V) \longrightarrow \mathbf{U}.(V,W)$ exhibits $\mathbf{U}.(V,W)$ as a finite flat $\mathbf{S}.(V)$-algebra and that it is hence a duality morphism of finite type as defined in (7.6.2.) . Extending this notion and the result (7.6.3.) to the "graded commutative" case - which is left to the reader - the term "linearly perfect morphism" is justified by

<u>Lemma 9.2.1.</u>: Let $F = (F^+,F^-)$ be a pair of K-linear maps as above. Then the following properties are equivalent :

(i) F is linearly perfect.

(ii) $\mathbf{D}(F)$ is linearly perfect.

(iii) $\mathbf{U}.(F)_*(\mathbf{U}.(V_2,W_2))$, the underlying "complex" of $\mathbf{U}.(V_1,W_1)$-modules obtained from $\mathbf{U}.(V_2,W_2)$ by restriction of scalars along $\mathbf{U}.(F)$, is perfect.

(iv) The complex $\mathbf{U}.(F)^!(\mathbf{U}.(V_1,W_1)) = \mathrm{RHom}_{\mathbf{U}.(V_1,W_1)}(\mathbf{U}.(V_2,W_2),\mathbf{U}.(V_1,W_1))$ of $\mathbf{U}.(V_2,W_2)$-modules is perfect.

(v) $\mathbf{U}.(F)$ is a duality morphism of finite type (and its virtual codimension equals $-\mathbf{ind}^+(F)$).

(vi) $\mathbf{U}.(\mathbf{D}(F))$ is a duality morphism of finite type (and its virtual codimension equals $-\mathbf{ind}^-(F)$).

<u>The proof</u> is obvious.

<u>Remark</u>: (a) For a linearly perfect morphism F as above, the relative dualizing module of $\mathbf{U}.(F)$ is

$$\omega_{\mathbf{U}.(F)} = H^{\mathbf{ind}^+(F)}(\mathbf{U}.(F)^!(\mathbf{U}.(V_1,W_1))) \quad \text{according to (7.6.3.)}$$

which yields by a simple computation :

$$\omega_{\mathbf{U}.(F)} = \mathbf{U}.(V_2,W_2) \otimes_K \mathbf{det}(F) \quad .$$

As in (8.7.5.(v)), we define the <u>canonical module</u> of $\mathbf{U}.(V,W)$ to be

$$\omega_{\mathbf{U}.(V,W)} = \mathbf{U}.(V,W) \otimes_K \mathbf{det}(V,W) \quad ,$$

so that one has as usual

- for $W = 0$ the ordinary canonical module $\omega_{S.(V)} = S.(V) \otimes_K \det(V)$, which may and will be identified with the module Ω_V^V of differential forms of maximal degree over $S.(V)$,
- for a linearly perfect morphism F as before,

$$\omega_{\mathbf{U}.(V_2,W_2)} = H^{\mathbf{ind}^+(F)}(\mathbf{U}.(F)^!(\omega_{\mathbf{U}.(V_1,W_1)})) \quad .$$

If the graded vectorspace (V,W) is to be understood, we will set

$$\mathbf{U}. = \mathbf{U}.(V,W) \quad \text{and} \quad \mathbf{UD}. = \mathbf{U}.(\mathbf{D}(V,W)) = \mathbf{U}.(W^*,V^*) \quad ,$$

and similarly we write simply $\omega_{\mathbf{U}.}$ and $\omega_{\mathbf{UD}.}$ for the corresponding canonical modules.

If we need to separate the two degrees, we will write $\mathbf{U}_i^j$ for the homogeneous component $S_i(V) \otimes_K \Lambda^j(W)$ of bidegree (i,j) . Analogously, for any graded $\mathbf{U}.$-module $N = \oplus N_i^j$, N_i^j denotes the component of degree (i,j) .

The key observation by J. Bernstein, S.I. Gelfand and I.M. Gelfand in [BGG], slightly generalized, was now that homological algebra over $\mathbf{U}.$ or $\mathbf{UD}.$ is essentially the "same" - but by a non-trivial equivalence.

To formulate it precisely, let us (re-)introduce the following module categories :

- Mod_{lf}-$\mathbf{U}.$ will denote the category of graded $\mathbf{U}.$-modules N which are <u>locally finite for the total degree</u>, that is, for any integer k , the K-vectorspace

$$(\textstyle\int_* N:)_k =_{\mathrm{def}} \displaystyle\bigoplus_{i+j=k} N_i^j$$

is finite dimensional. In particular, each component N_i^j is of finite dimension over K .

As $\mathbf{U}.$ itself is certainly locally finite for the total degree and as it is furthermore "positively graded", namely concentrated in degrees $\mathbb{N}\times[0,\dim_K W]$, it follows that

- $\text{mod-}\mathbf{U}.$, the category of all finitely generated graded $\mathbf{U}.$-modules, is a full subcategory of $\text{Mod}_{lf}\text{-}\mathbf{U}.$, whose objects N satisfy :
  - for all but finitely many j , $N^j_{\cdot} = 0$ and
  - for all but finitely many negative i , $N_i^{\cdot} = 0$.

  In particular, the associated $\mathbb{Z}$-graded module $\int_* N$ is essentially positively graded.
- $\text{art-}\mathbf{U}.$ will denote the full subcategory of $\text{mod-}\mathbf{U}.$ which consists of all graded artinian $\mathbf{U}.$-modules N . Equivalently, each component N_i^j is finite dimensional over K and there are only finitely many pairs (i,j) such that $N_i^j \neq 0$. The artinian modules constitute a Serre-subcategory of $\text{mod-}\mathbf{U}.$, so that we may form the (abelian) quotient category
- $\text{Proj-}\mathbf{U}. = \text{mod-}\mathbf{U}./\text{art-}\mathbf{U}.$. "Forgetting" the $\Lambda^{\cdot}(W)$-structure, objects in it can be identified with $\mathbb{Z}$-graded, coherent sheaves of modules on the projective space $\mathbf{P}(V)$, the action of the exterior algebra being recovered through the $\mathcal{O}_{\mathbf{P}(V)}$-linear maps which represent the multiplication with the "1-forms" in W .

  The "typical object" in $\text{Proj-}\mathbf{U}.$ is hence a finite family $(F^j)_j$ of coherent sheaves of $\mathcal{O}_{\mathbf{P}(V)}$-modules together with $\mathcal{O}_{\mathbf{P}(V)}$-linear maps

$$\lambda^j : F^{j-1} \otimes_K W \longrightarrow F^j$$

  satisfying $\lambda^{j+1}(\lambda^j(f\otimes w)\otimes w) = 0$ for all local sections f of F^{j-1} and any w in W .

  In still different terms, $\text{Proj-}\mathbf{U}.$ can be identified as the category of coherent sheaves of modules over the topological space $\mathbf{P}(V)$, ringed by the $\mathbb{Z}$-graded sheaf $\mathcal{O}_{\mathbf{P}(V)} \otimes_K \Lambda^{\cdot}(W)$.

  We will denote the projection functor from $\text{mod-}\mathbf{U}.$ to $\text{Proj-}\mathbf{U}.$ by $\mathbf{a}$ and think of it as "sheafification" as in the classical case where $W = 0$.

Passing now to the underlying $\mathbb{Z}/2\mathbb{Z}$-graded structures, we are in the realm of "super-geometry" :

$\mathbf{U}^{\cdot} = S.(V) \otimes_K (\Lambda^+(W) \oplus \Lambda^-(W))$ is the ($\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$-graded) coordinate ring of the affine superspace $\mathbf{A}(V,W)$ attached to the linear superspace (V,W) , the projective space $\mathbf{P}(V)$ ringed by the $\mathbb{Z}/2\mathbb{Z}$-graded sheaf of K-algebras $\mathcal{O}_{\mathbf{P}(V)} \otimes_K(\Lambda^+(W) \oplus \Lambda^-(W))$ representing the underlying projective super-space $\mathbf{P}(V,W)$.

Remark: (b) $\mathbf{P}(V,W)$ is actually the "algebraic" projective super-space as the structure sheaf on its underlying reduced scheme, which is the usual projective space $\mathbf{P}(V)$, is given by the (germs of) algebraic - rather than holomorphic - functions. Observe that we did not restrict our field of coefficients ! For more details in the real or complex case, see for example [Lei].

Denoting $\mathcal{O}_{\mathbf{P}(V,W)}$ the just introduced structure sheaf on $\mathbf{P}(V,W)$ and mod-$\mathbf{P}(V,W)$ the category of coherent sheaves of modules over it, by definition the sheaf cohomology of a coherent $\mathbb{Z}/2\mathbb{Z}$-graded $\mathcal{O}_{\mathbf{P}(V,W)}$-module $\underline{F}$ is the $\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$-graded K-vectorspace which is given in degree $(i,\underline{j})$ by

$$H^i(\underline{F})_{\underline{j}} = \operatorname{Ext}^i_{\text{mod-}\mathbf{P}(V,W)}(\mathcal{O}_{\mathbf{P}(V,W)},\underline{F}(\underline{j})) \quad .$$

If now $(F^j)_j$ is an object in Proj-$\mathbf{U}$. as above, set $\underline{F}^{\cdot} = F^+ \oplus F^-$, with even component $F^+ = \bigoplus_j F^{2j}$ and odd component $F^- = \bigoplus_j F^{2j+1}$.

It represents naturally an object in mod-$\mathbf{P}(V,W)$. For any two objects $F = (F^j)_j$ and $G = (G^j)_j$ in Proj-$\mathbf{U}$. one has then obviously

$$\operatorname{Ext}^i_{\text{mod-}\mathbf{P}(V,W)}(\underline{F}^{\cdot},\underline{G}^{\cdot})_{\underline{j}} = \bigoplus_{l=\underline{j} \bmod 2} \operatorname{Ext}^i_{\text{Proj-}\mathbf{U}.}(F,G(l))$$

and, as $\underline{\mathbf{a}(\mathbf{U}.)}^{\cdot} = \mathcal{O}_{\mathbf{P}(V,W)}$, in particular

$$H^i(\underline{F}^{\cdot})_{\underline{j}} = \bigoplus_{l=\underline{j} \bmod 2} \operatorname{Ext}^i_{\text{Proj-}\mathbf{U}.}(\mathbf{a}(\mathbf{U}.),F(l)) \quad .$$

Hence, passing to the underlying $\mathbb{Z}/2\mathbb{Z}$-grading yields an exact embedding of Proj-$\mathbf{U}$. into mod-$\mathbf{P}(V,W)$ and in this sense, the study of objects in Proj-$\mathbf{U}$. equals the study of special sheaves on projective super-space, namely of those whose $\mathbb{Z}/2\mathbb{Z}$-grading can be "refined" to a $\mathbb{Z}$-grading. Similar remarks apply to the corresponding natural functor from mod-$\mathbf{U}$. into mod-$\mathbf{A}(V,W)$, the category of graded coherent sheaves of modules on the affine "super-cone" over $\mathbf{P}(V,W)$.

In all, we have a commutative diagram of abelian categories and exact functors between them

$$\begin{tikzcd}
\text{mod-}\mathbf{U}. \arrow[r, "\text{underlying } \mathbb{Z}/2\mathbb{Z}\text{-grading}"] \arrow[d] \arrow[dr, phantom, "\longleftarrow \text{ ``sheafification'' } \mathbf{a}(-) \longrightarrow"] & \text{mod-}\mathbf{A}(V,W) \arrow[d] \\
\text{Proj-}\mathbf{U}. \arrow[r, "\text{underlying } \mathbb{Z}/2\mathbb{Z}\text{-grading}"] & \text{mod-}\mathbf{P}(V,W)
\end{tikzcd} ,$$

so that one may be tempted to think of $\mathbf{A}(V,W)$ as the <u>even affine cone</u>, of $\mathbf{U}.$ as the "homogeneous coordinate ring" of the "<u>total cone</u>" and of $(\mathbf{P}(V), \mathbf{a}(\mathbf{U}.))$ as the "<u>odd projective cone</u>" associated to the projective super-space defined by (V,W) .

Coming back to our original subject, recall that <u>MCM(**U**.)</u> denotes the category of graded maximal Cohen-Macaulay modules over $\mathbf{U}.$.
Deriving the abelian categories introduced, we are led to study

9.3. <u>The associated triangulated categories</u>

To investigate various triangulated categories canonically attached to a linear superspace (V,W) , we will need the following criterion observed by A.A. Beilinson :

<u>Lemma and Definition 9.3.1.</u>, [Bei;Lemma 1]:

(i) Let C be any triangulated category, $(X_i)_i$ a family of objects in C . Then there is a smallest full and triangulated subcategory of C which contains this family of objects. Let us call it the <u>triangulated span</u> of $(X_i)_i$ in C . If this category is already equivalent to C with respect to the embedding, we say that $(X_i)_i$ <u>generates</u> C (as a triangulated category).

(ii) Let $F : C \longrightarrow D$ be an exact functor between triangulated categories. Then the restriction of F to the subcategory spanned by $(X_i)_i$ is full, faithful or an exact equivalence onto the subcategory of D which is spanned by $(F(X_i))_i$ if the maps

$$F^k(X_i,X_j) : \mathrm{Hom}_C(X_i,X_j[k]) \longrightarrow \mathrm{Hom}_D(F(X_i),F(X_j)[k])$$

are injective, surjective or isomorphisms (of abelian groups) for all k and any pair (X_i,X_j) of objects in the family.

<u>Remark</u>: (a) If the triangulated category C carries further gradings, say defined by shift functors $-(\delta)$, a family $(X_i)_i$ of objects in C will be said <u>to generate</u> (a subcategory of) C <u>up to shifts</u>, if the family $(X_i(\delta))_{i,\delta}$ generates it in the aforementioned sense.

As now $\mathbf{U}.$ is already bigraded, its derived categories will be triply graded by the functors $-[k](i,j)$, with k,i,j any integers.
What now simplifies considerably the homological algebra over $\mathbf{U}.$, is the fact that most interesting triangulated categories attached to it can

be generated up to shifts by single modules - considered as complexes in the usual way. A formal indication, why this makes life easier, is obtained by just considering the associated Grothendieck groups :

Remark: (b) If C is any triangulated category which can be generated by a family of objects $(X_i)_i$, the associated Grothendieck group $K(C)$ is generated as an abelian group by the classes of these objects. If now C carries further gradings, defined by shift functors $-(\delta)$, δ in some group Δ of degrees, then $K(C)$ becomes in a natural way a module over the integral group ring of Δ and if the family $(Y_j)_j$ generates C up to shifts, the classes $cl(Y_j)$ generate $K(C)$ as such a module.

Hence, if C can be generated by a single object (up to shifts), its Grothendieck group is a cyclic group (resp. module over the group ring).

In case of $\mathbf{U}.$, some naturally occurring triangulated categories are the following ones :

(9.3.2.) $D^b_{art}(\mathbf{U}.)$, the full triangulated subcategory of $D^b(\mathbf{U}.)$ whose objects are those complexes with artinian cohomology, is equivalent to $D^b(\text{art-}\mathbf{U}.)$.

(i) It can be obtained as the thick hull of the image of the forgetful functor

$$\mathbf{U}.(p_W)_* : D^b(\Lambda^\cdot(W)) \longrightarrow D^b(\mathbf{U}.) \quad ,$$

associated to the projection $p_W : (V,W) \longrightarrow (0,W)$ onto the odd quotient space. (Of course, it is also the thick hull of the image of the forgetful functor associated to the zero-map $0 : (V,W) \longrightarrow (0,0)$, but this map is not linearly perfect, whence we prefer p_W !).

(ii) $D^b(\text{art-}\mathbf{U}.)$ is generated up to shifts by the single $\mathbf{U}.$-module K . These assertions are obvious.

(iii) For the Grothendieck groups one has hence :

$$K(D^b_{art}(\mathbf{U}.)) = K(D^b(\text{art-}\mathbf{U}.)) = K_0(\text{art-}\mathbf{U}.) = cl(K).\mathbb{Z}[s,s^{-1},\lambda,\lambda^{-1}] \ ,$$

with $cl(K(i,j)) = (-1)^j cl(K) s^i \lambda^j$. (The sign has been introduced just to keep track of the "parity" of even and odd part. It makes formulas look more natural.)

(iv) From the explicit knowledge of the natural projective resolution of K over $\mathbf{U}.$, (8.5.7.), we get :

$$\bigoplus_{i,j} \mathrm{RHom}_{\mathbf{U}.}(K,K(i,j)) = \bigoplus_{i,j} \mathbf{S}_i(W^*) \otimes_K \Lambda^j(V^*)[-i-j](j,i)$$

as complexes of bigraded K-vectorspaces.

Accordingly, $\mathrm{Ext}^k_{\mathbf{U}.}(K,K)_{(i,j)} = 0$ for $k+i+j \neq 0$ and

$$\mathrm{Ext}^{i+j}_{\mathbf{U}.}(K,K)_{(-j,-i)} = \mathbf{S}_i(W^*) \otimes_K \Lambda^j(V^*) = \mathbf{UD}^j_i \quad .$$

The multiplicative structure on the Yoneda-Ext-algebra is precisely the <u>opposite</u> one of $\mathbf{UD}.$, due to our conventions on shifts, (8.4.), and our preference for right modules. As $\Lambda^{\cdot}(V^*)^{op}$ is canonically isomorphic as a K-algebra to $\Lambda^{\cdot}(V)$, [ALG III.§11.5], and $\mathbf{S}.(W^*)$ is commutative, the Yoneda-Ext-algebra of K over $\mathbf{U}.$ "is" $\mathbf{U}.(W^*,V)$ - but regraded.

(9.3.3.) $D^b_{perf}(\mathbf{U}.)$ denotes, as always, the full triangulated subcategory of graded perfect complexes in $D^b(\mathbf{U}.)$.

(i) It can be obtained as the thick hull of either the image of the left adjoint $\mathbf{U}.(i_V)^*$ or the right adjoint $\mathbf{U}.(i_V)^!$ of the forgetful functor associated to the inclusion $i_V : (V,0) \longrightarrow (V,W)$ of the even subspace. These functors

$$\mathbf{U}.(i_V)^* \ , \ \mathbf{U}.(i_V)^! : D^b(\mathbf{S}.(V)) \longrightarrow D^b(\mathbf{U}.) \quad ,$$

are well defined by (9.2.1.) as i_V is linearly perfect and their images span the perfect subcategory as $\mathbf{S}.(V)$ is of finite global dimension, see (1.3.(b)) and remark that

$$\mathbf{U}.(i_V)^!(-) = \mathbf{U}(i_V)^*(-) \otimes_{\mathbf{U}.} \omega_{\mathbf{U}./\mathbf{S}.(V)} = \mathbf{U}(i_V)^*(-) \otimes_K (\det(W))^{-1} \quad ,$$

by Remark (a) in (9.2.).

(ii) According to these choices, there are - at least - two canonical generators for $D^b_{perf}(\mathbf{U}.)$ imposing themselves, namely either $\mathbf{U}. = \mathbf{U}.(i_V)^*(\mathbf{S}.(V))$ or else $\omega_{\mathbf{U}.} = \mathbf{U}.(i_V)^!(\omega_{\mathbf{S}.(V)})$.

(iii) For the Grothendieck group, remark that a finitely generated projective and graded $\mathbf{U}.$-module is necessarily a direct sum of modules of the form $\mathbf{U}.(i,j)$ for various (i,j) . It follows that

$$K(D^b_{perf}(\mathbf{U}.)) = K_0(\mathbf{U}.) = \mathrm{cl}(\mathbf{U}.).\mathbb{Z}[s,s^{-1},\lambda,\lambda^{-1}] \quad ,$$

and that, using the same conventions as in (9.3.2.(iii)), one has

$$\mathrm{cl}(\omega_{\mathbf{U.}}) = \mathrm{cl}(\mathbf{U.}\otimes_K \mathbf{det}(V,W)) = \mathrm{cl}(\mathbf{U.}(-v,w)) = \mathrm{cl}(\mathbf{U.}).(-1)^w s^{-v}\lambda^w \quad .$$

(iv) As concerns the graded endomorphism rings of these generators, one obtains as in (8.4.) :

$$\bigoplus_{i,j} \mathrm{RHom}_{\mathbf{U.}}(\mathbf{U.},\mathbf{U.}(i,j)) = \bigoplus_{i,j} \mathrm{RHom}_{\mathbf{U.}}(\omega_{\mathbf{U.}},\omega_{\mathbf{U.}}(i,j)) = \mathbf{U}^{op}_{\cdot}[0] \quad ,$$

whence the corresponding Ext-algebra is isomorphic to $\mathbf{U}^{op}_{\cdot}$, again conveniently regraded. By the same argument as in (9.3.2.(iv)), $\mathbf{U}^{op}_{\cdot}$ is isomorphic as a graded K-algebra to $\mathbf{U.}(V,W^*)$.

(9.3.4.) $D^b_{a/p}(\mathbf{U.})$ will denote the full triangulated subcategory of all perfect complexes with artinian cohomology in $D^b(\mathbf{U.})$. Hence it is the intersection of $D^b_{art}(\mathbf{U.})$ and $D^b_{perf}(\mathbf{U.})$.

(i) Accordingly, it can be obtained as the thick hull of the image of $D^b_{perf}(\Lambda^\cdot(W))$ under $\mathbf{U.}(p_W)_*$ or also as the thick hull of the image of $D^b_{art}(\mathbf{S.}(V))$ under either functor $\mathbf{U.}(i_V)^*$ or $\mathbf{U.}(i_V)^!$.

(ii) It follows that this category can be generated up to shifts by either single module $\Lambda^\cdot(W) = \mathbf{U.}(p_W)_*(\Lambda^\cdot(W)) = \mathbf{U.}(i_V)^*(K)$ or also $\omega_{\Lambda^\cdot(W)} = \Lambda^\cdot(W)\otimes_K (\det(W))^{-1} = \mathbf{U.}(i_V)^!(K)$, as K (resp. $\Lambda^\cdot(W)$) generate $D^b_{art}(\mathbf{S.}(V))$, (respectively $D^b_{perf}(\Lambda^\cdot(W))$).

(iii) For the Grothendieck group one finds hence

$$K(D^b_{a/p}(\mathbf{U.})) = \mathrm{cl}(\Lambda^\cdot(W)).\mathbb{Z}[s,s^{-1},\lambda,\lambda^{-1}] \quad ,$$

and, filtering $\Lambda^\cdot(W)$ by the powers of its natural augmentation ideal, it follows that the image of the generator $\mathrm{cl}(\Lambda^\cdot(W))$ in $K(D^b_{art}(\mathbf{U.}))$ is given by

$$\begin{aligned}\mathrm{cl}(\Lambda^\cdot(W)) &= \mathrm{cl}(\bigoplus_j \Lambda^j(W)) = \mathrm{cl}(\bigoplus_j K(0,-j)^{\oplus\binom{w}{j}})\\ &= \mathrm{cl}(K).(1-\lambda^{-1})^w \quad .\end{aligned}$$

As $\Lambda^\cdot(W)$ over $\mathbf{U.}$ can be resolved by an appropriate Koszul complex, one obtains for the class of $\Lambda^\cdot(W)$ in $K(D^b_{perf}(\mathbf{U.})) = K_0(\mathbf{U.})$:

$$\begin{aligned}\mathrm{cl}(\Lambda^\cdot(W)) &= \sum_i (-1)^i \mathrm{cl}(\mathbf{U.}\otimes_K \Lambda^i(V))\\ &= \sum_i (-1)^i \mathrm{cl}(\mathbf{U.}(-i,0)^{\oplus\binom{v}{i}}) = \mathrm{cl}(\mathbf{U.}).(1-s^{-1})^v \quad .\end{aligned}$$

As $cl(\omega_{\Lambda^\cdot(W)}) = cl(\Lambda^\cdot(W)(0,w)) = cl(\Lambda^\cdot(W)).(-\lambda)^w$, the above formulas, if rewritten exclusively using dualizing modules, become :

$$cl(\omega_{\Lambda^\cdot(W)}) = cl(K).(\lambda-1)^w = cl(\omega_{\mathbf{U}.}).(1-s)^v \quad .$$

(iv) Using once again the resolution of $\Lambda^\cdot(W)$ over $\mathbf{U}.$ by a Koszul complex, one obtains

$$\begin{aligned}\bigoplus_{i,j} \mathrm{RHom}_{\mathbf{U}.}(\Lambda^\cdot(W),\Lambda^\cdot(W)(i,j)) &= \bigoplus_{i,j} \mathrm{RHom}_{S.(V)}(K,\Lambda^\cdot(W)(i,j)) \\ &= \bigoplus_{i} \Lambda^\cdot(W) \otimes_K \Lambda^i(V)^*[-i](i,0) \quad ,\end{aligned}$$

so that the Yoneda-Ext-algebra of $\Lambda^\cdot(W)$ over $\mathbf{U}.$ is easily seen to be isomorphic to $\Lambda^\cdot(W) \otimes_K (\Lambda^\cdot(V)^*)^{op} = \Lambda^\cdot(W) \otimes_K \Lambda^\cdot(V^*)$ - with the same change in grading as in the other cases considered.

Of course, the result for $\omega_{\Lambda^\cdot(W)}$ is the same.

(9.3.5) Now let us consider $D^b(\mathbf{U}.)$ itself.

(i) It can be generated up to shifts by either single module $S.(V)$ or $\omega_{S.(V)}$. For this, remark that any finitely generated graded $\mathbf{U}.$-module admits a finite filtration by submodules such that the associated subquotients are annihilated by $W = S_0(V) \otimes_K \Lambda^1(W)$ in $\mathbf{U}.$.

Hence these subquotients are naturally $S.(V)$-modules and, as the polynomial ring $S.(V)$ is of finite global dimension, they admit finite resolutions by graded free $S.(V)$-modules. This shows that $S.(V)$ as a $\mathbf{U}.$-module generates mod-$\mathbf{U}.$, hence also $D^b(\text{mod-}\mathbf{U}.)$.

As $\omega_{S.(V)}$ is a faithfully projective $S.(V)$-module, it generates as well.

(ii) The Grothendieck group of $D^b(\mathbf{U}.)$ equals $K_0'(\mathbf{U}.)$ - using the notation of [Qu 1] . It is freely generated over $\mathbb{Z}[s,s^{-1},\lambda,\lambda^{-1}]$ by the class of $S.(V)$.

To see this, recall first that by Remark (b) above, this class necessarily generates the Grothendieck group. That there is no non-trivial relation follows from the obvious fact that $\mathbf{U}.(i_V)_*$, the forgetful functor associated to i_V , "preserves" the class of $S.(V)$, which is known to generate freely $K_0'(S.(V))$ by [Qu 1;Thm. 6] .

(iii) Using the natural projective resolution of K over $\Lambda^\cdot(W)$, exhibited in (8.5.7.), one obtains

$$\bigoplus_{i,j} \mathrm{RHom}_{\mathbf{U}.}(S.(V),S.(V)(i,j)) = \bigoplus_{i,j} \mathrm{RHom}_{\Lambda^\cdot(W)}(K,S.(V)(i,j))$$

$$= \bigoplus_i S.(V) \otimes_K \Gamma^i(W)^*[-i](0,i) = \bigoplus_i S.(V) \otimes_K S_i(W^*)[-i](0,i) \quad .$$

Hence the Ext-algebra of $S.(V)$ - or also $\omega_{S.(V)}$ - over $\mathbf{U}.$ is given by $S.(V) \otimes_K S.(W^*) = S.(V \oplus W^*)$ - regraded as always.

To state our main theorem below, it is useful to consider still another generator of $D^b(\mathbf{U}.)$, namely the "true" dualizing complex of $S.(V)$, which is given by $\mathbf{K}_V = \omega_{S.(V)}[v]$. Its graded endomorphism ring in the derived category $D^b(\mathbf{U}.)$ is still the same as that of $S.(V)$, and its class in the Grothendieck group $K_0'(\mathbf{U}.) = K(D^b(\mathbf{U}.))$ is apparently

$$cl(\mathbf{K}_V) = cl(S.(V)[v](-v,0)) = cl(S.(V)).(-s)^{-v} \quad .$$

Remark that $\mathbf{K}_V = 0_V^!(K)$, $0_V : (0,0) \longrightarrow (V,0)$ the zero map considered also as the structure morphism from the affine space $\mathbf{A}(V)$ to the point $\mathrm{Spec}(K)$, $0_V^!$ the usual functor from the duality theory for smooth morphisms, Ha;III .

To finish consideration of $D^b(\mathbf{U}.)$ for the moment, observe still how the classes of $\mathbf{U}.$, $\Lambda^\cdot(W)$ and K are expressed in $K_0'(\mathbf{U}.)$ - using the same arguments as in (9.3.4.(iii)) :

$$cl(\mathbf{U}.) = cl(S.(V)).(1-\lambda^{-1})^w \quad ,$$

$$cl(K) = cl(S.(V)).(1-s^{-1})^v \quad , \text{ and}$$

$$cl(\Lambda^\cdot(W)) = cl(S.(V)).(1-\lambda^{-1})^w(1-s^{-1})^v \quad .$$

Now let us name, respectively identify the quotient categories of $D^b(\mathbf{U}.)$ which are obtained by factoring out the various thick subcategories introduced.

(9.3.6.) $D^b(\mathbf{U}.)/D^b_{art}(\mathbf{U}.)$ is naturally equivalent to $D^b(\mathrm{Proj}\text{-}\mathbf{U}.)$. Hence this derived category of $\mathrm{Proj}\text{-}\mathbf{U}.$ can be generated - up to shifts - by the single object $\mathbf{a}(S.(V))$, the sheafification of the $\mathbf{U}.$-module $S.(V)$. It follows that the Grothendieck group of $\mathrm{Proj}\text{-}\mathbf{U}.$ is given by

$$K_0(\mathrm{Proj}\text{-}\mathbf{U}.) = K(D^b(\mathrm{Proj}\text{-}\mathbf{U}.)) = K(D^b(\mathbf{U}.))/K(D^b_{art}(\mathbf{U}.)) \quad ,$$

whence $K_0(\text{Proj-}\mathbf{U}.) = cl(\mathbf{a}(S(V))).\mathbb{Z}[s,s^{-1},\lambda,\lambda^{-1}]/(1-s^{-1})^v$.

Remark: (c) Passing to the projective superspace $\mathbf{P}(V,W)$, one may deduce that its Grothendieck group is obtained from the above by adding the relation $\lambda^2 = 1$. Remark also that the sheaf on $\mathbf{P}(V,W)$ which is the image of $\mathbf{a}(\mathbf{S}.(V))$ is just the structure sheaf of the underlying reduced space $\mathbf{P}(V,W)_{red}$, which in turn is nothing but ordinary projective space $\mathbf{P}(V)$. Hence, the structure sheaf of the underlying reduced space generates the Grothendieck group, but not the structure sheaf itself, which is the image of $\mathbf{a}(\mathbf{U}.)$. More precisely, one has in that group $K_0(\mathbf{P}(V,W))$:

$$\begin{aligned} cl(\mathcal{O}_{\mathbf{P}(V,W)}) &= cl(\mathcal{O}_{\mathbf{P}(V)}).(1-\lambda^{-1})^w \\ &= cl(\mathcal{O}_{\mathbf{P}(V)}).2^w(1-\lambda) \mod (\lambda^2-1) . \end{aligned}$$

This shows a significant difference between "ordinary" and "super" projective geometry. This explains, for example, the phenomenon observed by O.V. Ogievetskii and I.B.Penkov, [O-P], that,extending naturally the theory of Serre duality to projective supermanifolds, the top cohomology group $H^v(\omega_{\mathbf{P}(V,W)})$ of the dualizing sheaf $\omega_{\mathbf{P}(V,W)}$ - which is just the sheaf obtained from $\omega_{\mathbf{U}.}$ - carries a natural trace, but is not one-dimensional. In fact, the "natural" domain of definition for the trace is the group

$$\mathrm{Ext}^v_{\mathcal{O}_{\mathbf{P}(V,W)}}(\mathcal{O}_{\mathbf{P}(V)},\omega_{\mathbf{P}(V,W)}) = H^v(\mathbf{P}(V),\omega_{\mathbf{P}(V)}) = K .$$

Rephrasing the above, the image of $\mathbf{U}.$ in Proj-$\mathbf{U}.$ - or mod-$\mathbf{P}(V,W)$ - generates only the "perfect" coherent sheaves :

(9.3.7.) $D^b_{perf}(\text{Proj-}\mathbf{U}.) =_{def} D^b_{perf}(\mathbf{U}.)/D^b_{a/p}(\mathbf{U}.)$ can be identified - up to an exact equivalence - with the full triangulated subcategory of $D^b(\text{Proj-}\mathbf{U}.)$ which is generated up to shifts by $\mathbf{a}(\mathbf{U}.)$, the structure sheaf of the ringed space associated to Proj-$\mathbf{U}.$. It is also the thick hull of the image of either induced functor

$$i^*_V \, , \, i^!_V : D^b(\mathbf{P}(V)) \longrightarrow D^b(\text{Proj-}\mathbf{U}.) ,$$

represents hence those objects in $D^b(\text{Proj-}\mathbf{U}.)$ which are obtainable from the underlying (ordinary) projective space $\mathbf{P}(V)$ by "lifting".

Accordingly, we set

(9.3.8.) <u>Prim-**U**.</u> = $D^b(\text{Proj-}\mathbf{U}.)/D^b_{perf}(\text{Proj-}\mathbf{U}.)$, and call it the category of (classes of) <u>primitive objects</u> over Proj-**U**. , in analogy to the theory of primitive cycles for projective schemes.

This quotient category is still generated up to shifts by the class of $\mathbf{a}(\mathbf{S}.(V))$. To abbreviate notations, let σ denote the class of this object in the Grothendieck group, to get

$$K(\underline{\text{Prim-}\mathbf{U}.}) = \sigma.\,\mathbb{Z}[s,s^{-1},\lambda,\lambda^{-1}]/(1-s^{-1})^v,(1-\lambda^{-1})^w$$

$$= \sigma.\,\mathbb{Z}[h,g]/h^v,g^w$$

where we have set $h = 1-s^{-1}$, which is the class of a hyperplane in the underlying (reduced or "even") space $\mathbf{P}(V)$ of "geometric K-rational points", (9.1.(i)), whereas $g = 1-\lambda^{-1}$ represents the class of a "line" in the odd quotient space W - a corresponding module of class g is given by $\mathbf{S}.(V) \otimes_K \Lambda^{\cdot}(W)/(W')$, W' a hyperplane in W , which can be identified with the class of a hyperplane in $\mathbf{P}(W^*)$, the projective space of "odd K-rational points", (9.1.(ii)).

Remark that this Grothendieck group is a finitely generated free abelian group of rank vw .

<u>Remark</u>: (d) Passing once again to the projective superspace $\mathbf{P}(V,W)$, we can as well form the category of primitive objects whose Grothendieck group, obtained from the above by adding the relation $g(g+2) = 0$, is the direct sum of a free group of rank v and a "large" 2-torsion group.

Observe that <u>Prim-**U**.</u> can be obtained directly from $D^b(\mathbf{U}.)$ - up to a natural equivalence - by factoring out the thick hull of both objects **U**. and K . Hence, we may also think of <u>Prim-**U**.</u> as representing the primitive classes (of complexes) of modules over **U**. . Going the other way round, we could factor out first the perfect complexes, i.e. the hull of **U**. , to obtain - up to an exact equivalence - by Theorem (4.4.1.) :

(9.3.9.) <u>MCM(**U**.)</u> , the category of graded maximal Cohen-Macaulay modules over **U**. , or, equivalently, <u>APC(**U**.)</u> , the homotopy category of graded complete resolutions over **U**. .

Remark that $\mathbf{S}.(V)$, the generator of $D^b(\mathbf{U}.)$ is already a MCM over **U**. , (8.3.1.) or (8.5.7.), so that it generates <u>MCM(**U**.)</u> as a triangulated category up to shifts - there is no need to pass to the MCM-approximation first.

: The natural complete resolution of $S.(V)$ over $\mathbf{U}.$, described in (8.5.7), generates then - up to shifts - the triangulated category $\underline{APC(\mathbf{U}.)}$.

It follows that the Grothendieck group of MCM's over $\mathbf{U}.$ modulo projectives is

$$K(\underline{MCM(\mathbf{U}.)}) = K(\underline{APC(\mathbf{U}.)}) = \sigma.\ \mathbb{Z}[s,s^{-1},\lambda,\lambda^{-1}]/(1-\lambda^{-1})^{w}$$

$$= \sigma.\ \mathbb{Z}[s,s^{-1},g]/g^{w}$$

where σ and g are defined as before, (remark that the module representing g is also already MCM, so that g keeps its original meaning).

(9.3.10.) $\underline{MCM_{art}(\mathbf{U}.)}$ will denote the thick hull in $\underline{MCM(\mathbf{U}.)}$ of the MCM-approximation of K. It is equivalent to both the full subcategory of MCM-approximations of artinian $\mathbf{U}.$-modules as well as the quotient category $D^{b}_{art}(\mathbf{U}.)/D^{b}_{a/p}(\mathbf{U}.)$.

As forming subsequent quotients of triangulated categories is - up to exact equivalences - independent of the order, it follows

Proposition 9.3.11.: The quotient categories of primitive classes of complexes of coherent sheaves on $Proj$-$\mathbf{U}.$, $\underline{Prim\text{-}\mathbf{U}.}$, and of MCM's modulo MCM-approximations of artinian modules, $\underline{MCM(\mathbf{U}.)}/\underline{MCM_{art}(\mathbf{U}.)}$, are equivalent as triangulated categories.

To resume the situation, we have the following commutative diagram of triangulated categories and exact functors, whose rows and columns in plein arrows are exact sequences of triangulated categories.

A dotted arrow A ---> B indicates that B is the thick hull of A under a suitable exact functor :

$$\begin{tikzcd}
 & D^b_{perf}(\Lambda^{\cdot}(W)) \arrow[r] \arrow[d, dashed] & D^b(\Lambda^{\cdot}(W)) \arrow[r] \arrow[d, dashed] & \underline{MCM(\Lambda^{\cdot}(W))} \arrow[r] \arrow[d, dashed] & 0 \\
D^b_{art}(\mathbf{S}.(V)) \arrow[r, dashed] \arrow[d] & D^b_{a/p}(\mathbf{U}.) \arrow[r] \arrow[d] & D^b_{art}(\mathbf{U}.) \arrow[r] \arrow[d] & \underline{MCM}_{art}\underline{(\mathbf{U}.)} \arrow[r] \arrow[d] & 0 \\
D^b(\mathbf{S}.(V)) \arrow[r, dashed] \arrow[d] & D^b_{perf}(\mathbf{U}.) \arrow[r] \arrow[d] & D^b(\mathbf{U}.) \arrow[r] \arrow[d] & \underline{MCM(\mathbf{U}.)} \arrow[r] \arrow[d] & 0 \\
D^b(\mathbf{P}(V)) \arrow[r, dashed] \arrow[d] & D^b_{perf}(Proj\text{-}\mathbf{U}.) \arrow[r] \arrow[d] & D^b(Proj\text{-}\mathbf{U}.) \arrow[r] \arrow[d] & \underline{Prim\text{-}\mathbf{U}.} \arrow[r] \arrow[d] & 0 \\
0 & 0 & 0 & 0 &
\end{tikzcd}$$

(9.3.12.)

Now we come to the main result of this chapter :

9.4. The Bernstein-Gelfand-Gelfand-correspondence

To abbreviate the formulation of the theorem, recall that the base field K can be interpreted as $\mathbf{U}.(0)$, the enveloping algebra of the zero-space $0 = (0,0)$.

Passing to the cohomology of an object in $D^b(\mathbf{U}.(0))$ identifies this latter category with the category of triply $\mathbb{Z}$-graded K-vectorspaces of finite total dimension.

This category is obviously generated up to shifts by the single object K , and its Grothendieck group is $\mathbb{Z}[s,s^{-1},\lambda,\lambda^{-1}]$, the class of the graded vectorspace $K[k](i,j)$ being represented by $(-1)^{k+j}s^i\lambda^j$ in view of our general convention in (9.3.3.(iii)).

Denote by γ the functor which regrades a triply graded vectorspace according to

$$\gamma(K[k](i,j)) = K[k+i+j](-j,-i) \quad .$$

This functor is exact - as $\gamma(-[k]) = \gamma(-[k](0,0)) = -[k](0,0) = -[k]$ - with respect to the triangulated structure and involutive. The induced automorphism of the Grothendieck group sends s into λ^{-1} and λ into s^{-1} .

For any given linear superspace (V,W) , we will denote - in analogy to (9.3.5.(iii)) - by

$$^{!}0_{(V,W)} : D^b(\mathbf{U}.(0)) \longrightarrow D^b(\mathbf{U}.(V,W))$$

the unique exact functor which commutes with the shifts $-(i,j)$ and satisfies

$$^{!}0_{(V,W)}(K) = \mathbf{K}_V = \dot{\omega}_V[v] \quad ,$$

where $\mathbf{K}_V$ is considered a complex of $\mathbf{U}.(V,W)$-modules as usual.

Beilinson's criterion (9.3.1.) , applied to $D^b(\mathbf{U}.(V,W))$ can then be expressed as follows :

Assume that F is any exact functor from $D^b(\mathbf{U}.(V,W))$ into another triangulated category. Then it is essentially uniquely determined by its composition with $^{!}0_{(V,W)}$.

If now $\mathbf{D}(V,W) = (W^*,V^*)$ is the "dual" linear superspace to (V,W) , one has from (9.3.5.(iii)) :

$$\mathrm{Ext}^k_{\mathbf{U}.(V,W)}(\mathbf{K}_V,\mathbf{K}_V)_{(i,j)} = \mathrm{Ext}^{k+i+j}_{\mathbf{U}.(\mathbf{D}(V,W))}(\mathbf{K}_{W*},\mathbf{K}_{W*})_{(-j,-i)}$$

for all integers k,i,j . (Namely, the indicated vectorspace is zero, except for $k+j = 0$, in which case it equals $S_i(V) \otimes_K S_k(W^*)$.)

In other words, the identity on $\mathbf{S}.(V \oplus W^*)$ furnishes a natural identification of triply graded rings

$$\mathrm{Ext}^{\cdot}_{\mathbf{U}.(V,W)}(\mathbf{K}_V,\mathbf{K}_V)_{(.,.)} = \gamma(\mathrm{Ext}^{\cdot}_{\mathbf{U}.(\mathbf{D}(V,W))}(\mathbf{K}_{W*},\mathbf{K}_{W*})_{(.,.)}) \quad . \tag{9.4.1.}$$

This is the key to :

<u>Theorem 9.4.2.</u>: Let (V,W) be a linear superspace over K . Then there exists a (covariant) exact functor

$$\mathbf{b}_{(V,W)} : D^b(\mathbf{U}.(V,W)) \longrightarrow D^b(\mathbf{U}.(\mathbf{D}(V,W)))$$

together with a natural isomorphism of functors on $D^b(\mathbf{U}.(0))$

$$^{!}0_{\mathbf{D}(V,W)}\circ\gamma \xrightarrow{\sim} \mathbf{b}_{(V,W)}\circ {}^{!}0_{(V,W)} \quad ,$$

which induces the isomorphism (9.4.1.) of triply graded rings if evaluated on the shifts and translates of K .

The functor $\mathbf{b}_{(V,W)}$ is essentially unique, necessarily an exact equivalence and admits $\mathbf{b}_{\mathbf{D}(V,W)}$ as its inverse.

Proof: The uniqueness of $\mathbf{b}_{(V,W)}$ follows immediately from Beilinson's criterion :

$$ {}^!O_{\mathbf{D}(V,W)} \circ \gamma(K[k](i,j)) = {}^!O_{\mathbf{D}(V,W)}(K[k+i+j](-j,-i)) $$
by definition of γ ,
$$ = \mathbf{K}_{W^*}[k+i+j](-j,-i) $$
by definition of ${}^!O_{\mathbf{D}(V,W)}$,

whereas

$$ \mathbf{b}_{(V,W)} \circ {}^!O_{(V,W)}(K[k](i,j)) = \mathbf{b}_{(V,W)}(\mathbf{K}_V[k](i,j)) \quad , $$

by definition of ${}^!O_{(V,W)}$. Hence $\mathbf{b}_{(V,W)}$ is determined on the generating family $\mathbf{K}_V(i,j)$ by the theorem, whence the uniqueness.

As its image is precisely a generating family of $D^b(\mathbf{U}.(\mathbf{D}(V,W))$ and as by hypothesis the induced maps on the morphism groups are isomorphisms, it follows also that $\mathbf{b}_{(V,W)}$ is an exact equivalence. Replacing (V,W) by its "dual" $\mathbf{D}(V,W) = (W^*,V^*)$, the same criterion shows that $\mathbf{b}_{\mathbf{D}(V,W)}$ is an inverse of $\mathbf{b}_{(V,W)}$.

The non-trivial part is the <u>existence</u>. Modifying slightly the sketch of proof in [BGĜ], a functor $\mathbf{b}$ as claimed can be obtained in the following way, (where we suppress once again (V,W) from the notations) :

First, let $N^\cdot_\cdot = \oplus N_i^j$ be any $\mathbf{U}.$-module which is locally finite for the total degree. Associate to it a double complex over $\mathbf{UD}.$:

Its term in (complex-)bidegree (j,i) is given by $N_i^j \otimes_K \mathbf{UD}.(j,i)$, and its differentials

$$ {}'d_i^j : N_i^j \otimes_K \mathbf{UD}.(j,i) \longrightarrow N_i^{j+1} \otimes_K \mathbf{UD}.(j+1,i) $$

and

$$ {}''d_i^j : N_i^j \otimes_K \mathbf{UD}.(j,i) \longrightarrow N_{i+1}^j \otimes_K \mathbf{UD}.(j,i+1) $$

are given by those $\mathbf{UD}.$-linear maps in

$$ \mathrm{Hom}_{\mathbf{UD}.}(N_i^j \otimes_K \mathbf{UD}.(j,i), N_i^{j+1} \otimes_K \mathbf{UD}.(j+1,i)) = \mathrm{Hom}_K(N_i^j, N_i^{j+1} \otimes_K W^*) $$

and

$$ \mathrm{Hom}_{\mathbf{UD}.}(N_i^j \otimes_K \mathbf{UD}.(j,i), N_{i+1}^j \otimes_K \mathbf{UD}.(j,i+1)) = \mathrm{Hom}_K(N_i^j, N_{i+1}^j \otimes_K V^*) $$

respectively, which correspond by adjunction to the action of the "odd 1-forms" in W and the action of the "even 1-forms" in V respectively on the $\mathbf{U}.$-module $N^\cdot_\cdot$.

The verification that $(N_i^j \otimes_K \mathbf{UD}.(j,i), {}'d, {}''d)$ constitutes indeed a double complex is immediate. Also, a $\mathbf{U}.$-linear morphism of modules yields obviously a morphism of the corresponding double complexes.

As $N^{\cdot}_{\cdot}$ is by assumption locally finite for the total degree, (which was not needed yet), the associated total complex, denoted $\mathbf{d}(\dot{N})$, consists of finitely generated, graded free $\mathbf{UD}.$-modules, such that the term in complex-degree k can be generated by homogeneous elements of degree $(-j,-i)$ with $j+i=k$. In other words, this complex is <u>linear</u> for the total degree. (Conversely, one sees immediately, that such a linear complex determines a unique $\mathbf{U}.$-module, locally finite for the total degree, of which it is the image under $\mathbf{d}$. As furthermore two homomorphisms of linear complexes can only be homotopic if they are equal, it follows that $\mathbf{d}$ establishes in fact an equivalence of categories between $\mathrm{Mod}_{lf}\text{-}\mathbf{U}.$, (9.2.), and the homotopy category of linear complexes over $\mathbf{UD}.$.)

Now $\mathbf{d}$ can be extended to a functor on finite complexes over $\mathbf{U}.$ by applying it to every term and the differentials - resulting in a double complex whose "rows" are linear complexes - and then passing to the associated total complex again.

This extension is compatible with translation and carries homotopies into homotopies. As $\mathbf{d}$ transforms evidently short exact sequences of $\mathbf{U}.$-modules into short exact sequences of complexes over $\mathbf{UD}.$, it maps necessarily finite acyclic complexes over $\mathbf{U}.$ into acyclic complexes over $\mathbf{UD}.$ and passes hence trivially to the corresponding derived categories, so that we end up with an exact functor - again denoted $\mathbf{d}$ - from $D^b(\mathrm{Mod}_{lf}\text{-}\mathbf{U}.)$ into $D(\mathrm{Mod}\text{-}\mathbf{UD}.)$.

Now let us consider $\mathbf{d}(\mathbf{S}.(V))$. By construction, this complex is isomorphic (as a complex of graded K-vectorspaces) to

$$\left(\bigoplus_i \mathbf{S}_i(V) \otimes_K \Lambda^{\cdot}(V^*)(i)[-i] \otimes_K \mathrm{S}.(W^*)\ ,\ \delta\right)$$

which is nothing but the usual Koszul complex $(\mathbf{S}.(V) \otimes_K \Lambda^{\cdot}(V^*),\ d)$ - up to a sign in the differential - tensored with $\mathrm{S}.(W^*)$ over K.

This shows :

(i) The inclusion of the $\mathbf{UD}.$-submodule $1 \otimes_K \det(V^*) \otimes_K \mathrm{S}.(W^*)$ in complex-degree zero - which is isomorphic to $\mathrm{S}.(W^*)[0](0,-v)$ as a "complex" - into $\mathbf{d}(\mathbf{S}.(V))$ is a quasi-isomorphism of complexes.

By inspection, it is also clear that

(ii) $\mathbf{d}(-[k](i,j)) = \mathbf{d}(-)[k+i+j](-j,-i)$ for all integers k,i,j and

(iii) The induced morphisms of vectorspaces from

$$\mathrm{Hom}_{D^b(\mathrm{Mod}_{lf}\text{-}\mathbf{U}.)}(\mathbf{S}.(V),\mathbf{S}.(V)[k](i,j)) = \mathrm{Ext}^k_{\mathbf{U}.}(\mathbf{S}.(V),\mathbf{S}.(V))_{(i,j)}$$

into

$$\mathrm{Ext}_{\mathbf{UD}.}^{k+i+j}(\mathbf{d}(\mathbf{S}.(V)),\mathbf{d}(\mathbf{S}.(V)))_{(-j,-i)} = \mathrm{Ext}_{\mathbf{UD}.}^{k+i+j}(\mathbf{S}.(W^*),\mathbf{S}.(W^*))_{(-j,-i)}$$

can be identified, by (9.3.5.(iii)), (9.4.1.), as the natural isomorphism $S_i(V) \otimes_K S_k(W^*) = S_k(W^*) \otimes_K S_i(V)$ for $k+j = 0$, (and as the isomorphism of zero-objects if $k+j \neq 0$).

As $K_V = \omega_V[v] = \mathbf{S}.(V)[v](-v,0)$ in $D^b(\mathbf{U}.)$, "renormalizing" $\mathbf{d}$ to

$$\mathbf{b}(-) = \mathbf{d}(- \otimes_K (\det W)^{-1}) = \mathbf{d}(-(0,w)) = \mathbf{d}(-)[w](-w,0)$$

will then yield an exact functor from $D^b(\mathrm{Mod}_{lf}\text{-}\mathbf{U}.)$ into $D(\mathrm{Mod}\text{-}\mathbf{UD}.)$ which satisfies

$$\begin{aligned}\mathbf{b}(K_V) &= \mathbf{b}(\mathbf{S}.(V)[v](-v,0)) = \mathbf{d}(\mathbf{S}.(V)[v](-v,w)) \\ &= \mathbf{d}(\mathbf{S}.(V))[w](-w,v) \qquad \text{by (ii)} .\end{aligned}$$

Into this complex

$$K_{W^*} = \omega_{W^*}[w] = \mathbf{S}.(W^*)[w](-w,0)$$

embeds quasi-isomorphically by (i), so that there is in fact a natural isomorphism in $D(\mathrm{Mod}\text{-}\mathbf{UD}.)$ from K_{W^*} into $\mathbf{b}(K_V)$.

From (ii) and (iii) it follows that the induced morphism on the graded endomorphism ring of K_V coincides with the identification of (9.4.1.), so that the restriction of $\mathbf{b}$ to $D^b(\mathbf{U}.)$ satisfies the requirements of the Theorem.

qed

Remark: (a) The functors $\mathbf{b}$ or $\mathbf{d}$ do not embed $D^b(\mathrm{Mod}_{lf}\text{-}\mathbf{U}.)$ into $D(\mathrm{Mod}\text{-}\mathbf{UD}.)$: As can be easily deduced either from the following or the explicit construction, both functors "vanish" precisely on those complexes which are $\mathrm{Ext}^{\cdot}_{\mathbf{U}.}(K,-)_{(.,.)}$-acyclic.

Hence, the point of the Theorem is essentially Nakayama's Lemma : It is equivalent - as $\mathbf{U}.$ is strongly Gorenstein - to the statement that no complex X in $D^b(\mathbf{U}.)$ - which is not acyclic - can satisfy

$$\mathrm{Ext}^k_{\mathbf{U}.}(K,X)_{(i,j)} = 0 \quad \text{for all integers } k,i,j .$$

Namely, $\mathrm{Ext}^k_{\mathbf{U}.}(K,X)_{(i,j)} = \mathrm{Ext}^k_{\mathbf{U}.}(\mathrm{RHom}_{\mathbf{U}.}(X,\mathbf{U}.),\mathrm{RHom}_{\mathbf{U}.}(K,\mathbf{U}.))_{(i,j)}$, as $\mathbf{U}.$ is strongly Gorenstein, cf. (4.6.), and graded commutative,

$$= \operatorname{Ext}^k_{\mathbf{U}.}(\operatorname{RHom}_{\mathbf{U}.}(X,\mathbf{U}.),\mathbf{det}(V,W)^{-1}[-v])_{(i,j)}$$

by (8.5.1.(i))

$$= \operatorname{Ext}^{k-v}_{\mathbf{U}.}(\operatorname{RHom}_{\mathbf{U}.}(X,\mathbf{U}.),K)_{(i+v,j-w)}$$

as $\mathbf{det}(V,W) = K(-v,w)$ by definition,

and this last vectorspace is canonically isomorphic to the K-dual of

$$\operatorname{Tor}^{\mathbf{U}.}_{k-v}(K,\operatorname{RHom}_{\mathbf{U}.}(X,\mathbf{U}.))_{(-i-v,w-j)} .$$

As with X also $\operatorname{RHom}_{\mathbf{U}.}(X,\mathbf{U}.)$ is a non-acyclic object in $D^b(\mathbf{U}.)$, the Lemma of Nakayama applies.

To identify more precisely the image of mod-$\mathbf{U}.$ under **b** or **d** , we make the

Definition 9.4.3.: Let C be any complex of finitely generated, graded free modules over $\mathbf{U}.$ (or $\mathbf{UD}.$) . Then

(i) C is almost linear (for the total degree), if there exists an integer l such that C^{-k} can be generated by homogeneous elements of bidegree (i,j) with $k-l \leqq i+j \leqq k+l$ for all k .
In this case, we say that the deviation (from linearity) of C is (uniformly) bounded by l .

(ii) C is a (generalized) monad if it is an almost linear complex which is bounded below and of bounded cohomology. The homotopy category of all monads over $\mathbf{U}.$ will be denoted $\operatorname{Mon}(\mathbf{U}.)$, its full subcategory of linear monads by $\operatorname{Mon}^0(\mathbf{U}.)$.

Remark that, by definition, a (generalized) monad is just a special projective co-resolution, (5.6.), whence (generalized) monads could as well be called almost linear projective co-resolutions.

It is clear, that being almost linear or a monad are properties of a complex which are stable under translation and which are preserved by forming mapping cones. Hence, almost linear complexes or monads form each a full triangulated subcategory of the homotopy category of "all" complexes.

Using the terms just defined, the proof of the Theorem above yields

Corollary 9.4.4.: With the notations as before, one has :

(i) There exists an exact equivalence from $D^b(\mathbf{U}.)$ to Mon($\mathbf{U}.$) which associates to any object in $D^b(\mathbf{U}.)$ an almost linear projective co-resolution.

(ii) The functor $\mathbf{b}$ establishes an exact equivalence from $D^b(\mathbf{U}.)$ to Mon($\mathbf{UD}.$) . It maps mod-$\mathbf{U}.$ isomorphically onto the abelian category $\mathrm{Mon}^0(\mathbf{UD}.)$ of linear monads over $\mathbf{UD}.$.

(iii) There exists a t-structure, [BBD;1.3.], on $D^b(\mathbf{U}.)$ whose heart is given by all such complexes which admit a linear projective co-resolution.

The proof of (ii) follows immediately from the proof of (9.4.2.). Namely, it has been observed there, that $\mathbf{d}$ (or $\mathbf{b}$) transforms any $\mathbf{U}.$-module, which is locally finite for the total degree, into a linear complex. But if such a module is finitely generated over $\mathbf{U}.$, the corresponding linear complex is necessarily bounded below, (9.2.). That the cohomology of this complex is bounded (and also once again that the complex itself is bounded below) is then a formal consequence of the fact that this is true for the complexes associated to a generating family, say $\mathbf{S}.(V)$ up to shifts, as established directly during the proof of (9.4.2.). This shows that $\mathbf{d}$ (or $\mathbf{b}$) , if restricted to $D^b(\mathbf{U}.)$ or mod-$\mathbf{U}.$, take the images claimed. As on the other hand Mon($\mathbf{UD}.$) is - by (6.2.1.(i)(b)) - a full subcategory of D(Mod-$\mathbf{UD}.$) , which lies by definition "inside" $D^b(\mathbf{UD}.)$, the Theorem implies that Mon($\mathbf{UD}.$) is already equivalent to $D^b(\mathbf{U}.)$ under either functor $\mathbf{d}$ or $\mathbf{b}$.

(i) is now obtained from (ii) by exchanging the roles of $\mathbf{U}.$ and $\mathbf{UD}.$: Given a complex X in $D^b(\mathbf{U}.)$, represent its image under $\mathbf{b}$ by some finite complex in $D^b(\mathbf{UD}.)$ and apply $\mathbf{b}^{-1}$. Making these choices "coherently" - without asking a logician - yields the exact equivalence.
The t-structure claimed in (iii) is obtained from the natural one on $D^b(\mathbf{UD}.)$ by applying $\mathbf{b}^{-1}$.

Remark: (b) As $\mathbf{U}.$ is graded and strongly Gorenstein, it follows from (5.6.) a priori that any object in $D^b(\mathbf{U}.)$ admits a graded projective co-resolution. The additional information provided here is that such a projective co-resolution can be chosen to be almost linear !

Conversely, although the proof of (9.4.2.) did not use explicitely the fact that $\mathbf{U}.$ is strongly Gorenstein, one of the significant properties of such rings, namely the existence of projective co-resolutions - and then also automatically of complete resolutions, see (3.1.) and (5.6.), - follows from the (proof of the) Theorem.

: Now we can compile a dictionnary between data over $\mathbf{U}.(V,W)$ and the corresponding data over $\mathbf{U}.(\mathbf{D}(V,W))$ under the equivalence $\mathbf{b}_{(V,W)}$ for any linear superspace (V,W) . The entries in the following list have to be read "up to isomorphism or natural equivalence" whatever applies :

(9.4.5.)

$$\text{data over } \mathbf{U}. = \mathbf{U}.(V,W) \;\underset{\longleftarrow}{\xrightarrow{\mathbf{b}}}\; \text{data over } \mathbf{UD}. = \mathbf{U}.(W^*,V^*)$$

complexes and modules	K_V	K_{W*}
	$\mathbf{U}.$	$\det(V^*) = K(0,-v)$
	$\det(W) = K(0,-w)$	$\mathbf{UD}.$
	$\Lambda^{\cdot}(W)$	$\Lambda^{\cdot}(V^*)$
gradings on $D^b(-)$ and the map on $K_0'(-)$	$[k](i,j)$	$[k+i+j](-j,-i)$
	$cl(\mathbf{S}.(V)).s^i\lambda^j$	$cl(\mathbf{S}.(W^*)).(-1)^{v+w}s^{-w-j}\lambda^{-v-i}$
subcategories of $D^b(-)$	$D^b_{perf}(\mathbf{U}.)$	$D^b_{art}(\mathbf{UD}.)$
	$D^b_{art}(\mathbf{U}.)$	$D^b_{perf}(\mathbf{UD}.)$
	$D^b_{a/p}(\mathbf{U}.)$	$D^b_{a/p}(\mathbf{UD}.)$
quotient-categories of $D^b(-)$	$\underline{MCM(\mathbf{U}.)}$	$D^b(\text{Proj-}\mathbf{UD}.)$
	$D^b(\text{Proj-}\mathbf{U}.)$	$\underline{MCM(\mathbf{UD}.)}$
	$\underline{Prim(\mathbf{U}.)}$	$\underline{Prim(\mathbf{UD}.)}$
cohomological functors on $D^b(-)$	$H^k(-)_{(i,j)}$	$Ext^{k+i+j}_{\mathbf{UD}.}(K,\mathbf{b}(-))_{(-j,-i+v)}$
	$Ext^k_{\mathbf{U}.}(-,\mathbf{U}.)_{(i,j)}$	$Ext^{k+i+j}_{\mathbf{UD}.}(\mathbf{b}(-),K)_{(-j,-i-v)}$

<u>The proof</u> is left to the reader. It follows immediately from the explicit description of $\mathbf{b}$ and (9.3.).

Remark that $\mathbf{b}$ "reflects" the diagram (9.3.12.) at the diagonal.

Without even knowing $\mathbf{b}$ explicitely, one may deduce the above dictionary just from Beilinson's criterion and the following, easily established

Lemma 9.4.6.: Let $F = (F^+, F^-) : (V,W) \longrightarrow (V',W')$ be a linearly perfect morphism of linear superspaces over K. Then

(i) F admits a unique factorization $F = f \circ \mathbf{D}(g)$, such that the even components of both f and g are identities.

(ii) If F^+ is an isomorphism, $\mathbf{U}.(F)_*(K_{V'})$ is isomorphic to K_V.

(iii) If F^- is an isomorphism, then $\mathbf{U}.(F)^!(K_V)$ is isomorphic to $K_{V'}$.

For the proof of (i) just observe that necessarily $f = (id_{V'}, F^-)$ and $g = (id_{W^*}, (F^+)^*)$. The remaining assertions are once again easy exercises on Koszul complexes.

Corollary 9.4.7.: Let F be a linearly perfect morphism as above, $F = f \circ \mathbf{D}(g)$ its decomposition as in (9.4.6.(i)). Then the following diagram of functors commutes - up to a natural isomorphism of functors :

$$\begin{tikzcd}
D^b(\mathbf{U}.(V,W)) \arrow[r, "\mathbf{b}_{(V,W)}"] & D^b(\mathbf{U}.(\mathbf{D}(V,W))) \\
D^b(\mathbf{U}.(V',W)) \arrow[u, "\mathbf{U}.(\mathbf{D}(g))_*"] \arrow[r, "\mathbf{b}_{(V',W)}"] & D^b(\mathbf{U}.(\mathbf{D}(V',W))) \arrow[u, "\mathbf{U}.(g)^*"'] \\
D^b(\mathbf{U}.(V',W')) \arrow[u, "\mathbf{U}.(f)_*"] \arrow[r, "\mathbf{b}_{(V',W')}"] \arrow[uu, bend left=60, "\mathbf{U}.(F)_*"] & D^b(\mathbf{U}.(\mathbf{D}(V',W'))) \arrow[u, "\mathbf{U}.(\mathbf{D}(f))^!"']
\end{tikzcd}$$

Proof: The commutativity of the upper square follows from the commutativity of the lower one by adjunction.

For the lower square just use Beilinson's criterion, (9.4.2.) and (9.4.6.(ii),(iii)).

Now take for example $F = p_W : (V,W) \longrightarrow (0,W)$, the projection onto the odd quotient of (V,W). Then the Corollary implies immediately that $\mathbf{b}_{(V,W)}$ induces an exact equivalence between $D^b_{art}(\mathbf{U}.)$ and $D^b_{perf}(\mathbf{UD}.)$. Similar arguments establish the other entries of (9.4.5.).

For further use, let us also record the following which cannot be obtained directly from the Theorem (9.4.2.) but rather from the explicit construction of $\mathbf{d}$ in its proof :

: Let $F : (V,W) \longrightarrow (V',W')$ be an <u>arbitrary</u> morphism of linear superspaces. Then the forgetful functor - or "extension of scalars" - associated to $\mathbf{U}.(F)$ is the identity on the underlying graded K-vectorspaces, hence preserves trivially the property of being locally finite for the total degree. As it is also exact, it passes "as such" to the derived categories, yielding an exact functor $\mathbf{U}.(F)_*$ from $D^b(\mathrm{Mod}_{lf}\text{-}\mathbf{U}.(V',W'))$ into $D^b(\mathrm{Mod}_{lf}\text{-}\mathbf{U}.(V,W))$.

On the other hand, tensor-product with $\mathbf{U}.(\mathbf{D}(V,W))$ over $\mathbf{U}.(\mathbf{D}(V',W'))$ - or the "change-of-rings" along $\mathbf{U}.(\mathbf{D}(F))$ - transforms almost linear complexes over $\mathbf{U}.(\mathbf{D}(V',W'))$ into such complexes over $\mathbf{U}.(\mathbf{D}(V,W))$ and preserves homotopies. It induces consequently an exact functor $\mathbf{U}.(\mathbf{D}(F))^*$ from the homotopy category of almost linear complexes over $\mathbf{U}.(\mathbf{D}(V',W'))$ with its natural triangulated structure, (9.4.3.), into the corresponding category over $\mathbf{U}.(\mathbf{D}(V,W))$.

Now the explicit construction of $\mathbf{d}_{(V,W)}$ shows that one has a natural equality of functors

(9.4.8.) $$\mathbf{d}_{(V,W)} \circ \mathbf{U}.(F)_* = \mathbf{U}.(\mathbf{D}(F))^* \circ \mathbf{d}_{(V',W')} \quad ,$$

and this implies (9.4.7.) in case that F is linearly perfect.

Similarly, passing to the graded K-dual of a (complex of) graded module(s) is exact and transforms modules which are locally finite for the total degree into such, hence induces a functor - denoted $(-)^\#$ as in (8.5.5.) - from $D^b(\mathrm{Mod}_{lf}\text{-}\mathbf{U}.(V,W))$ into $D^b(\mathrm{Mod}_{lf}\text{-}\mathbf{U}.(V,W))^{op}$.

Using then the appropriate sign convention for the dual of a complex - as explained for example in [SGA XVII;1.1.5.1.] - , it follows that there is a natural equality of functors

(9.4.9.) $$\mathbf{d}^{op}_{(V,W)}(-^\#) = \mathrm{Hom}^{\cdot}_{\mathbf{U}.(\mathbf{D}(V,W))}(\mathbf{d}_{(V,W)}(-),\mathbf{U}.(\mathbf{D}(V,W)))$$

from $D^b(\mathrm{Mod}_{lf}\text{-}\mathbf{U}.(V,W))$ into the opposite of the homotopy category of almost linear complexes over $\mathbf{U}.(\mathbf{D}(V,W))$.

Finally, if M and N are two graded $\mathbf{U}.(V,W)$-modules such that their tensor-product over K is locally finite for the total degree, one has the following equality, natural in both arguments :

(9.4.10.) $$\mathbf{d}(M \otimes_K N) = \mathbf{d}(M) \otimes_{\mathbf{U}.(\mathbf{D}(V,W))} \mathbf{d}(N) \quad .$$

Now we come back to our main theme and apply the correspondence to

9.5. Maximal Cohen-Macaulay modules and coherent sheaves

First, let us state explicitely the meaning of the functor $\mathbf{b}$ for maximal Cohen-Macaulay modules over a ring of the form $S.(V) \otimes_K \Lambda^{\cdot}(W)$:

Proposition 9.5.1.: The functor $\mathbf{b}$ induces an exact equivalence between the categories $\underline{MCM(\mathbf{U}.(V,W))} = D^b(\mathbf{U}.(V,W))/D^b_{perf}(\mathbf{U}.(V,W))$ and $D^b(\text{Proj-}\mathbf{U}.(\mathbf{D}(V,W))) = D^b(\mathbf{U}.(\mathbf{D}(V,W)))/D^b_{art}(\mathbf{U}.(\mathbf{D}(V,W)))$ with their respective natural triangulated structure. In particular :

(i) $\underline{MCM(\mathbf{U}.(V,W))}$ comes equipped with a t-structure whose heart is equivalent to the abelian category $\text{Proj-}\mathbf{U}.(W^*,V^*)$. The associated cohomological functor, see [BBD;Thm.1.3.6.], is given by

$$H^\circ : \underline{MCM(\mathbf{U}.(V,W))} \longrightarrow \text{Proj-}\mathbf{U}.(W^*,V^*) \quad ,$$

$$H^\circ(M) = \mathbf{a}\Big(\bigoplus_{i,j} \mathrm{Ext}^{i+j}_{\mathbf{U}.(V,W)}(K,M)_{(-j,-i+w)}\Big) \ ,$$

where the direct sum on the right is considered a (right) graded module over $\mathbf{U}.(W^*,V^*) = \mathrm{Ext}^{\cdot}_{\mathbf{U}.(V,W)}(K,K)_{(\cdot,\cdot)}$ with respect to the Yoneda-product, cf. (9.3.2.).

(ii) Under this equivalence, Tate-cohomology over $\mathbf{U}.(V,W)$ becomes transformed into coherent sheaf cohomology over $\text{Proj-}\mathbf{U}.(W^*,V^*)$. More precisely, for any graded maximal Cohen-Macaulay $\mathbf{U}.(V,W)$-module M , (or any complex in $\underline{D^b(\mathbf{U}.(V,W))}$), we have

$$\begin{aligned}\hat{H}^i_{(V,W)}(M \otimes_K (\det W)^{-1}) &=_{def} \underline{\mathrm{Ext}}^i_{\mathbf{U}.(V,W)}(K, M \otimes_K (\det W)^{-1}) \\ &= \mathrm{Ext}^i_{\text{Proj-}\mathbf{U}.(W^*,V^*)}(\mathbf{a}(\mathbf{U}.(W^*,V^*)), \mathbf{a}\circ\mathbf{b}(M)) \\ &=_{def} H^i(\text{Proj-}\mathbf{U}.(W^*,V^*), \mathbf{a}\circ\mathbf{b}(M)) \quad .\end{aligned}$$

(iii) The duality for Tate-cohomology over $\mathbf{U}.(V,W)$ - see (7.7.5.) and (8.5.5.&6.) - translates into Grothendieck-Serre duality for the cohomology of (complexes of) coherent sheaves of modules on the ringed space $\text{Proj-}\mathbf{U}.(W^*,V^*)$ - formulated in [O-P] for the analogous $\mathbb{Z}/2\mathbb{Z}$-graded case.

The proof consists just in a reformulation of (part of) the dictionary (9.4.5.) and we restrict ourselves to explain (iii) . (The fixed linear superspace (V,W) is again dropped from the notation.)

From Remark (a) in (9.2.) recall that the canonical module of $\mathbf{UD}.$ over K is given by $\omega_{\mathbf{UD}.} = \mathbf{UD}. \otimes_K \mathbf{det}(W^*,V^*) = \mathbf{UD}.(-w,v)$. The table (9.4.5.) shows then that one has an isomorphism in $D^b(\mathbf{UD}.)$:

$$\omega_{\mathbf{UD}.} \longrightarrow \mathbf{b}((\det W)[v-w](-v,w)) = \mathbf{b}((\det V)[v-w]) \quad .$$

Let $\omega = \mathbf{a}(\omega_{\mathbf{UD}.})$ denote the sheafification of the dualizing module. Not surprisingly, it will serve as the dualizing sheaf on Proj-$\mathbf{UD}.$ and the induced sheaf on the "underlying" projective superspace $\mathbf{P}(W^*,V^*)$, see (9.2.) and Remark (c) after (9.3.6.), is just the "Berezinian sheaf", [Lei],[O-P].

If now F is any complex in $D^b(\text{Proj-}\mathbf{UD}.)$, M any MCM over $\mathbf{U}.$, (or, as well, any complex in $D^b(\mathbf{U}.)$), such that F is isomorphic to the sheafification of $\mathbf{b}(M)$, we get from (ii) above isomorphisms :

$$\mathrm{Ext}^{w-1}_{\text{Proj-}\mathbf{UD}.}(F,\omega) = \underline{\mathrm{Ext}}^{v-1}_{\mathbf{U}.}(M,\det V)$$

and

$$H^0(\text{Proj-}\mathbf{UD}.,F) = \hat{H}^0(M \otimes_K (\det W)^{-1}) \ .$$

Recall, (7.7.5.(iii)&(iv)) and (8.5.5.&6.), that the duality for Tate-cohomology over $\mathbf{U}.$ is obtained by combining the Yoneda-product with the trace τ_K , see also (8.5.7.(v)), which hence yields in view of the just exhibited isomorphisms of cohomology groups the diagram

$$\begin{tikzcd}
\underline{\mathrm{Ext}}^{v-1}_{\mathbf{U}.}(M,\det V) \times \hat{H}^0(M \otimes_K (\det W)^{-1}) \arrow[r, "-\circ-"] \arrow[d, leftrightarrow, "\simeq"] & \mathbf{det}(V,W) \otimes_K \hat{H}^{v-1} \arrow[r, "\tau_K"] \arrow[d, leftrightarrow, "\simeq"] & K \arrow[d, equal] \\
\mathrm{Ext}^{w-1}_{\text{Proj-}\mathbf{UD}.}(F,\omega) \times H^0(\text{Proj-}\mathbf{UD}.,F) \arrow[r, "-\circ-"] & H^{w-1}(\text{Proj-}\mathbf{UD}.,\omega) \arrow[r] & K
\end{tikzcd} ,$$

in which the upper row describes the duality for Tate-cohomology over $\mathbf{U}.$ whereas the lower row is the expected form of Grothendieck-Serre duality on Proj-$\mathbf{UD}.$. Expliciting the trace on $H^{w-1}(\text{Proj-}\mathbf{UD}.,\omega)$ in terms of "super"-differential forms, it becomes then justified - as mentioned - to interpret τ_K as "the integration of differential forms over the fundamental cycle of Proj-$\mathbf{UD}.$ ".

<u>Remark:</u> (a) If $i_{W^*} : (W^*,0) \longrightarrow (W^*,V^*)$ denotes as usual the inclusion of the even subspace of (W^*,V^*) , (9.1.(iii)), and - abusively, as in (9.3.7.) - $(i_{W^*})^*$ the "inverse image functor" associated to the morphism of ringed spaces from Proj-$\mathbf{U}.(W^*,V^*)$ to Proj-$\mathbf{U}.(W^*,0)$ induced by i_{W^*} , then one has obviously that $\mathbf{a}(\mathbf{U}.(W^*,V^*))$, the structure sheaf

of Proj-$\mathbf{U}.(W^*,V^*)$, is the inverse image $(i_{W^*})^*(\mathcal{O}_{\mathbf{P}(W^*)})$ of the usual structure sheaf on the projective space $\mathbf{P}(W^*)$.

Hence, there is an equality of cohomological functors

$$H^i(\text{Proj-}\mathbf{U}.(W^*,V^*),-) = H^i(\mathbf{P}(W^*),(i_{W^*})_*(-)) \quad ,$$

so that indeed Tate-cohomology over $\mathbf{U}.(V,W)$ "is" nothing but coherent sheaf cohomology on $\mathbf{P}(W^*)$, the projective space of all "odd points" of the linear superspace (V,W) , (9.1.(ii)).

Given the natural duality between "geometric" and "odd" points, this indicates once again in which sense Tate-cohomology "complements" sheaf-cohomology on linear super-spaces.

As a further (?) "application", remark that sheaf-cohomology together with Tate-cohomology just detect those complexes over $\mathbf{U}.(V,W)$ which are at the same time perfect and of artinian cohomology. Now, considering the "geometric" points only, hence forgetting the structure over the exterior algebra, the "Lemme d'acyclicité" implies that any perfect representative of such a complex must have at least $\dim_K V$ many non-zero terms. Dually, that is considering the "odd points" or applying the same argument to the image under $\mathbf{b}$, the (total) length of the cohomology of such a complex is bounded below by $\dim_K W$. This kind of argument can be found for example in the work of G. Carlsson, [Car], where this dictionary is used - although not in the same language - to bound the cohomology of spaces admitting free actions of elementary abelian 2-groups.

Coming back to the interpretation of Tate-cohomology as cohomology on projective space, it should be an interesting exercise to translate the fundamental theorems on this cohomology - like Serre's Theorems A and B or the Theorem of Riemann-Roch - into statements about the Tate-cohomology of modules over rings of the form $\mathbf{S}.(V) \otimes_K \Lambda^{\cdot}(W)$.

Here, we will instead restrict ourselves to just emphasize the consequences of the existence of a (natural) t-structure on $\underline{\mathrm{MCM}(\mathbf{U}.(V,W))}$ for any linear superspace (V,W) .

To begin with, this allows one to define higher algebraic K-groups for MCM's over $\mathbf{U}.(V,W)$ and these groups may serve as "stabilized" K-groups for $\mathbf{U}.(V,W)$ as asked for in (4.9.) :

<u>Definition and Proposition 9.5.2.</u>: Let D be any t-category with heart C. Then define the higher algebraic K-groups $K_i(D)$ to be the K-groups $K_i(C)$ as defined by D. Quillen, [Qu 1].

For any linear superspace (V,W) set $\underline{K}_i(\mathbf{U}.(V,W)) = K_i(\underline{MCM(\mathbf{U}.(V,W))})$, the category of maximal Cohen-Macaulay modules over $\mathbf{U}.(V,W)$ modulo projectives endowed with the t-structure from (9.5.1.(i)).

Then one has :

(i) $K_i(\mathbf{U}.(V,W)) = K_i'(\mathbf{U}.(V,W)) = K_i(K) \otimes_{\mathbb{Z}} \mathbb{Z}[s,s^{-1},\lambda,\lambda^{-1}]$,

(ii) There are short exact sequences of abelian groups

$$0 \longrightarrow K_i(\mathbf{U}.(V,W)) \xrightarrow{(1-\lambda^{-1})^W} K_i'(\mathbf{U}.(V,W)) \longrightarrow \underline{K}_i(\mathbf{U}.(V,W)) \longrightarrow 0 ,$$

and , consequently,

(iii) $\underline{K}_i(\mathbf{U}.(V,W)) = K_i(K) \otimes_{\mathbb{Z}} \mathbb{Z}[s,s^{-1},g]/(g^W)$, where g represents the class of $(1-\lambda^{-1})$ as in (9.3.8.) . This K-group is naturally isomorphic to $K_i(\mathbf{P}(W^*))[\lambda,\lambda^{-1}]$, the isomorphism induced by the functor $\mathbf{b}_{(V,W)}$.

<u>Proof</u>: In view of the Bernstein-Gelfand-Gelfand-correspondence and the dictionary deduced from it, the assertions above just come down to the calculation of higher K-groups of projective space as done by D. Quillen in [Qu 1;Prop.4.3.]. For the last statement in (iii), remark that the "direct image" functor $(i_{W*})_*$ is t-exact with respect to the natural t-structures on $D^b(\text{Proj-}\mathbf{U}.(W^*,V^*))$ and $D^b(\text{Proj-}\mathbf{U}.(W^*,0))$ and that it admits $(i_{W*})^*$ as an exact left adjoint. Combining now the foregoing with [BBD;1.3.17.(iii)] and [Qu 1;§3] , it follows the existence of natural isomorphisms of abelian groups

$$\underline{K}_i(\mathbf{U}.(V,W)) \xrightarrow{K_i(\mathbf{b})} K_i(\text{Proj-}\mathbf{U}.(W^*,V^*)) \xrightarrow{K_i((i_{W*})_*)} K_i(\text{Proj-}\mathbf{U}.(W^*,0))$$

and
$$K_i(\mathbf{P}(W^*)) \otimes_{\mathbb{Z}} \mathbb{Z}[\lambda,\lambda^{-1}] \longrightarrow K_i(\text{Proj-}\mathbf{U}.(W^*,0)) .$$

<u>Remark</u>: (b) Analogously, one may define in a natural way higher K-groups for the category <u>Prim-U.(V,W)</u> of primitive objects, (9.3.8.), and show that these groups are canonically isomorphic to the corresponding groups for $\mathbf{P}(V)\times\mathbf{P}(W^*)$, the scheme of all "points" of the linear superspace (V,W) .

(c) Above, we have reduced the (definition and) calculation of K-groups for MCM's to the one for projective spaces by means of the Bernstein-Gelfand-Gelfand-correspondence $\mathbf{b}$. In fact, analyzing D. Quillen's result, or, even more instructively, its extension by R.G.Swan, [Sw], to encompass the case of non-singular quadrics, one observes that this correspondence is used there - implicitely - and that the determination of the K-groups of the projective scheme in question is deduced from the explicit knowledge of "all" MCM's over the Yoneda-Ext-algebra of the residue class field of the associated homogeneous coordinate ring. (In case of projective space $\mathbf{P}(V)$, this Ext-algebra is $\Lambda^{\cdot}(V^*)$, see (9.3.2.(iv)), and the category <u>$MCM(\Lambda^{\cdot}(V^*))$</u> is isomorphic to <u>$\text{mod-}\Lambda^{\cdot}(V^*)$</u> by (8.1.). For a non-singular quadric, the Ext-algebra is the homogenized Clifford-algebra of the associated quadratic form, and a module over it is MCM iff it is finitely generated and the homogenizing parameter is a non-zero-divisor on it. For more details along these lines, see also the appendix to [BEH].)

Next, we will investigate a little bit closer the t-structure on the category of maximal Cohen-Macaulay $\mathbf{U}.(V,W)$-modules modulo projectives by using only local information at the "odd points" of (V,W).

This will result in a generalization of the "monadic" description of vector bundles on usual projective space as given in [BGG] to yield a representation of coherent sheaves of modules in terms of monads satisfying certain "perversity-conditions". Finally, we will give a "normal form" for such a monad, making precise the statement that "any coherent sheaf of modules on projective space is entirely determined - up to isomorphism - by a finite number of cohomology groups".

To begin with, we have to recall some elementary facts on the natural t-structure on $D^b(\text{Proj-}\mathbf{U}.(V,W))$ - or just on $D^b(\mathbf{P}(V))$.

As was done already in case of i_V, for any morphism of linear superspaces $F : (V,W) \longrightarrow (V',W')$ - with F^+ surjective - we will denote by $F_*: D^b(\text{Proj-}\mathbf{U}.(V',W')) \longrightarrow D^b(\text{Proj-}\mathbf{U}.(V,W))$ the "direct image functor" associated to the morphism $\text{Proj}(F)$ of ringed spaces. F^* or $F^!$ will denote its left or right adjoint - if either exists.

We introduce further the following

<u>Notations 9.5.3.</u>: Let (V,W) be a linear superspace over K, λ a non-zero linear form on V, corresponding to a geometric point $\langle\lambda\rangle$ of (V,W) in $\mathbf{P}(V)$. Let $K(\langle\lambda\rangle)$ denote the residue class field at that point of $\mathbf{P}(V)$.

Then we set for any complex C in $D^b(\text{Proj-}\mathbf{U}.(V,W))$:

$$T_i(\langle\lambda\rangle;C) = H^{-i}(\lambda^*\circ(i_V)_*(C)) = \text{Tor}_i^{\mathcal{O}_{\mathbf{P}(V)}}((i_V)_*(C),K(\langle\lambda\rangle))$$

and

$$S_i(C) = \{ \mathbf{p} \in \text{Proj-}\mathbf{S}.(V) \ : \ \mathrm{Tor}_i^{\mathcal{O}_{\mathbf{P}(V)}}((i_V)_*(C), K(\mathbf{p})) \neq 0 \} \quad .$$

(Here it is to be understood that we have summed up over the still existing "exterior degrees" for $(i_V)_*(C)$, so that the "correct" definition of $T_i(\langle\lambda\rangle;C)$ should read $\bigoplus_j H^{-i}(\lambda^{*}\circ(i_V)_*(C(0,j)))$.)

The functors $T_i(\langle\lambda\rangle;-)$ are easily described in terms of the equivalence $\mathbf{b}$: Let M be a maximal Cohen-Macaulay module over $\mathbf{U}.(W^*,V^*)$, $z : K \longrightarrow V^*$ the odd point of (W^*,V^*) which corresponds to λ by duality. Then (9.4.7.) shows that one has

$$\bigoplus_j H^{-i}(\lambda^*\circ(i_V)_*(\mathbf{b}(M)(0,j))) = \bigoplus_j \hat{H}^{-i+j}(z_*\circ\mathbf{U}.(p_{V^*})^!(M(-j,0))) \quad ,$$

where $p_{V*} = \mathbf{D}(i_V) : (W^*,V^*) \longrightarrow (0,V^*)$ is the projection onto the odd quotient space of $\mathbf{D}(V,W)$.

The right hand side can be evaluated explicitely. If $K\langle z\rangle$ denotes the sub-algebra of $\Lambda^{\cdot}(V^*)$ generated by the image of z , the cohomology group in question is just

$$\bigoplus_j H^{-i+j}(\mathrm{Hom}^{\cdot}_{K\langle z\rangle}(\mathbf{CR}_{K\langle z\rangle}(K), z_*\mathrm{RHom}_{\mathbf{U}.(W^*,V^*)}(\Lambda^{\cdot}(V^*),M(-j,0))) \quad ,$$

which looks worse than it is :

- a complete resolution of K over $K\langle z\rangle$ is given by the "periodic" complex

  $$\ldots \xrightarrow{z} K\langle z\rangle(0,k) \xrightarrow{z} K\langle z\rangle(0,k+1) \xrightarrow{z} \ldots \quad ,$$

  (where z is also identified with $z(1)$),
- $\Lambda^{\cdot}(V^*)$ is resolved over $\mathbf{U}.(W^*,V^*)$ by the Koszul complex

  $$(\Lambda^k(W^*) \otimes_K \mathbf{S}.(W^*) \otimes_K \Lambda^{\cdot}(V^*)[k](-k,0) \ , \ d_{W^*} \otimes 1) \quad ,$$

  where d_{W^*} represents the ordinary Koszul differential associated to W^* .

As a simple example, which appears already in [BGG], assume $W = 0$ - in which case one may forget about i_V and the Koszul complex above - , so that $M = \oplus M_i^j$ is just a finite direct sum (indexed by i) of graded $\Lambda^\cdot(V^*)$-modules. Identifying z - or rather $z(1)$ - with a 1-form in $\Lambda^\cdot(V^*)$, multiplication with it turns M into a complex of vectorspaces

$$\ldots \xrightarrow{z} M_\cdot^j \xrightarrow{z} M_\cdot^{j+1} \xrightarrow{z} \ldots$$

whose cohomology is the desired object.

The behaviour of the homological functors $T_i(\langle - \rangle;-)$ is well-known :

<u>Lemma 9.5.4.</u>: Let V be a K-vectorspace of dimension v, x any point in $\mathbf{P}(V)$ and F a coherent sheaf of $\mathcal{O}_{\mathbf{P}(V)}$-modules. Denote F_x the localization of F at x. Then the following assertions are equivalent for any non-negative i:

(i) $T_i(x;F) \neq 0$ (ii) $\operatorname{projdim}_{\mathcal{O}_{\mathbf{P}(V),x}} F_x \geq i$

(iii) $\operatorname{depth}_{\mathcal{O}_{\mathbf{P}(V),x}} F_x \leq v-1-i$ (iv) $H^j_{\{x\}}(F_x) \neq 0$ for some $j \leq v-1-i$

(v) $\bigoplus_{j=0}^{v-1-i} \operatorname{Ext}^{v-1-j}_{\mathcal{O}_{\mathbf{P}(V),x}}(F_x,\omega_{\mathbf{P}(V),x}) \neq 0$

(vi) x is in the support of $\bigoplus_{j=0}^{v-1-i} \mathcal{E}xt^{v-1-j}_{\mathcal{O}_{\mathbf{P}(V)}}(F,\omega_{\mathbf{P}(V)})$.

<u>Proof</u>: (i) <==> (ii) is the existence of a minimal projective resolution of F_x over the local ring $\mathcal{O}_{\mathbf{P}(V),x}$.
(ii) <==> (iii) is the theorem of Auslander-Buchsbaum-Serre : $\mathcal{O}_{\mathbf{P}(V),x}$ is a regular local ring of dimension $v-1$ and hence

$$\operatorname{projdim}_{\mathcal{O}_{\mathbf{P}(V),x}} F_x + \operatorname{depth}_{\mathcal{O}_{\mathbf{P}(V),x}} F_x = v-1$$

(iii) <==> (iv) is the characterization of depth by local cohomology.
(iv) <==> (v) is Grothendieck-Serre-duality : The K-dual of $H^j_{\{x\}}(F_x)$ is isomorphic to $\operatorname{Ext}^{v-1-j}_{\mathcal{O}_{\mathbf{P}(V),x}}(F_x,\omega_{\mathbf{P}(V),x})$.
(v) <==> (vi) is the fact that localization and forming $\mathcal{E}xt$'s commute.

<u>Corollary 9.5.5.</u>: With notations as before,
(i) $S_i(F)$ contains $S_{i+1}(F)$ and $S_0(F)$ equals the support of F.
(ii) Any $S_i(F)$ is a closed subset of $\mathbf{P}(V)$.
(iii) The codimension of $S_i(F)$ in $\mathbf{P}(V)$ is at least i .
(iv) F satisfies Serre's condition S_i, iff $S_{v-i}(F) = \emptyset$.
(v) If $\dim F \leq r$, then $S_{v-1-r}(F) = \cdots = S_0(F) = \operatorname{Supp}(F)$.

In other words, if we consider the function $c_F(i) = \mathrm{codim}_{\mathbf{P}(V)} S_i(F)$, it has the following properties :

(i') It is non-decreasing and its "initial value" $c_0(F)$ equals the codimension of the support of F, by (i).

(ii') It stays above or on the diagonal by (iii).

(iii') It hits the diagonal at least once, namely for $i = v-1-\dim F$, by (v).

These properties are enough to prove

<u>Proposition 9.5.6.</u>: Let C be a complex in $D^b(\text{Proj-}\mathbf{U}.(V,W))$. Then C is quasi-isomorphic to $H^0(C)$, that is, there is no other cohomology than possibly in degree zero or, equivalently, C belongs to the heart of the natural t-structure on $D^b(\text{Proj-}\mathbf{U}.(V,W))$, if and only if

(i) $S_i(C) = \emptyset$ for $i < 0$ and

(ii) $\mathrm{codim}_{\mathbf{P}(V)} S_i(C) \geqq i$ for $i \geqq 0$.

<u>Proof</u>: It follows from the foregoing that the two conditions are necessary. To prove that they are sufficient, we show that with

- $i(C) = \min(i : H^i(C) \neq 0)$ and $m(C) = \max(i : H^i(C) \neq 0)$,
- $r = \dim H^{i(C)}(C)$,

one has :

(a) $S_{-m(C)}(C) \neq \emptyset$ and (b) $\mathrm{codim}_{\mathbf{P}(V)} S_{v-1-r-i(C)}(C) \leqq v-1-r$,

which evidently will suffice.

(a) and (b) both follow from an examination of the spectral sequences

$$E_2^{i,j}(x;C) = T_{-i}(x;H^j(C)) \Longrightarrow T_{-i-j}(x;C),$$

x any point in $\mathbf{P}(V)$.

By definition of $m(C)$, for any x in $\mathbf{P}(V)$ the terms $E_2^{-i,j}(x;C)$ vanish if either

- $j-i \geqq m(C)+1$ or
- $j-i = m(C)$, but $i \neq 0$, or
- $j-i = m(C)-1$ and $i \neq 0,1$.

Hence, $T_{-m(C)}(x;C) = E_2^{0,m(C)}(x;C) = T_0(x;H^{m(C)}(C))$ which proves (a).

To establish (b), remark first that by (9.5.5.(v)) one has

$$S_{v-1-r}(H^{i(C)}(C)) = \mathrm{Supp}(H^{i(C)}(C)),$$

which by assumption is of dimension r. As on the other hand for any $k > 0$ one obtains from (9.5.5.(iii)) that

$$\operatorname{codim}_{\mathbf{P}(V)} S_{v-1-r+k+1}(H^{i(C)+k}(C)) \geqq v-1-r+k+1 > v-1-r \quad ,$$

the above spectral sequences imply that $S_{v-1-r-i(C)}(C)$ contains all r-dimensional components of $\operatorname{Supp}(H^{i(C)}(C))$ - whence (b).

To simplify our next statement, we translate as above the notions $T_i(< >;-)$ and $S_i(-)$ into Tate-cohomology :

<u>Definition 9.5.7.</u>: Let $w : K \longrightarrow W$ be a non-zero odd point of a linear superspace (V,W) . Then we set

$$\hat{H}^i_w(M) = \bigoplus_j \hat{H}^{i+j}(w_* \circ \mathbf{U}.(p_W)^!(M(-j,0)))$$

for any object M in $D^b(\mathbf{U}.(V,W))$.

The schematic support of this family of functors $\hat{H}^i_-(-)$ in $\mathbf{P}(W^*)$ will then be denoted $\hat{S}_{-i}(M) = \operatorname{Supp}_{\mathbf{P}(W^*)} \hat{H}^{+i}_-(M)$.

More precisely, it contains all those homogeneous primes $\mathbf{q}$ in Proj-$\mathbf{S}.(W^*)$, for which there exists a finite field extension L of K , a L-valued odd point $w_L : L \longrightarrow W \otimes_K L$, such that the corresponding point in $\mathbf{P}(W^* \otimes_K L)$ lies over $\mathbf{q}$ and such that $\hat{H}^{+i}_{w_L}(M \otimes_K L) \neq 0$.

Just by construction, one has

$$\hat{H}^i_w(M) = T_{-i}(<\mathbf{D}(w)>;\mathbf{ab}(M)) \quad \text{and} \quad \hat{S}_{-i}(M) = S_{-i}(\mathbf{ab}(M)) \quad .$$

Hence we get the following description of the t-structure on MCM's :

<u>Theorem 9.5.8.</u>: Let (V,W) be a linear superspace over K , M any complex of modules in $D^b(\mathbf{U}.(V,W))$. Then the following are equivalent :

(i) The image of M in $\underline{D^b(\mathbf{U}.(V,W))}$ is in the heart of the natural t-structure.

(ii) M admits an essentially linear projective co-resolution.

(iii) $\mathbf{ab}(M)$ has only cohomology in (complex-)degree zero.

(iv) $\hat{S}_i(M) = \emptyset$ for $i < 0$ and $\operatorname{codim}_{\mathbf{P}(W^*)} \hat{S}_i(M) \geqq i$ for $i \geqq 0$.

<u>Proof</u>: (i) <==> (iii) is the definition, (9.5.1.(i)).
(ii) <==> (iii) as by the properties of the correspondence $\mathbf{b}$, to say that the sheafification of $\mathbf{b}(M)$ has only cohomology in (complex-)degree zero just means that up to a perfect complex M lies in the pre-image of mod-$\mathbf{U}.(\mathbf{D}(V,W))$. But that signifies, by (9.4.4.), that M admits a projective co-resolution which becomes linear after a finite number of terms.

(iii) <==> (iv) just translates the characterization of the natural t-structure on $D^b(\text{Proj-}\mathbf{U}.(W^*,V^*))$, which was obtained in (9.5.6.), by means of $\mathbf{b}$.

To complete further our dictionary, let us point out the significance of the existence of MCM-approximations over $\mathbf{U}.(V,W)$ for the study of complexes of coherent sheaves of modules over $\text{Proj-}\mathbf{U}.(\mathbf{D}(V,W))$.

Recall that, given a complex X in $D^b(\mathbf{U}.(V,W))$, a MCM-approximation $\mathbf{M}(X)$ is by definition, (5.6.), a maximal Cohen-Macaulay $\mathbf{U}.(V,W)$-module which, considered as a complex, is quasi-isomorphic to X up to a perfect complex.

Now a complex C in $D^b(\mathbf{U}.(V,W))$ is isomorphic to a single MCM if and only if it satisfies

(i) $\text{Ext}^k_{\mathbf{U}.(V,W)}(\mathbf{U}.(V,W),C(i,j)) = 0$ except for $k = 0$, i.e. the cohomology of C is concentrated in (complex-)degree zero, and

(ii) $\text{Ext}^k_{\mathbf{U}.(V,W)}(C(i,j),\mathbf{U}.(V,W)) = 0$ except for $k = 0$, which means that the only cohomology module, which may exist by (i), is MCM.

Applying our dictionary, this translates into the following conditions for $\mathbf{b}(C)$:

(i') $\mathbf{b}(C)$ admits a linear projective co-resolution, or, equivalently, is isomorphic to a linear monad in $D(\mathbf{U}.(W^*,V^*))$, by (9.4.4.(ii)),

and

(ii') $\text{Ext}^{k+i+j}_{\mathbf{U}.(W^*,V^*)}(\mathbf{b}(C)(-j,-i),\det(V^*)) = 0$ for any $k \neq 0$, or, equivalently :

$\mathbf{b}(C) \otimes_K (\det V^*)^{-1}$ admits a linear projective resolution.

Remember furthermore, that MCM-approximations are unique up to projective summands, hence over $\mathbf{U}.(V,W)$ up to direct sums of graded free modules of the form $\mathbf{U}.(V,W)(i,j)$.

This says that we may add to $\mathbf{b}(C)$ complexes of the form

$$\mathbf{b}(\mathbf{U}.(V,W)(i,j)) = (\det V^*)[i+j](-j,-i) = K[i+j](-j,-i-v) ,$$

or, connecting (i') and (ii') to a single complete resolution of $\mathbf{b}(C)$, the resulting acyclic complex of finitely generated graded free modules determined by C is modified by direct summands which constitute shifted complete resolutions of K itself. Remark that "the" complete resolution of K was determined explicitely in (8.5.7.).

Hence, translating the existence of MCM-approximations yields

Proposition 9.5.9.: Let Y be any complex in $D^b(\text{Proj-}\mathbf{U}.(W^*,V^*))$. Then there exist :

(i) A complex of graded $\mathbf{U}.(W^*,V^*)$-modules $\mathbf{C}(Y)$, bounded below and with bounded cohomology, whose term in (complex-)degree k is a finite direct sum of modules of the form $\mathbf{U}.(W^*,V^*)(i,j)$ with $i+j = k$.

(ii) A complex of graded $\mathbf{U}.(W^*,V^*)$-modules $\mathbf{P}(Y)$, bounded above and with bounded cohomology, whose term in (complex-)degree k is a finite direct sum of modules of the form $\mathbf{U}.(W^*,V^*)(i,j)$ with $i+j = k-v$.

(iii) A morphism of complexes $\mathbf{N}(Y) : \mathbf{P}(Y) \longrightarrow \mathbf{C}(Y)$ which induces an isomorphism in cohomology and such that the sheafification of the image of $\mathbf{N}(Y)$ is a (necessarily finite) complex isomorphic to Y in $D^b(\text{Proj-}\mathbf{U}.(W^*,V^*))$.

(iv) The morphism $\mathbf{N}(Y)$ is unique up to isomorphism if one imposes the "minimality-condition" that the mapping-cone of $\mathbf{N}(Y)$ does not contain a shifted complete resolution of K as a direct summand.

One may hence call the morphism $\mathbf{N}(Y) : \mathbf{P}(Y) \longrightarrow \mathbf{C}(Y)$ the minimal monadic representation of the given complex of sheaves Y.

The proof is immediate from the foregoing : If Y' denotes any representative of Y in $D^b(\mathbf{U}.(W^*,V^*))$, set $X' = \mathbf{b}_{(W^*,V^*)}(Y')$, $\mathbf{M}(X')$ its unique MCM-approximation over $\mathbf{U}.(V,W)$ which does not contain any free summand. Then $\mathbf{C}(Y)$ is the linear monad associated to $\mathbf{b}_{(V,W)}(\mathbf{M}(X'))$, by (i') above, $\mathbf{P}(Y)$ its minimal projective resolution, which has the required properties in view of (ii') above. $\mathbf{N}(Y)$ is then the natural map, whose image is by construction quasi-isomorphic to $\mathbf{b}_{(V,W)}(\mathbf{M}(X'))$ and differs from $Y' = \mathbf{b}_{(V,W)}(X')$ only by a complex which has artinian cohomology, hence its sheafification is isomorphic to Y again.

Starting with another lifting Y'' of Y, it differs from Y' only by a complex with artinian cohomology, hence the corresponding objects X' and X'' differ only by a perfect complex, whence they have the same minimal MCM-approximation. This shows the uniqueness of the construction.

Remark: (d) If $\dim_K V \neq 1$, the (isomorphism class of the) complex Y (in $D^b(\text{Proj-}\mathbf{U}.(W^*,V^*))$) can be recovered from the (isomorphism class of the) mapping cone over $\mathbf{N}(Y)$ (in the category of complexes) : By construction, we can then distinguish the linear subcomplex $\mathbf{C}(Y)$ and the corresponding quotient complex $\mathbf{P}(Y)[1]$ just from the occurring degrees.

Let us emphasize that we deal indeed with isomorphism classes of complexes - and not with homotopy classes. The reader is asked to supplement (9.5.9.) by convincing himself that in fact the homotopy classes of the occurring complete resolutions are in bijection with the isomorphism classes of objects in the (equivalent) categories of primitive objects Prim-$\mathbf{U}.(V,W)$ or Prim-$\mathbf{U}.(W^*,V^*)$.

As an example of the foregoing - and to justify the title of this chapter - let us mention the

9.6. Applications to projective geometry

(9.6.1.) Considering the last result first, assume $V = 0$ in (9.5.9.). Then $\mathbf{U}.(W^*,V^*) = \mathbf{S}.(W^*)$, the polynomial ring over W^* . For any object Y in $D^b(\mathbf{P}(W^*))$, by construction both $\mathbf{P}(Y)$ and $\mathbf{C}(Y)$ are linear complexes, which are furthermore finite as $\mathbf{S}.(W^*)$ is of finite global dimension. It follows that $\mathbf{N}(Y)$ is necessarily an isomorphism and the minimality condition just ensures that neither $\mathbf{P}(Y)$ nor $\mathbf{C}(Y)$ contain a shifted Koszul complex as a direct summand. In particular, the mapping cone over such a morphism $\mathbf{N}(Y)$ is certainly contractible - in accordance with the fact that the category of primitive objects over a polynomial ring is trivial.

To put it differently, if we consider in general a linear superspace (V,W) in which either the even or odd summand is zero, the diagram in (9.3.12.) degenerates, leaving essentially a unique row or column, and the "minimal monadic representation" of (9.5.9.) is already completely determined by either $\mathbf{P}(Y)$ or $\mathbf{C}(Y)$.

Now let us summarize the various descriptions of complexes of coherent sheaves over a projective space $\mathbf{P}(W^*)$ which are contained in the above - and which all already appeared in the literature, at least in some disguise :

(9.6.2.) Assume given a finite dimensional vectorspace W over some field K . Then the derived category $D^b(\mathbf{P}(W^*))$ of complexes of sheaves of modules on the projective space $\mathbf{P}(W^*)$ whose non-zero cohomology sheaves are finite in number and coherent, is naturally equivalent to any of the following categories :

(i) $D^b(\mathbf{S}.(W^*))/D^b_{art}(\mathbf{S}.(W^*))$, which is (equivalent to) Serre's classical description.

(ii) $\underline{\mathrm{Mon}^0(S.(W^*))}/(\text{Koszul-complexes})$, that is, the full subcategory of the category in (i) which is generated by all finite complexes of graded free $S.(W^*)$-modules, <u>linear</u> in the sense that the module in (complex-)degree k is of the form $A_k \otimes_K S.(W^*)(k)$ with A_k a finite dimensional K-vector-space.

<u>Remark</u> that direct sums of appropriately shifted Koszul complexes are the only such linear complexes with artinian cohomology. As morphisms between linear complexes are homotopic iff they are equal, it is clear a priori that the morphism groups in this category consist precisely of all morphisms of such complexes modulo those which factor over a direct sum of Koszul complexes. The point of the Bernstein-Gelfand-Gelfand-correspondence, if considered from this point of view, is hence that any complex in $D^b(\mathbf{P}(W^*))$ is quasi-isomorphic to (the sheafification of) such a linear complex. For single sheaves, this is precisely the main ingredient established and used by D. Quillen in his calculation of the higher K-groups for projective bundles, [Qu 1;§8], see also Remark (c) in (9.5.).

Dismantling such a linear complex as to keep only the generating sets in each (complex-)degree - and tensoring with $\det(W)$ - we get the incarnation of $D^b(\mathbf{P}(W))$ as provided by the Bernstein-Gelfand-Gelfand-correspondence :

(iii) $\underline{\text{mod-}\Lambda^\cdot(W)} = \underline{\mathrm{MCM}(\Lambda^\cdot(W))}$, the category of all finitely generated (= maximal Cohen-Macaulay) graded modules over the exterior algebra generated by W in degree one, modulo all finitely generated projective (= free = injective) modules over this local quasi-Frobenius algebra.

(iv) $\underline{\mathrm{APC}(\Lambda^\cdot(W))}$, the homotopy category of all acyclic complexes of finitely generated, graded free $\Lambda^\cdot(W)$-modules.

(v) $D^b(\Lambda^\cdot(W))/D^b_{perf}(\Lambda^\cdot(W))$, the stabilized derived category of the exterior algebra over W .

The equivalence of the descriptions in (i)-(iii) and (v) is the original result in [BGG]. The equivalence with (iv) has then been remarked by A.A. Beilinson, as S.I. Gelfand informed me - see also [Gel].

10. - *Applications to Singularities and Hypersurfaces*

In the following R will always denote a commutative (noetherian) ring of finite Krull dimension, $r = \dim R$. For simplicity (only), R will also always be assumed to have connected prime spectrum.

10.1. Local Tate-cohomology

Let (R,m,k) be a local commutative Gorenstein ring with maximal ideal m and residue class field $k = R/m$. Then, by analogy with (8.2.) and (8.5.), for any integer i, we call

$$\hat{H}^i_R(M) = \underline{\mathrm{Ext}}^i_R(k,M)$$

the i-th local Tate-cohomology of R with values in M, which may be any (complex of) R-module(s) (in $D^-(\mathrm{Mod}\text{-}R)$, (7.4.1.)).

As in general, $\hat{H}^{\cdot}_R(M)$ will be a graded module over the local Tate-cohomology ring of R, $\hat{H}^{\cdot}_R = \hat{H}^{\cdot}_R(k)$, with respect to Yoneda-product.

As R is local, any finitely generated R-module M admits - up to isomorphism - a unique decomposition $M = M' \oplus R^m$ where M' does not contain any free summand, (see [Sw 1;Thm.2.6.]).

If now M is already a MCM over R, the groups $\hat{H}^i_R(M)$ yield nothing new :

Lemma 10.1.1.: Let M be a MCM over a local Gorenstein ring (R,m,k), $M = M' \oplus R^m$ the decomposition in which M' contains no free summand. Then one gets

(i) $\hat{H}^i_R(M) = \mathrm{Ext}^i_R(k,M)$ for any $i > r = \dim R$,

(ii) $\hat{H}^r_R(M) = \mathrm{Ext}^r_R(k,M') = \mathrm{Ext}^r_R(k,M)/k^m$,

(iii) $\hat{H}^{r-1}_R(M) = M' \otimes_R k$, and

(iv) $\hat{H}^i_R(M) = \mathrm{Tor}^R_{r-1-i}(M,k)$ for any $i < r-1$.

The proof follows immediately from (6.2.5.(3)) - or also directly from the construction of a complete resolution for M as in (4.5.1.) -, once it is observed that $\mathrm{RHom}_R(k,R) = k[-r]$ for any local Gorenstein ring and that $\mathrm{Ext}^i_R(k,M) = 0$ for any $i < r$ and all MCM over R.

Usually, it is a rather difficult problem to determine (the dimension of) the k-vectorspaces occurring in this Lemma. Sometimes, to know that they are necessarily modules over $\hat{H}^0_R$ will yield restrictions, for example if there are no "small" simple modules over the ring $\hat{H}^0_R$.

The size of $\hat{H}^0_R$ is essentially known from (5.5.2.). More generally, we get, as in (8.5.6.), the following information on the local Tate-cohomology ring of R:

<u>Proposition 10.1.2.</u>: Let (R,m,k) be a local Gorenstein ring. Then

(i) The $\mathbb{Z}$-graded k-algebra $\hat{H}^{\cdot}_R$ carries a natural involution, with respect to which it becomes isomorphic to its graded opposite k-algebra.

(ii) There exists a non-degenerate k-bilinear pairing, associative for the (Yoneda-)product on $\hat{H}^{\cdot}_R$:

$$\hat{H}^{\cdot+r-1}_R \otimes_k \hat{H}^{-\cdot}_R \longrightarrow k \quad ,$$

in particular, $\hat{H}^{\cdot}_R$ is a self-injective k-algebra.

(iii) The k-algebra $\hat{H}^0_R = \underline{\mathrm{Hom}}_R(\mathbf{M}(k),\mathbf{M}(k))$ is naturally isomorphic to its own opposite algebra. Its dimension over k is given by

(a) $\hat{H}^0_R = 0$ if and only if R is regular,

(b) $\dim \hat{H}^0_R \geq 2^{\dim R}$ if R is not regular, and in this case

(c) $\dim \hat{H}^0_R = 2^{\dim R}$ if and only if either $\dim R \leq 1$ or R is an (abstract) hypersurface ring. In these cases, $\hat{H}^0_R$ is a quasi-Frobenius algebra, if $\dim R$ is odd.

<u>Proof</u>: (i), (ii) and the first assertion in (iii) follow from the general theory developed in chapter 7. Also (iii,a) is obvious, as $\hat{H}^0_R = 0$ iff k is of finite projective dimension over R iff R is regular. The given bounds on the dimension encode well-known results on the ranks of the free modules in a minimal free resolution of k over R. More precisely, one has in general:

<u>Lemma 10.1.3.</u>: Let (R,m,k) be a local Gorenstein ring, M any MCM over R and $\underline{x} = (x_1,\ldots,x_d)$ a regular sequence in m. If the classes of the elements x_i are linearly independent over K in m/m^2, then there is an isomorphism of graded k-vectorspaces, natural in M:

$$\underline{\mathrm{Ext}}^{\cdot}_R(k,M) \longrightarrow \Lambda^{\cdot}((\underline{x}/\underline{x}^2)^*) \otimes_{R/\underline{x}R} \underline{\mathrm{Ext}}^{\cdot}_{R/\underline{x}R}(k,M/\underline{x}M) \quad .$$

The proof of the Lemma extends the corresponding one for the ordinary Ext's . The ring homomorphism $R \longrightarrow R/\underline{x}R$ is perfect as $\underline{x}$ is a regular sequence, and the resolution of $R/\underline{x}R$ is given by a Koszul complex. Now apply the spectral sequence $"E_\cdot^{i,j}$ from (7.2.3.) and remark that it degenerates completely at the E_2-level iff the classes of the x_i are linearly independent modulo m^2 .

To conclude the proof of the Proposition, one may assume - after eventually extending the residue class field - that there is a maximal regular sequence of elements whose classes in m/m^2 are linearly independent, so that $R/\underline{x}R$ is a zero-dimensional Gorenstein ring. Then (10.1.1.) shows that $\mathrm{Ext}^{-\bar{i}}_{R/\underline{x}R}(k,k) = \mathrm{Tor}^{R/\underline{x}R}_{i-\bar{1}}(k,k)$ for any $i > 0$. Hence

$$\hat{H}^0_R = \underline{\mathrm{Ext}}^0_R(k,k) = k \oplus \Big(\bigoplus_{j=1}^{r} \Lambda^j((\underline{x}/\underline{x}^2)^*) \otimes_{R/\underline{x}R} \mathrm{Tor}^{R/\underline{x}R}_{j-\bar{1}}(k,k)\Big)$$

which implies (b) . It also follows that the equality in (c) holds iff all the k-vectorspaces $\mathrm{Tor}^{R/\underline{x}R}_j(k,k)$ are one-dimensional for $j < r$, which happens if and only if either r is at most 1 or the dimension of m/m^2 is precisely $\dim R + 1$.

The last assertion in (iii) is clear (from (ii)) if $\dim R \leq 1$. In case of a hypersurface, it follows from the explicit knowledge of the ordinary Yoneda-Ext-algebra of k over R .

Namely, one has explicitely

Example 10.1.4.: (i) If (R,m,k) is a one-dimensional local ring which is Gorenstein, then $\hat{H}^0_R$ is one of the following k-algebras :
(a) $k \times k$ (b) a quadratic field extension $k(a)$ of k , or
(c) the local k-algebra $k[x]/x^2$.
Case (a) occurs for example for a non-singular quadric, (b) for a regular but singular quadric - see [BEH] - and (c) is the generic case which occurs if the multiplicity of R is at least 3 .
(ii) If (R,m,k) is an (abstract) hypersurface ring, the natural map of vectorspaces $S_2(m/m^2) \longrightarrow m^2/m^3$ has at most a one-dimensional kernel. Let $Q : \Gamma_2((m/m^2)^*) \longrightarrow k$ be the quadratic form determined by this kernel (which may hence be zero !).
Then $\hat{H}^0_R = C^+(Q)$, the even Clifford-algebra of this quadratic form - and the involution on $\hat{H}^0_R$ is the main involution of the Clifford-algebra. If $Q = 0$, $C^+(Q) = \Lambda^+((m/m^2)^*)$ of course.

If $Q \neq 0$, we get restrictions as mentioned above : The dimension of a simple module over $C^+(Q)$ is of the form 2^l , where l depends on Q - and the field k . It follows then that for any module M over R

the dimension of any Tate-cohomology group has to be divisible by 2^l. It follows then - for example - that an indecomposable MCM over R has to have a rank which is divisible by 2^{l-1} - see again [BEH] for details.

We conclude this section by restating the Duality Theorem (7.7.5.) in the commutative context :

Proposition 10.1.5.: Let (R,m,k) be a local Gorenstein ring, M a maximal Cohen-Macaulay R-module which is locally free on the punctured spectrum $\mathrm{Spec}(R) - \{m\}$. If then E denotes an injective envelope of the R-module $\mathrm{Ext}_R^{\dim R}(k,R)$, there exists a natural R-linear trace

$$\tau_M : \underline{\mathrm{Ext}}_R^{\dim R-1}(M,M) \longrightarrow E \quad ,$$

such that for any complex N in $D^b(R)$, the pairings obtained from composing the Yoneda-product with this trace are R-bilinear and non-degenerate :

$$\underline{\mathrm{Ext}}_R^{i-1+\dim R}(N,M) \otimes_R \underline{\mathrm{Ext}}_R^{-i}(M,N) \xrightarrow{-\circ-} \underline{\mathrm{Ext}}_R^{\dim R-1}(M,M) \xrightarrow{\tau_M} E \ .$$

In particular, the two modules which are paired into E have the same (finite) length, and the stable Yoneda-Ext-algebra $\underline{\mathrm{Ext}}_R^{\cdot}(M,M)$ is self-injective.

For the proof take $R = \omega_S = S = T = \omega_T$ and $M = {}^*N = N^*$ in (7.7.5.) and remark that M locally free off the maximal ideal implies that it is stably transversal to any complex N.

Remark: (a) If $R_{\mathfrak{p}}$ is regular for any prime different from m, any maximal Cohen-Macaulay R-module is free off the maximal ideal.
(b) Of course, $\mathrm{Ext}_R^{\dim R}(k,R) = k$ as a R-module, so that the module E is just an injective envelope of the residue class field. In particular, as explained in (7.7.), if R contains a copy of k, one obtains the trace above as the R-linear extension of a k-linear trace into k.
The reason why we introduced E as an envelope of $\mathrm{Ext}_R^{\dim R}(k,R)$ - and not just of k - is furnished by the graded case :

If R, instead of being local, is non-negatively graded and if the irrelevant ideal generated by all elements of positive degree is maximal, then the assertion above still holds for graded MCMs and the graded Tate-cohomology. But in this case, as before in (8.5.), we have to keep track of the grading of $\mathrm{Ext}_R^{\dim R}(k,R)$ to obtain a homogeneous pairing.
Let us give an explicit

Example 10.1.6.: Let $S. = k[x_0,\ldots,x_{n+d}]$ be a polynomial ring over k, $R. = S./\underline{f}$ a homogeneous complete intersection of dimension $n+1$, defined by the regular sequence $f_1,\ldots,f_d$ of homogeneous polynomials with $\deg(f_i) = a_i$. Then the canonical module of $R.$ over k is given by $\omega_{R.} = R.(-n-d-1 + \Sigma\, a_i)$ - and satisfies hence $\operatorname{Ext}^{n+1}_{R.}(k,\omega_{R.}) = k(0)$.

It follows that for any graded MCM R-module M which is locally free off the irrelevant ideal and for any complex N of graded R.-modules in $D^b(R.)$, there are perfect pairings

$$\underline{\operatorname{Ext}}^{i+n}_{R.}(N,M(-n-d-1 + \Sigma\, a_i)) \times \underline{\operatorname{Ext}}^{-i}_{R.}(M,N) \longrightarrow k \quad ,$$

homogeneous for the total degree.

In particular, taking $M = N$, the "centre of symmetry" is in degree $[\frac{n}{2}](\frac{-n-d-1+\Sigma\, a_i}{2})$ for the self-dual algebra $\underline{\operatorname{Ext}}^{\cdot}_{R.}(M,M)$.

Finally, we come to the example which motivated our study here :

10.2. Maximal Cohen-Macaulay modules on hypersurfaces

Considering the classical case of integral group representations, the theory of hypersurfaces as compared to general (commutative) Gorenstein rings should show similar particular features as the cohomological theory of cyclic groups if compared to that of arbitrary finite groups.

After all, integral group rings of cyclic groups are "hypersurfaces" relative to the regular ring $\mathbb{Z}[t]$, and the key fact - namely that complete resolutions can be chosen to be periodic of period two - holds for general hypersurfaces by [Eis].

As an instance of this analogy, we will introduce in the next section <u>Herbrand-differences</u> for any two modules over a hypersurface with only isolated singularities.

But first, we recall some essential facts and study the structure of the Tate-cohomology over general hypersurfaces.

To fix the notations, P will henceforth denote a regular domain, f a non-zero element in P, $p : P \longrightarrow R = P/fP$ the projection onto the hypersurface ring R defined by f. We denote $\operatorname{Ker}(p) = fP$ by I and hence I/I^2 is the conormal module of p - or of R over P. It is a free R-module of rank one.

If $P = k[\underline{x}]$ is a polynomial ring, f a homogeneous element of degree d, then $I/I^2 = R(-d)$ as a graded free R-module.

As before, $p_* : D^b(R) \longrightarrow D^b(P)$ denotes the forgetful functor, p^* its left, $p^!$ its right adjoint. As P is regular, the thick hull of p^* is precisely $D^b_{perf}(R)$, the category of perfect complexes over R.

The structure theorem for modules over hypersurface rings, due to D. Eisenbud, can be expressed as follows :

Theorem 10.2.1.: With notations as just introduced, there is a distinguished triangle of endo-functors on $D^b(R)$ - or even on $D^-(\text{Mod-}R)$:

$$- \otimes_R I/I^2[1] \longrightarrow p^*p_* \xrightarrow{\pi} \text{id} \xrightarrow{s} - \otimes_R I/I^2[2] ,$$

where π is the co-unit of the adjunction (p^*, p_*).
(Remark that there is no need to derive the tensor-product with I/I^2 as this module is already free over R.)

From the abstract point of view, this triangle just expresses that R is a P-module of projective dimension one, resolved by the Koszul complex $0 \longrightarrow I \longrightarrow P \longrightarrow 0$.

Namely, if X is any complex of R-modules, p_*X is still the same complex but regarded as consisting of P-modules, and p^*p_*X is represented by the total complex associated to $X \otimes_P (0 \longrightarrow I \longrightarrow P \longrightarrow 0)$.

Now the obvious "spectral sequence" associated to this double complex shows that the projection onto

$$X = (X \otimes_R R) \otimes_P R = X \otimes_P H^0(0 \longrightarrow I \longrightarrow P \longrightarrow 0)$$

- which represents $\pi(X)$ - is surjective and that its kernel is quasi-isomorphic to $X \otimes_P I[1] = X \otimes_R P \otimes_P I[1] = X \otimes_R I/I^2[1]$.

But this is precisely the statement of the theorem.
The difficult part, examined in [Eis], is to give s explicitely for a complex X.

Here we will only show how this theorem implies immediately that resolutions over R become eventually periodic of period two :

If X is any bounded above complex of projective R-modules with finitely generated and bounded cohomology, p^*p_*X is perfect. Hence there exists a finite free complex representing it. But this means that s becomes a quasi-isomorphism after truncating those finitely many degrees in which this representative of p^*p_*X is concentrated.

If P is local, X a minimal complex of finitely generated free R-modules, then X is determined up to isomorphisms of complexes in its isomorphism class in $D^{-}(\text{Mod-}R)$ - and similarly in case that P is non-negatively graded with a local regular ring in degree zero.

In these cases, $X \otimes_R I/I^2\, 2$ will also be minimal and hence $s(X)$ an isomorphism of complexes - except at a finite number of terms. This proves that X - up to the twist with the free module I/I^2 - can be chosen to be eventually periodic of period two. Remark that if X is the resolution of a single R-module M, it can be chosen to be periodic after at most $\text{projdim}_P p_* M$ many steps.

Remark: (a) The complex $I/I^2\, 1$ is the natural representative of the relative cotangent complex $\mathbb{L}_{R/P}$ of R over P, so that s can be thought of as a natural transformation from the identity on $D^{-}(\text{Mod-}R)$ into the functor $- \otimes_R \mathbb{L}_{R/P}\, 1$. But, following Illusie's treatment of Atiyah-classes, Ill;IV.2.3., the relative Atiyah-class $at_{R/P}$ provides also such a transformation, and indeed one sees easily from the explicit construction of s in Eis that they are the same.

Now we will show that (10.2.1.) determines completely the natural transformation $c^{\cdot} : \text{Ext}^{\cdot}_R(-,-) \longrightarrow \underline{\text{Ext}}^{\cdot}_R(-,-)$ from the ordinary to the stabilized Ext-groups, (6.2.5.(3)).

As $p^* p_*$ maps $D^b(R)$ into $D^b_{perf}(R)$, the class of the transformation s becomes an isomorphism of functors in the stabilized category $\underline{D^b(R)}$.

If e denotes a (homogeneous) generator of $(I/I^2)^*$, the normal module of R over P - which is still free of rank one -, we may "normalize" s to the morphism of functors $s' = (1 \otimes_R e) \circ s : id \longrightarrow T^2$.

Then (10.2.1.) yields homomorphisms of graded rings - $\deg s' = 2$:

$$(10.2.2.) \qquad \mathbb{Z}\, s' \longrightarrow \oplus \operatorname{Hom}_{D^b(R)}(id, T^i) \longrightarrow \operatorname{Hom}_{\underline{D^b(R)}}(id, T^i) \quad ,$$

which, when evaluated in a complex X, become the ring homorphisms

$$\mathbb{Z}\, s' \longrightarrow \text{Ext}^{\cdot}_R(X,X) \xrightarrow{c\,(X,X)} \underline{\text{Ext}}^{\cdot}_R(X,X) \quad ,$$

sending s' into the class of $s'(X)$.

To say that s' is a natural transformation of functors, amounts to the fact that the image of s' is <u>central</u> and that the induced right or left actions on $\text{Ext}^{\cdot}_R(X,Y)$ for any two complexes X, Y in $D^{-}(\text{Mod-}R)$ by the Yoneda-product with $s'(X)$ and $s'(Y)$ resp. are the same, so that any of the graded Ext-groups becomes a symmetric $Z\, s'$ -bimodule.

As $c^{\cdot} \circ s'$ is invertible and $c^{\cdot}$ is an isomorphism in large degrees :

, <u>Theorem 10.2.3.</u>: For any two (complexes of) R-modules M and N (in $D^b(R)$), one has

$$\underline{\mathrm{Ext}}^{\cdot}_R(M,N) = \mathrm{Ext}^{\cdot}_R(M,N) \otimes_{\mathbb{Z}[s']} \mathbb{Z}[s',s'^{-1}] \quad ,$$

and the natural transformation $c^{\cdot}(M,N)$ is the inclusion of the first factor. Its kernel is hence precisely the s'-torsion in $\mathrm{Ext}^{\cdot}_R(M,N)$.

<u>Remark</u>: (b) As s' is of degree 2 , $\underline{\mathrm{Ext}}^{\cdot}_R(M,N)$ is in fact already completely determined by the homogeneous localizations

$$\underline{\mathrm{Ext}}^{+}_R(M,N) = (\bigoplus_i \mathrm{Ext}^{2i}_R(M,N))/(s'-1)$$

which equals any $\underline{\mathrm{Ext}}^{2j}_R(M,N)$, and

$$\underline{\mathrm{Ext}}^{-}_R(M,N) = (\bigoplus_i \mathrm{Ext}^{2i+1}_R(M,N))/(s'-1) \quad ,$$

which equals any $\underline{\mathrm{Ext}}^{2j+1}_R(M,N)$, whence we will call these groups also the "even" and "odd" Tate-cohomology respectively.

In other words, $\underline{D^b(R)}$ is in fact a $\mathbb{Z}/2\,\mathbb{Z}$-graded category.

Now assume (for simplicity) that P contains a field k and that f has only isolated singularities (non-regular points will suffice).

Then $\underline{\mathrm{Ext}}^{\cdot}_R(M,N)$ will be a finitely generated $\mathbb{Z}/2\,\mathbb{Z}$-graded module over $k[s']$ - as $\deg s' = 2$ - and can hence be decomposed into the even and odd submodule, E^+ and E^- respectively. Furthermore, the s'-torsion of either $E^{+/-}$ can be characterized in the usual way, say by means of the Jordan-blocs of the action of s' .

Remain the free $k[s']$-modules $E^{+/-}$/torsion , which are classified by their ranks. The difference of these ranks behaves like a "stable" Euler-Poincare-characteristic and seems hence to be a very reasonable invariant - in particular if compared to the case of cyclic groups - whence we give it a name :

10.3. <u>The Herbrand-difference</u>

<u>Definition 10.3.1.</u>: Let (R,m,k) be a local hypersurface ring with m the only prime at which R is not regular.

For any two (complexes of) R-modules M , N set :

$$h(M,N) = \mathrm{length}(\underline{\mathrm{Ext}}^{+}_R(M,N)) - \mathrm{length}(\underline{\mathrm{Ext}}^{-}_R(M,N))$$

and call this number the <u>Herbrand-difference</u> of (M,N) over R .

<u>Remarks</u>: (a) If R is not local, but f still has only isolated non-regular points, one may define these differences at any non-regular point (and sum up) to get <u>local</u> (and <u>global</u>) Herbrand-differences.

This applies in particular for homogeneous hypersurfaces, which are only non-regular at the origin. In this case, it is to be understood that we sum up over all the graded components of the corresponding $\underline{\mathrm{Ext}}$'s .

(b) The name is apparently motivated by the <u>Herbrand-quotient</u> for cyclic groups, [Ser 1;VIII,§5], defined as

$$\mathbf{h}(M) = |\hat{H}^0(G,M)|/|\hat{H}^1(G,M)| \quad ,$$

the quotient of the orders of the Tate-cohomology groups for a given representation M of the cyclic group G . Comparing with our definition,

$$\mathbf{h}(M) = \prod_{\mathbf{p}} \mathbf{p}^{\mathbf{h}_{\mathbf{p}}(\mathbb{Z},M)}$$

where $\mathbf{h}_{\mathbf{p}}$ is the local Herbrand-difference at the prime $\mathbf{p}$.
(Of course only those primes are non-regular - and hence contribute - which divide the order of G .)

(10.3.2.) Due to the periodicity of Tate-cohomology, for any R-module M and any short exact sequence $0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow N_3 \longrightarrow 0$ there is an <u>exact hexagon</u>

$$\begin{array}{ccccc}
\underline{\mathrm{Ext}}^+_R(M,N_1) & \longrightarrow & \underline{\mathrm{Ext}}^+_R(M,N_2) & \longrightarrow & \underline{\mathrm{Ext}}^+_R(M,N_3) \\
\uparrow & & & & \downarrow \\
\underline{\mathrm{Ext}}^-_R(M,N_3) & \longleftarrow & \underline{\mathrm{Ext}}^-_R(M,N_2) & \longleftarrow & \underline{\mathrm{Ext}}^-_R(M,N_1)
\end{array}$$

and similarly for exact sequences in the first argument.

It follows :

(i) $\mathbf{h}(-,-)$ is additive on short exact sequences in each argument, and also on distinguished triangles in $D^b(R)$.

This implies that $\mathbf{h}$ induces a $\mathbb{Z}$-valued, bilinear pairing on the Grothendieck group $K_0'(R)$ of all finitely generated R-modules. This pairing will be denoted by the same symbol.

(ii) $\mathbf{h}(U,-) = \mathbf{h}(-,U) = 0$ for all finitely generated R-modules U of finite projective dimension (or U in $D^b_{perf}(R)$).

This is clear from the definition of Tate-cohomology and shows that **h** as a pairing on $K_0'(R)$ contains $K_0(R)$, the Grothendieck group of all finitely generated projective R-modules, in its radical.

Hence it induces a pairing on $\underline{K}_0(R) = K_0'(R)/K_0(R)$, (4.9.), still to be denoted **h**.

Next we want to apply the Duality Theorem in connection with the periodicity. Let us first state this as a separate

Corollary 10.3.3.: Let (R,m,k) be a hypersurface ring with m as its only non-regular prime. Then

(i) If $\dim R$ is even, $\underline{\mathrm{Ext}}_R^+(M,N)$ and $\underline{\mathrm{Ext}}_R^-(N,M)$ are dual to each other. In particular, they have the same length.

(ii) If $\dim R$ is odd, $\underline{\mathrm{Ext}}_R^+(M,N)$ is dual to $\underline{\mathrm{Ext}}_R^+(N,M)$ and the same holds for the odd Tate-cohomology groups.

In particular,

(iii) For any complex M in $D^b(R)$, the $\mathbb{Z}/2\mathbb{Z}$-graded stable Ext-algebra $\underline{\mathrm{Ext}}_R^+(M,M) \oplus \underline{\mathrm{Ext}}_R^-(M,M)$ is quasi-Frobenius and carries a graded involution. If $\dim R$ is odd, already $\underline{\mathrm{Ext}}_R^+(M,M)$ is a quasi-Frobenius subalgebra.

This is obvious from (10.1.5.).

In the graded case, we can still do a little bit better :

Corollary 10.3.4.: Let R be homogeneous, quotient of a polynomial ring in an even number of variables by a polynomial of odd degree d - and with the origin as only non-regular point.

Then for any graded (complex of) R-module(s) M (in $D^b(R)$), the length of $\underline{\mathrm{Ext}}_R^{+/-}(M,M)$ is even.

Just substitute the given data in (10.1.6.) to obtain that the symmetry is centered at half an integer.

Applying all this to the Herbrand-differences, we get

(iii) $\mathbf{h}(M,N) = (-1)^{\dim R-1}\mathbf{h}(N,M)$ for all (complexes of) R-modules M and N (in $D^b(R)$).

In particular, the pairing given by **h** is

- alternating if $\dim R$ is even,
- symmetric if $\dim R$ is odd, and, if R is defined by a polyno-

- the quadratic form $Q_h(X) = \mathbf{h}(X,X)$ is <u>even</u>.

Before we go on to deduce further properties of $\mathbf{h}$, let us make the

<u>Remark</u>: (a) The specialization of the Duality Theorem to hypersurfaces shows that for <u>any</u> complex (or just MCM) on a hypersurface the stable $\mathbb{Z}/2\mathbb{Z}$-graded endomorphism ring shares the characteristic properties of the Clifford-algebra of a quadratic form - and it is a Clifford-algebra in case of the (MCM-approximation of the) residue class field as mentioned in (10.1.4.(ii)). A satisfactory explanation is missing - except when the hypersurface itself is a regular quadric.

Coming back to Herbrand-differences, let us translate the existence of the "trivial" duality, (4.6.):

(iv) Let M, N be two maximal Cohen-Macaulay R-modules (or complexes in $D^b(R)$) and set $(-)^* = \mathrm{Hom}_R(-,R)$, (resp. $= \mathrm{RHom}(-,R)$). Then one has obviously

$$\mathbf{h}(M,N) = \mathbf{h}(N^*,M^*) \quad .$$

(v) Let L be a perfect complex of R-modules, l its Euler-Poincaré characteristic. Then it follows by induction on the number of non-zero components of L that

$$\mathbf{h}(-\overset{\mathbb{L}}{\otimes}_R L,-) = l.\mathbf{h}(-,-) \quad .$$

(vi) If A is (a syzygy module of) some <u>artinian</u> R-module, then

$$\mathbf{h}(A,-) = \mathbf{h}(-,A) = 0 \ .$$

This follows for artinian modules themselves from the fact that their classes in $K_0'(R)$ are torsion - as for any local Cohen-Macaulay ring - , but $\mathbf{h}(A,-)$ is an additive function into the torsion-free group of integers. For syzygy-modules, the statement follows from (ii).

Finally let us mention a different description of $\mathbf{h}$ in case that R is <u>normal</u>:

(vii) Let N and M be two MCMs over a normal hypersurface ring. Then

$$\mathbf{h}(M,N) = h^1_m(N \otimes_R M^*) - h^0_m(N \otimes_R M^*) \quad ,$$

where $h^i_m(-)$ denotes the length of the local cohomology module $H^i_m(-)$.

The proof is obtained from the exact sequence in (6.4.1.(iii)), as under the assumptions made $\mathrm{Hom}_R(M,N)$ is reflexive, whence

$$H^i_m(\mathrm{Hom}_R(M,N)) = 0 \quad \text{for} \quad i = 0,1$$

and hence

$$H^i_m(N \otimes_R M^*) = \underline{\mathrm{Ext}}_R^{i-1}(M,N) \quad \text{for} \quad i = 0,1 .$$

To summarize the properties of the Herbrand-difference, let us introduce - aside $K_0^{(\prime)}(R)$ - also the Grothendieck group $K(\text{art-}R)$ of all artinian R-modules. Then summing up over all the non-regular points we get :

Theorem 10.3.5.: Let R be a hypersurface ring which has only isolated non-singular points and set $\varepsilon = (-1)^{\dim R-1}$.

Then the Herbrand-difference $\mathbf{h}$ induces an ε-symmetric pairing on

$$K(R) =_{\mathrm{def}} K_0'(R)/K_0(R) + K(\text{art-}R) .$$

This pairing is ε-invariant under the duality induced by $\mathrm{RHom}_R(-,R)$ and $K_0(R)$-bilinear with respect to the natural $K_0(R)$-module structure on $K_0'(R)$.

Formulating it separately in the graded case, we get

Theorem 10.3.6.: Let R be a homogeneous hypersurface ring defined by a polynomial of degree d in $k[x_0,\dots,x_n]$. Let X be the underlying projective hypersurface in $\mathbf{P}^n$ and assume that it is regular.

Then the Herbrand-difference induces a bi-linear pairing on

$$K^0(X) =_{\mathrm{def}} K_0(X)/K_0(\mathbf{P}^n) \quad ,$$

which is symmetric if $\dim X = n-1$ is even and the associated quadratic form is even if furthermore the degree d is odd. In case that $\dim X$ is odd, the pairing is alternating.

This Theorem suggests that the Herbrand-difference should serve as the (?) intersection form on the (co-)primitive cycles of the projective hypersurface X .

Let us consider some examples :

Example 10.3.7.: Let X be a non-singular quadric. Then it is known, by [Knö], see also [BEH], that there is precisely one non-free and indecomposable graded MCM - up to shifts and isomorphisms - if $\dim X$ is odd. In this case, $\underline{K}(R) = \mathbb{Z}/2\mathbb{Z}$ and hence $\mathbf{h}$ is trivial.

If $\dim X$ is even, there are precisely two such MCMs and the group $\underline{K}_0(R)$ is free of rank one, generated by either one of them. For any of these generating modules M, one has $\mathbf{h}(M,M) = 1$.

Example 10.3.8.: Let R be a simple analytic hypersurface singularity which is even-dimensional. Then $\underline{K}_0(R)$ is finite and hence $\mathbf{h}$ trivial again.

In case of odd dimension - and characteristic different from 2 - one is reduced to the curve case by H. Knörrer's periodicity result, [Knö].

But for a reduced plane curve singularity, the Grothendieck group $K_0'(R)$ is generated - at least up to torsion - by the structure sheaves of the irreducible branches and it is left as an exercise to calculate the pairing induced by $\mathbf{h}$ on these modules.

Finally we consider:

10.4. The 27 lines on a cubic - once again

Let $P = k[x,y,u,v]$ and let f be a homogeneous polynomial of degree three such that the underlying surface X is smooth.

(10.4.1.) A line on X is apparently given by two linear forms l_1, l_2 such that f can be written as

$$f = l_1 q_1 + l_2 q_2$$

for two quadratic polynomials q_1, q_2 in P.

Set $A = P/(l_1, l_2)$ and $L = \operatorname{Proj} A$, and denote by $A' = P/(l_1', l_2')$ and $L' = \operatorname{Proj} A'$ a (module of a) second line.

(10.4.2.) Denote $(-.-)$ the intersection form on $\operatorname{Pic}(X)$. Then it is well-known that $\operatorname{Pic}(X)$ looks as follows:

$$\operatorname{Pic}(X) = \bigoplus_{i=0}^{6} \mathbb{Z}.l_i$$

with

$$(l_0 . l_0) = 1 \quad , \quad (l_i . l_i) = -1 \quad \text{for } i > 0 \quad \text{and} \quad (l_i . l_j) = 0 \quad \text{for}$$

$i \neq j$. Recall also that in this representation the class of the dualizing sheaf is $\omega = (-3,1,\ldots,1)$. It generates the image of $\operatorname{Pic}(\mathbf{P}^3)$ in

$\mathrm{Pic}(X)$. As X is two-dimensional, $K_0(X) = \mathbb{Z} \oplus \mathrm{Pic}(X)$ and hence the group on which the Herbrand-difference will be defined is

$$K^0(X) = K_0(X)/K_0(\mathbf{P}^3) = \mathrm{Pic}(X)/\mathrm{Im}(\mathrm{Pic}(\mathbf{P}^3) \longrightarrow \mathrm{Pic}(X)) = \mathrm{Pic}(X)/\mathbb{Z}.\omega \ .$$

As the classes $l_1,\dots,l_6$ correspond to lines, the classes of lines will generate $K^0(X)$ as an abelian group and it will be enough to determine the pairing $\mathbf{h}$ on pairs of the form (A,A') , and to compare it with the values $(\underline{L}.\underline{L}')$, $\underline{L}^{(')}$ the class of the corresponding line.

(10.4.3.) Now we have the following table :

position of the lines	skew	transversal	identical
$(\underline{L}.\underline{L}')$	0	1	-1
$\underline{\mathrm{Ext}}_R^+(A,A')$	k	0	$P/(l_1,l_2,q_1,q_2)$
$\underline{\mathrm{Ext}}_R^-(A,A')$	0	length 2	0
$\mathbf{h}(A,A')$	1	-2	4

This table can be readily verified by means of direct calculation. It follows :

<u>Proposition 10.4.5.</u>: The Herbrand-difference of the homogeneous coordinate ring of a smooth cubic satisfies :

$$\mathbf{h}(A,A') = (-\tfrac{1}{3})(3\underline{L} + \omega)(3\underline{L}' + \omega) \quad ,$$

whence $(K_0(X)/K_0(\mathbf{P}^3) \otimes_{\mathbb{Z}} \mathbb{R} \ , \ \mathbf{h} \otimes 1)$ is isometric as a Euclidean space to the orthogonal complement of ω in $(\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R} \ , \ -(\ .\ )\ )$.

Of course, there should exist a more conceptual proof for this.

References

Abbreviations

[ALG] Bourbaki,N.: "Algèbre", Chap. I-III : CCLS Paris 1970; Chap. IV-VII : Masson Paris 1981; Chap. X : Masson Paris 1980.

[EGA] Grothendieck,A.; Dieudonné,J.: Eléments de la Géométrie Algébrique I-IV, Publ. de l'IHES, 4,8,11,17,20,24,28,32; (1959 ff)

[SGA] Séminaire de la Géométrie Algébrique IV,VI,VII,4 1/2, Springer Lecture Notes in Math. 269,270,305;225;288,340;569, Springer Verlag Berlin-Heidelberg -New York.

Bibliography

[Aus 1] Auslander,M.: Anneaux de Gorenstein et torsion en algèbre commutative, Sém. Samuel, Paris 1966/67.

[Aus 2] Auslander,M.: Isolated singularities and existence of almost split sequences. In: Representation Theory II, Groups and Orders, Lecture Notes 1178, 194-241, Springer Verlag New York 1986.

[A-B] Auslander,M.; Bridger,M.: Stable Module Theory, Mem. of the AMS 94, AMS Providence 1969.

[Av] Avramov,L.: Local algebra and rational homotopy, astérisque 113/114, 15-43, (1984).

[Ba] Bass,H.: Algebraic K-Theory, Benjamin Inc. New York 1968.

[BBD] Beilinson,A.A.; Bernstein,J. ; Deligne,P.: Faisceaux pervers, astérisque 100, (1984).

[Bei] Beilinson,A.A.: The Derived Category of Coherent Sheaves on P^n , Sel.Math. Sov. 3(3), 233-237,(1983/84)

[Be] Berger,R.: Über verschiedene Differentenbegriffe, Ber. Heidelberger Akad. Wiss. 1960, I.Abh.(1960)

[BGG] Bernstein,I.N.; Gelfand,I.M.; Gelfand,S.I.: Algebraic Bundles on P^n and Problems of Linear Algebra, Funkt.Anal. 12(3), 66-67, (1978)

[BEH] Buchweitz,R.-O.; Eisenbud, D.; Herzog, J.: Cohen-Macaulay Modules on Quadrics, this volume.

[BGS] Buchweitz,R.-O.; Greuel,G.-M.; Schreyer,F.-O.: Cohen-Macaulay Modules on Hypersurface Singularities II, Inv.math. 88, 165-182, (1987)

[C-E] Cartan,H; Eilenberg,S.: Homological Algebra, Princeton Univ. Press, Princeton 1973.

[Eis] Eisenbud,D.: Homological Algebra on a Complete Intersection with an Application to Group Representations, Trans. AMS 260, 35-64, (1980).

[FHJ] Felix,Y. et al.: The Radical of the Homotopy Lie Algebra, preprint 1986.

[FGR] Fossum,R.; Griffith,P.A.; Reiten,I.: Trivial Extensions of Abelian Categories, Lecture Notes 456, Springer Verlag New York 1975.

[Fos] Fossum,R.: The Noetherian Different of Projective Orders, J. reine u. ang. Math. 224, 207-218, (1966).

[Gel] Gelfand,S.I.: Appendix to the russian translation of [OSS], Mir Moscow 1984.

[Hap] Happel,D.: Triangulated Categories in Representation Theory of Finite Dimensional Algebras, Habilitationsschrift, Bielefeld 1985.

[Ha] Hartshorne,R.: Residues and Duality, Lecture Notes 20, Springer Verlag New York 1966.

[He 1] Heller,A.: The loop-space functor in homological algebra, Trans. AMS 96, 382-394, (1960).

[He 2] Heller,A.: Homological Algebra in Abelian Categories, Ann. of Math. 68, 484-525, (1958).

[Her] Herzog,J.: Ringe mit nur endlich vielen Isomorphieklassen unzerlegbarer Cohen-Macaulay Moduln, Math.Ann. 233,21-34(1978)

[H-K] Herzog,J.; Kunz,E.: Der kanonische Modul eines Cohen-Macaulay Ringes, Lecture Notes 238, Springer Verlag New York 1971.

[Kap] Kaplansky,I.: Commutative Rings, rev. edition, Univ. of Chicago Press, Chicago 1974.

[Knö] Knörrer,H.: Cohen-Macaulay Modules on hypersurface singularities I, inv. math. 88, 153-164, (1987).

[Ku] Kunz,E.: Vollständige Durchschnitte und Differenten, Archiv der Math. 19, 47-58, (1968)

[O-P] Ogievetskii,O.V.; Penkov,I.B.: Serre Duality for Projective Supermanifolds, 18(1), 78-79, (1984).

[OSS] Okonek,Ch.; Schneider,M.; Spindler,H.: Vector Bundles on Complex Projective Space, Progress in Math. 3, Birkhäuser Verlag, New York 1980.

[Qu 1] Quillen,D.: Higher algebraic K-theory I, in: Algebraic K-Theory I, 85-147, Lecture Notes 341, Springer Verlag New York 1973.

[Qu 2] Quillen,D.: On the (co-)homology of commutative rings, Proc.Symp. AMS XVII, 65-87, (1970).

[Ro] Roos,J.-E.: Complĕments a l'ĕtude des quotients primitifs des algĕbres de Lie semi-simples, C.R.Acad.Sci. Paris, Sĕr. A 276, 447-450, (1973).

[Ser 1] Serre,J.-P.: Cohomologie Galoisienne, Lecture Notes 5, Springer Verlag New York 1973.

[Ser 2] Serre,J.-P.: Local Fields, Grad.Texts in Math 67, Springer Verlag New York 1979.

[Sta] Stafford,J.-T.: Homological properties of the enveloping algebra $U(sl_2)$, Math.Proc.Camb.Phil.Soc. 91, 29-37, (1982).

[Sw] Swan,R.G.: K-theory of quadric hypersurfaces, Ann. of Math. 122, 113-153, (1985).

[Sw 1] Swan,R.G.: Algebraic K-Theory, Lecture Notes 76, Springer Verlag New York 1968.

[Ver] Verdier,J.-L.: Catĕgories Dĕrivĕes; Etat O, in: SGA 4 1/2, 262-311.

[Z] Zaks, A.: Injective Dimensions of Semi-Primary Rings J. of Alg. 13, 73-86 (1969).