Eta-Products, BPS States and K3 Surfaces

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A Pair of Classical Functions

- Euler φ and Dedekind η: $\begin{cases} \varphi(q) &= \prod\limits_{n=1}^{\infty}(1-q^n)^{-1} = \sum\limits_{k=0}^{\infty}\pi_k q^k \\ \eta(q) &= q^{\frac{1}{24}}\prod\limits_{n=1}^{\infty}(1-q^n) = q^{\frac{1}{24}}\varphi(q)^{-1} \end{cases}$
- Notation: upper-half plane $\mathcal{H} := \{z : \mathrm{Im}(z) > 0\}$; nome $q = \exp(2\pi i z)$
- Remarks, 24 is special mathematically and physically
  - q-expansion $\pi_k = \#$ **integer partitions** of k
  - η is modular form of weight $\frac{1}{2}$: $q^{\frac{1}{24}}$ is crucial (24 comes from $\zeta(-1)$ through Bernoulli B_2 and Eisenstein $E_2(q)$)
  - Familiar to string theorists, *bosonic oscillator partition function* $G(q) := \mathrm{Tr}\, q^{\sum\limits_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n} = \sum\limits_{n=0}^{\infty} d_n q^n = \varphi(q)^{24} = q\eta(q)^{-24}$; Hardy-Ramanujan gives asymptotics $\rightsquigarrow$ Hagedorn (24 comes from conformal anomaly $\zeta(-1)$)

    Rmk: 24 iff modularity of 1-loop diagram

Elliptic Curves

- For elliptic curve $y^2 = 4x^3 - g_2\ x - g_3$, two (related) functions "discriminate" – test isomorphism/inequivalent modular forms

$$\begin{cases} \text{Modular Discriminant:} & \Delta = g_2^3 - 27 g_3^2 \\ \text{Klein j-Invariant:} & j = 1728 \frac{g_2^3}{\Delta} = \frac{\theta_{E_8}}{\Delta} \end{cases}$$

- In terms of modular parameter z, $(x, y) = (\wp(z), \wp'(z))$

$$\Delta(z) = \eta(z)^{24} := q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n$$

- Similarly (only 1980's! by Borcherds in his proof of Moonshine)

$$j(p) - j(q) = \left(\frac{1}{p} - \frac{1}{q}\right) \prod_{m,n=1} (1 - p^n q^m)^{c_{n*m}}$$

for $j(q) = \sum_n c_n q^n = \frac{1}{q} + 744 + 196884 q + \ldots$

Multiplicativity

- Multiplicative Function/Sequence $\{a_n\}$ (with $a_1 = 1$)

$$\text{(Completely) Multiplicative}: \quad a_{m*n} = a_m a_n \ , \quad m,n \in \mathbb{Z}_{>0} \ ;$$
$$\text{(Weakly) Multiplicative}: \quad a_{m*n} = a_m a_n \ , \quad \gcd(m,n) = 1 \ ;$$

- Rmk: Dirichlet transform $\rightsquigarrow$ interesting: $L(s) = \sum\limits_{n=1} \frac{a_n}{n^s}$, e.g. $a_n = 1 \rightsquigarrow L(s) = \zeta(s)$
- Ramanujan: $\Delta(q) = qG(q)^{-1} = \sum\limits_{n=1}^{\infty} \tau(n) q^n \rightsquigarrow$ Ramanujan tau-function $\tau(n)$ is (weakly) multiplicative:

n	1	2	3	4	5	6	7	8	9	10
$\tau(n)$	1	-24	252	-1472	4830	-6048	-16744	84480	-113643	-115920

- **24 is crucial** in $\Delta(z) = \eta(z)^{24}$
- Fun fact [YHH-McKay, 2014] $\sum\limits_{n=1}^{24} \tau(n)^2 \equiv \sum\limits_{n=1}^{24} c_n(j)^2 \equiv 42 (\mathrm{mod}\, 70)$

Multiplicative Eta-Products

- $\eta(q)^{24}$ is multiplicative, are there others made from η?
- Define Frame Shape [J. S. Frame, or cycle shape] (t: cycle length)

$$F(z) = [n_1, n_2, \ldots, n_t] := \prod_{i=1}^{t} \eta(n_i z) = \prod_{i=1}^{t} \eta(q^{n_i})$$

- Dummit-Kisilevsky-McKay (1982):
  - $a_1 = 1$ and $[n_1, n_2, \ldots, n_t]$ is partition of 24
  - Balanced: $n_1 > \ldots > n_t$, $n_1 n_t = n_2 n_{t-1} = \ldots$
  - there are precisely 30 which are multiplicative out of $\pi(24) = 1575$
  - each is a modular form of weight $k = t/2$, level $N = n_1 n_t$, Jacobi character χ

$$F(\frac{az+b}{cz+d}) = (cz+d)^k \chi^k F(z) \ , \quad \chi = \begin{cases} (-1)^{\frac{d-1}{2}} \left(\frac{N}{d}\right) & , \quad d \text{ odd} \\ \left(\frac{d}{N}\right) & , \quad d \text{ even} \end{cases} , \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$$

in summary:

The 30

k	N	eta-product	χ
12	1	$[1^{24}] = \Delta(q)$	1
8	2	$[2^8, 1^8]$	1
6	3	$[3^6, 1^6]$	1
	4	$[2^{12}]$	1
5	4	$[4^4, 2^2, 1^4]$	$\left(\frac{-1}{d}\right)$
4	6	$[6^2, 3^2, 2^2, 1^2]$	1
	5	$[5^4, 1^4]$	1
	8	$[4^4, 2^4]$	1
	9	$[3^8]$	1
3	8	$[8^2, 4, 2, 1^2]$	$\left(\frac{-2}{d}\right)$
	7	$[7^3, 1^3]$	$\left(\frac{-7}{d}\right)$
	12	$[6^3, 2^3]$	$\left(\frac{-3}{d}\right)$
	16	$[4^6]$	$\left(\frac{-1}{d}\right)$

k	N	eta-product	χ
2	15	$[15, 5, 3, 1]$	1
	14	$[14, 7, 2, 1]$	1
	24	$[12, 6, 4, 2]$	1
	11	$[11^2, 1^2]$	1
	20	$[10^2, 2^2]$	1
	27	$[9^2, 3^2]$	1
	32	$[8^2, 4^2]$	1
	36	$[6^4]$	1
1	23	$[23, 1]$	$\left(\frac{-23}{d}\right)$
	44	$[22, 2]$	$\left(\frac{-11}{d}\right)$
	63	$[21, 3]$	$\left(\frac{-7}{d}\right)$
	80	$[20, 4]$	$\left(\frac{-20}{d}\right)$
	108	$[18, 6]$	$\left(\frac{-3}{d}\right)$
	128	$[16, 8]$	$\left(\frac{-2}{d}\right)$
	144	$[12^2]$	$\left(\frac{-1}{d}\right)$

k	eta-product
"$\frac{3}{2}$"	$[8^3]$
"$\frac{1}{2}$"	$[24]$

Q: Do these show up as partition functions in physics?

Type II on $K3 \times T^2$

- 4-D, $\mathcal{N} = 4$ theory ($\simeq$ heterotic on T^6)
- Any other preserving $\mathcal{N} = 4$? cf. Aspinwall-Morrison: freely-acting quotients of K3, a total of 14, Nikulin Classification (preserves the $(2,0)$-form, 1979):

$$\mathbb{Z}_{n=2,\ldots,8} \ , \quad \mathbb{Z}^2_{m=2,3,4} \ , \quad \mathbb{Z}_2 \times \mathbb{Z}_4 \ , \quad \mathbb{Z}_2 \times \mathbb{Z}_6 \ , \quad \mathbb{Z}_2^3 \ , \quad \mathbb{Z}_2^4$$

- CHL orbifold [Chaudhuri-Hockney-Lykken, 1995]:
  - Type IIB on $K3 \times \tilde{S}^1 \times S^1$ with Nikulin involution and simultaneously $\mathbb{Z}_t \curvearrowright S^1$ by $\exp(2\pi i/t)$
  - Dual to het on $T^4 \times \tilde{S}^1 \times S^1$ with $\mathbb{Z}_t \curvearrowright \Gamma^{20,4}$ Narain lattice

Dyonic Spectrum and 1/2-BPS States

- Dyons (q_e, q_m) [cf. precision BH micro-state counting, Sen-David, 2006]:
  - D5-branes wrapping $K3 \times S^1$ and Q_1 D1-branes wrapping S^1
  - KK monopole for $\tilde{S}^1$ with $(2-k, J)$ units of $(S^1, \tilde{S}^1)$ momentum
  - $q_e^2 = 2(k-2)/t \ , \quad q_m^2 = 2(Q_1 - 1) \ , \quad q_e \cdot q_m = J$
- in the unorbifolded case (Het on T^6): 1/2-BPS states with charge $n = \frac{1}{2}q_e^2$ has degeneracy [Sen, Dabholkar-Denef-Moore-Pioline, 2007]

$$\eta(q)^{-24} = \Delta(q)^{-1} = [1^{24}]^{-1} = \frac{1}{16}\sum_{n=-1} d_n q^n$$

- In general [Govindarajan-Krishna, 2009]:

$$\left(\prod_{i=1}^{t} \eta(n_i z)\right)^{-1} = \frac{1}{16}\sum_{n=-1} d_n q^{n/t}$$

- in summary:

The 14

k	N	eta-product	χ	Nikulin K3
12	1	$[1^{24}] = \Delta(q)$	1	-
8	2	$[2^8, 1^8]$	1	$\mathbb{Z}_2$
6	3	$[3^6, 1^6]$	1	$\mathbb{Z}_3$
	4	$[2^{12}]$	1	$\mathbb{Z}_2 \times \mathbb{Z}_2$
5	4	$[4^4, 2^2, 1^4]$	$\left(\frac{-1}{d}\right)$	$\mathbb{Z}_4$
4	6	$[6^2, 3^2, 2^2, 1^2]$	1	$\mathbb{Z}_5$
	5	$[5^4, 1^4]$	1	$\mathbb{Z}_6$
	8	$[4^4, 2^4]$	1	$\mathbb{Z}_2 \times \mathbb{Z}_4$
	9	$[3^8]$	1	$\mathbb{Z}_3 \times \mathbb{Z}_3$
3	8	$[8^2, 4, 2, 1^2]$	$\left(\frac{-2}{d}\right)$	$\mathbb{Z}_7$
	7	$[7^3, 1^3]$	$\left(\frac{-7}{d}\right)$	$\mathbb{Z}_8$
	12	$[6^3, 2^3]$	$\left(\frac{-3}{d}\right)$	$\mathbb{Z}_2 \times \mathbb{Z}_6$
	16	$[4^6]$	$\left(\frac{-1}{d}\right)$	$\mathbb{Z}_4 \times \mathbb{Z}_4$
2	11	$[11^2, 1^2]$	1	$\mathbb{Z}_{11}$

Rmk: No $k \leq 1$ and only one $k = 2$

Special K3 Surfaces X

- Néron-Severi Lattice $NS(X) := \ker\left(\gamma \to \int_\gamma \Omega\right) = H^2(X;\mathbb{Z}) \cap H^{1,1}(X) =$ Divisors/(Alg. equiv.); Picard Number $\rho(X) = \mathrm{rk}(NS(X))$
- Mordell-Weil Lattice $MW(X) = \mathrm{rk}(X_\mathbb{Q})$ (cf. Birch-Swinerton-Dyer for $\mathbb{E}$)
- Generically

	K3	Alg. K3	Elliptic K3	...	Exceptional
$NS(X)$	$\{0\}$	$H\mathbb{Z}$	$B\mathbb{Z} \oplus F\mathbb{Z}$	...	$\Gamma^{20} \subset E_8(-1)^2 \oplus (U_2)^3$
$\rho(X)$	0	1	2	...	20

- Classification results (each a subset)
  - Exceptional ("singular") [Shioda-Inose, 1977]: top $\rho(X) = 20$; 1:1 with integral binary quadratic forms $\begin{pmatrix} a & b \\ b & c \end{pmatrix} \Big/ SL(2;\mathbb{Z})_{\text{similarity}}$;
  - Extremal non-Elliptic: ???
  - Extremal Elliptic [Shimada-Zhang]: + finite $MW(X)$, a total of 325;
  - Extremal Semi-Stable Elliptic [Miranda-Persson, 1988]: + Type I_n fibres only, a total of 112

Elliptic Semi-Stable Extremal K3

- $K3 \to \mathbb{P}^1$ with elliptic fibration: $\{y^2 = 4x^3 - g_2(s)\ x - g_3(s)\} \subset \mathbb{C}[x,y,s]$
- Only I_n sing. fibres, s.t. I_n **partition of 24**, i.e., EssE $\sim$ Frame Shape
  - Shioda-Tate: $\rho = \sum\limits_i (n_i - 1) + \mathrm{rk}(MW) + 2 = 26 + \mathrm{rk}(MW) - t \rightsquigarrow t \geq 6$
  - Extremal: $t = 2k = 6$
- Klein j-invariant is a rational function

  $J(s) = \frac{1}{1728} j(s) = \frac{g_2^3(s)}{\Delta(s)} = \frac{g_2^3(s)}{g_2^3(s) - 27 g_3^2(s)} \ : \ \mathbb{P}^1_s \longrightarrow \mathbb{P}^1,$ s.t.
  - 8 preimages of $J(s) = 0$ all multiplicity (ramification index) 3;
  - 12 preimages of $J(s) = 1$ with ramification index 2;
  - t preimages of $J(s) = \infty$, ramification indices $[n_1, \ldots, n_t]$;
  - ?$\exists$ ramification points $x_1, \ldots, x_m \neq (0, 1, \infty)$ but for $t = 6$, no such points.

Grothendieck's Dessin d'Enfant

- Belyǐ Map: rational map $\beta : \Sigma \longrightarrow \mathbb{P}^1$ ramified only at $(0, 1, \infty)$
  - **Theorem [Belyǐ]:** (1980) β exists $\Leftrightarrow$ Σ can be defined over $\overline{\mathbb{Q}}$
  - (β, Σ) Belyǐ Pair
- Dessin d'Enfants $= \beta^{-1}([0,1] \in \mathbb{P}^1) \subset \Sigma$
  - bi-partite graph on Σ: label all the preimages $\beta^{-1}(0)$ black and $\beta^{-1}(1)$ white, then $\beta^{-1}(\infty)$ lives one per face and $\beta^{-1}([0,1])$ gives connectivity
  - B blacks and W whites, with valency of each $=$ ramification index
  - Ramification data / Passport: $\left\{ \begin{array}{c} r_0(1), r_0(2), \ldots, r_0(B) \\ r_1(1), r_1(2), \ldots, r_1(W) \\ r_\infty(1), r_\infty(2), \ldots, r_\infty(I) \end{array} \right\}$
- Rmk: Dimer Models on T^2 = Quivers on Toric CY3 = Dessins [Hanany-YHH-Jejjala-Pasuconis-Ramgoolam-Rodriguez-Gomez]

Dessins: Permutation Triples and Cartography

- Equivalent description of dessin, Permutation Triple:
  - d edges in dessin, use cycle notation in symmetric group S_d

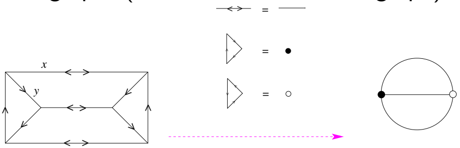
    $\sigma_B = (\ldots)_{r_0(1)}(\ldots)_{r_0(2)} \ldots (\ldots) r_0(B)$,

    $\sigma_W = (\ldots)_{r_1(1)}(\ldots)_{r_1(2)} \ldots (\ldots) r_1(W)$,

    $\sigma_B \sigma_W \sigma_\infty = \mathbb{I}$
  - encodes how the sheets are permuted at the ramification points; cf. Ramgoolam, de Mello Koch et al. relation to matrix models
  - Cartographic group: $\langle \sigma_B, \sigma_W \rangle \subset S_d$
- **Upshot:** Beukers-Montanus, 2008 j-invariants of EssE K3s are Belyi
- Grothendieck: "I do not believe that a mathematical fact has ever struck me quite so strongly as this one, nor had a comparable psychological impact ..."
- Dessins $\sim$ faithful rep of $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$

Modular Group & Cayley Graphs

- Modular Group: $\Gamma := PSL(2;\mathbb{Z}) \simeq \langle S, T \mid S^2 = (ST)^3 = I \rangle$
  - free product $C_2 \star C_3$, $C_2 = \langle x | x^2 = I \rangle$ and $C_3 = \langle y | y^3 = I \rangle$.
  - Cayley Graph: nodes = group elements, arrows = group multiplication $\rightsquigarrow$ free trivalent tree with nodes replaced by directed triangles
- Finite index subgroups of Γ
  - Finite number of cosets
  - each coset $\rightsquigarrow$ node, arrows = group multiplication $\Rightarrow$
  - coset graphs (finite directed trivalent graph): Schreier-Cayley Graphs



An Index = 6 Example

Finite Index, Genus 0, Torsion Free Congruence Subgroups

- Congruence: most important (everything so far)
- Torsion Free: nothing except $\mathbb{I}$ of finite order
- Genus Zero: upper half plane $\mathcal{H}$ can quotient $G \subset \Gamma \rightsquigarrow$ Modular Curve Σ_C
    - $\Gamma/\mathcal{H} \simeq \mathbb{P}^1$ (upto cusps)
    - so what subgroup also gives $\mathbb{P}^1$? i.e. genus$(\Sigma_C) = 0$?
    - RARE & relevant to Moonshine
    - Complete classification by Sebbar (2003): torsion-free, genus 0: only 33
- The 33 genus 0 torsion free subgroups of Γ, all are index 6, 12, 24, 36, 48, 60
- 9 of these are 6-partitions of 24

The 9

Group	Cusp Widths
Ia: $\Gamma(4)$	$[4^6]$
Ib: $\Gamma(8;4,1,2)$	$[2^2,4^3,8]$
IIa: $\Gamma_0(3)\cap\Gamma(2)$	$[2^3,6^3]$
IIb: $\Gamma_0(12)$	$[1^2,3^2,4,12]$
IIIa: $\Gamma_1(8)$	$[1^2,2,4,8^2]$
IIIb: $\Gamma_0(8)\cap\Gamma(2)$	$[2^4,8^2]$
IIIc: $\Gamma_0(16)$	$[1^4,4,16]$
IIId: $\Gamma(16;16,2,2)$	$[1^2,2^3,16]$
IV: $\Gamma_1(7)$	$[1^3,7^3]$

$$\Gamma(m) := \{A \in SL(2;\mathbb{Z}) \mid A \equiv \pm I \bmod m\}/\{\pm I\}$$
$$\Gamma_1(m) := \left\{A \in SL(2;\mathbb{Z}) \,\middle|\, A \equiv \pm \begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix} \bmod m\right\}/\{\pm I\}$$
$$\Gamma_0(m) := \left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \Gamma \,\middle|\, c \equiv 0 \bmod m\right\}/\{\pm I\}$$
$$\Gamma(m;\tfrac{m}{d},\epsilon,\chi) := \left\{\pm\begin{pmatrix}1+\frac{m}{\epsilon\chi}\alpha & d\,\beta\\ \frac{m}{\chi}\gamma & 1+\frac{m}{\epsilon\chi}\delta\end{pmatrix} \,\middle|\, \gamma \equiv \alpha \bmod \chi\right\}.$$

YHH-McKay-Read: The 112 EssE $\sim$ (not necessarily) Congruence subgroups

Modular Elliptic Surfaces

- Recall: $\gamma \in \Gamma \curvearrowright \mathcal{H}$ by $\tau \mapsto \gamma \cdot \tau = \frac{a\tau+b}{c\tau+d}$, with $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\det\gamma = 1$
- Extend and twist (Shioda, 1970's):

$$(\gamma, (m,n)) \in \Gamma \ltimes \mathbb{Z}^2 \curvearrowright \mathcal{H} \times \mathbb{C} : \quad (\tau, z) \mapsto \left(\frac{a\tau+b}{c\tau+d}, \frac{z+m\tau+n}{c\tau+d} \right)$$

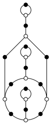
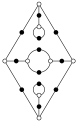
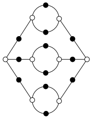
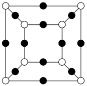
- Quotient: $(\mathcal{H} \times \mathbb{C})/(\Gamma \ltimes \mathbb{Z}^2)$ is a complex surface which is fibred
  - Base: $\mathcal{H}/\Gamma =$ the modular curve Σ_C
  - Fibre: (generically) $\mathbb{C}/\mathbb{Z}^2 \simeq$ a torus
  - Get a surface elliptically fibred over Σ_C: modular elliptic surface with complex parameter τ (torus $T^2 \simeq \mathbb{C}/(m\tau + n)$)
- Take finite index genus 0 subgroup of Γ: base is $\mathbb{P}^1 \Rightarrow$ elliptic surfaces over $\mathbb{P}^1$ of Euler number $=$ index of group; so 24 $\rightsquigarrow$ K3!

Correspondences

- elliptic models for Nikulin K3: Garbagnati-Sarti, 2008; EssE equations: Topp-Yui 2007, YHH-McKay 2012
- YHH-McKay 2013: They are the same (EssE, modular) K3 by explicitly showing the same j-invariants as Belyi maps, and $\sim$ congruence groups
- Extremal case: 6-partitions of 24

Eta Product	(k, N, χ)	Modular Subgroup	Nikulin Involution	J-Map
$[7^3, 1^3]$	$(3, 7, \left(\frac{-7}{d}\right))$	$\Gamma_1(7)$	$\mathbb{Z}_7$	$\frac{\left(s^8-12s^7+42s^6-56s^5+35s^4-14s^2+4s+1\right)^3}{(s-1)^7s^7\left(s^3-8s^2+5s+1\right)}$
$[8^2, 4, 2, 1^2]$	$(3, 8, \left(\frac{-2}{d}\right))$	$\Gamma_1(8)$	$\mathbb{Z}_8$	$-\frac{16\left(s^8-28s^6-10s^4+4s^2+1\right)^3}{s^4\left(s^2+1\right)^8\left(2s^2+1\right)}$
$[6^3, 2^3]$	$(3, 12, \left(\frac{-3}{d}\right))$	$\Gamma_0(3) \cap \Gamma(2)$	$\mathbb{Z}_2 \times \mathbb{Z}_6$	$\frac{\left(3s^2+8\right)^3\left(3s^6+600s^4-960s^2+512\right)^3}{8s^6\left(8-9s^2\right)^2\left(s^2-8\right)^6}$
$[4^6]$	$(3, 16, \left(\frac{-1}{d}\right))$	$\Gamma(4)$	$\mathbb{Z}_4^2$	$\frac{16(1+14s^4+s^8)^3}{s^4(s^4-1)^4}$

Explicit Equations, Congruence Groups, Dessins

Nikulin Inv	Dessin/Schreier Graph	Congruence Group	Equation
$\mathbb{Z}_7$		$\Gamma_1(7)$	$y^2 + (1+s-s^2)xy + (s^2-s^3)y = x^3 + (s^2-s^3)x^2$
$\mathbb{Z}_8$		$\Gamma_1(8)$	$(x+y)(xy-1) + \frac{4is^2}{s^2+1}xy = 0$
$\mathbb{Z}_2 \times \mathbb{Z}_6$		$\Gamma_0(3) \cap \Gamma(2)$	$(x+y)(x+1)(y+1) + \frac{8s^2}{8-s^2}xy = 0$
$\mathbb{Z}_4^2$		$\Gamma(4)$	$x(x^2+2y+1) + \frac{s^2-1}{s^2+1}(x^2-y^2) = 0$

Beyond Extremality, Beyond Modular Group

- YHH-McKay, 2013 non 6-partitions of 24 (Shioda: $t \geq 6$)

Eta Product	(k, N, χ)	Nikulin Involution	Equation
$[2^8, 1^8]$	$(8, 2, 1)$	$\mathbb{Z}_2$	$y^2 = x(x^2 + p_4 x + q_8)$
$[3^6, 1^6]$	$(6, 3, 1)$	$\mathbb{Z}_3$	$y^2 = x^3 + \frac{1}{3}x(2p_2 q_6 + p_2^4) + \frac{1}{27}(q_6^2 - p_2^6)$
$[2^{12}]$	$(6, 4, 1)$	$\mathbb{Z}_2^2$	$y^2 = x(x - p_4)(x - q_4)$
$[4^4, 2^2, 1^4]$	$(5, 4, \left(\frac{-1}{d}\right))$	$\mathbb{Z}_4$	$y^2 = x(x^2 + (p_2 - 2q_4)x + q_4^2)$
$[6^2, 3^2, 2^2, 1^2]$	$(4, 6, 1)$	$\mathbb{Z}_6$	$y^2 = x(x^2 + (-3p_2^2 + q_2^2)x + p_2^3(3p_2 + 2q_2))$
$[5^4, 1^4]$	$(4, 5, 1)$	$\mathbb{Z}_5$	$y^2 = x^3 + \frac{1}{3}x\left(-q_2^4 + p_2^2 q_2^2 - p_2^4 - 3p_2 q_2^3 + 3p_2^3 q_2\right) + $ $+\frac{1}{108}(p_2^2 + q_2^2)(19q_2^4 - 34p_2^2 q_2^2 + 19p_2^4 + 18p_2 q_2^3 - 18p_2^3 q_2)$
$[4^4, 2^4]$	$(4, 8, 1)$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$y^2 = x(x - p_2^2)(x - q_2^2)$
$[3^8]$	$(4, 9, 1)$	$\mathbb{Z}_3^2$	$y^2 = x^3 + 12x\left((s^2+1)(p_0 s^2 + q_0)^3 + (s^2+1)^4\right) +$ $+2\left((p_0 s^2 + q_0)^6 - 20(p_0 s^2 + q_0)^3(s^2+1)^3 - 8(s^2+1)^6\right)$

- YHH-Read, 2014 Hecke groups: beyond trivalency

Relation to Fermat

- Weight $t/2 = k = 2$, Taniyama-Shimura-Wiles: (Hasse-Weil) L-function of elliptic curve of conductor N $\underset{\text{Dirichlet}}{\overset{\text{Mellin}}{\rightleftarrows}}$ Weight 2, level N Modular form
- A Elliptic Curve/Eta Product Correspondence for $t = 4$

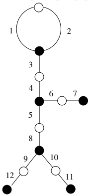
Tate form $y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 + a_6$

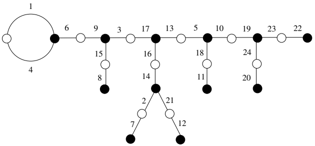
N	eta-product	$(a_1, a_2, a_3, a_4, a_6)$	j
15	$[15, 5, 3, 1]$	$(1, 1, 1, -10, -10)$	$13^3 \cdot 37^3 / 2^6 \cdot 3^7 \cdot 5^4$
14	$[14, 7, 2, 1]$	$(1, 0, 1, 4, -6)$	$5^3 \cdot 43^3 / 2^{12} \cdot 3^3 \cdot 7^3$
24	$[12, 6, 4, 2]$	$(0, -1, 0, -4, 4)$	$13^3 / 2^2 \cdot 3^5$
11	$[11^2, 1^2]$	$(0, -1, 1, -10, -20)$	$-2^6 \cdot 31^3 / 3^3 \cdot 11^5$
20	$[10^2, 2^2]$	$(0, 1, 0, 4, 4)$	$11^3 / 2^2 \cdot 3^3 \cdot 5^2$
27	$[9^2, 3^2]$	$(0, 0, 1, 0, -7)$	0
32	$[8^2, 4^2]$	$(0, 0, 0, 4, 0)$	1
36	$[6^4]$	$(0, 0, 0, 0, 1)$	0

Relation to Moonshine

- Ramanujan $\tau(n) = \{1, -24, 252, -1472, 4830, -6048, -16744, \ldots\}$
- Dim(Irreps) of Sporadic group Mathieu M_{24} = { 1, 23, 45, 45, 231, 231, 252, 253, 483, 770, 770, 990, 990, 1035, 1035, $\ldots$ }

  Mason, 1985: $-24 = -1 - 23, \quad 252 = 252, \quad -1472 = 1 + 23 - 231 - 1265, \ \ldots$
- Mukai, 1988: All K3 automorphisms $\subset M_{23}$
- Monsieur Mathieu et son chien:





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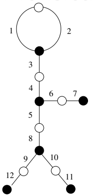

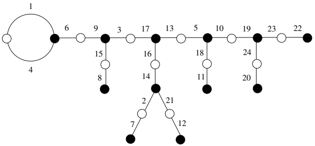

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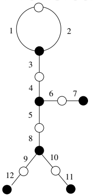

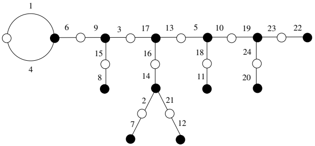

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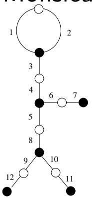
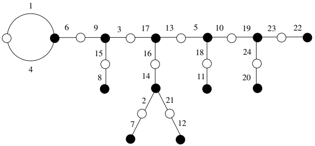




30: Revisited

Ramanujan & Mathieu

K3 (Nikulin)

k	N	eta-product	χ
12	1	$[1^{24}] = \Delta(q)$	1
8	2	$[2^8, 1^8]$	1
6	3	$[3^6, 1^6]$	1
	4	$[2^{12}]$	1
5	4	$[4^4, 2^2, 1^4]$	$\left(\frac{-1}{d}\right)$
4	6	$[6^2, 3^2, 2^2, 1^2]$	1
	5	$[5^4, 1^4]$	1
	8	$[4^4, 2^4]$	1
	9	$[3^8]$	1
3	8	$[8^2, 4, 2, 1^2]$	$\left(\frac{-2}{d}\right)$
	7	$[7^3, 1^3]$	$\left(\frac{-7}{d}\right)$
	12	$[6^3, 2^3]$	$\left(\frac{-3}{d}\right)$
	16	$[4^6]$	$\left(\frac{-1}{d}\right)$

Elliptic Curves

k	N	eta-product	χ
2	15	$[15, 5, 3, 1]$	1
	14	$[14, 7, 2, 1]$	1
	24	$[12, 6, 4, 2]$	1
	11	$[11^2, 1^2]$	1
	20	$[10^2, 2^2]$	1
	27	$[9^2, 3^2]$	1
	32	$[8^2, 4^2]$	1
	36	$[6^4]$	1
1	23	$[23, 1]$	$\left(\frac{-23}{d}\right)$
	44	$[22, 2]$	$\left(\frac{-11}{d}\right)$
	63	$[21, 3]$	$\left(\frac{-7}{d}\right)$
	80	$[20, 4]$	$\left(\frac{-20}{d}\right)$
	108	$[18, 6]$	$\left(\frac{-3}{d}\right)$
	128	$[16, 8]$	$\left(\frac{-2}{d}\right)$
	144	$[12^2]$	$\left(\frac{-1}{d}\right)$

?

k	eta-product
" $\frac{3}{2}$ "	$[8^3]$
" $\frac{1}{2}$ "	$[24]$

?

Attractors Ferrara-Gibbons-Kallosh, 1995 & Arithmetic Moore, 1998

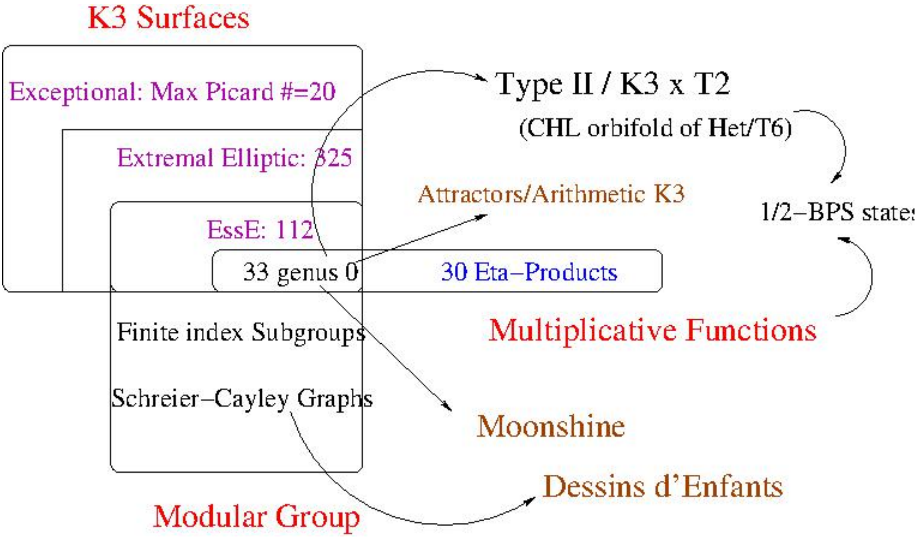
- Type IIB / $M_4 \times (X = CY_3)$, $\gamma \in H_3(X;\mathbb{Z})$
  - Abelian fieldstrength: $F \in \bigwedge^2(M_4;\mathbb{R}) \otimes H^3(X;\mathbb{R})$; dyonic charges: $\int F = \hat{\gamma} \in H^3(X;\mathbb{Z})$; central charge $|Z(z;\gamma)|^2 = |\int_\gamma \Omega|^2 / \int \Omega \wedge \overline{\Omega}$
  - $|Z(z;\gamma)|^2$ has stationary point z_* in complex structure moduli space with $Z(z_*;\gamma) \neq 0 \quad \Leftrightarrow \quad \hat{\gamma}$ has Hodge decomposition $\hat{\gamma} = \hat{\gamma}^{3,0} + \hat{\gamma}^{0,3}$ (i.e., $\hat{\gamma}^{1,2} = \hat{\gamma}^{2,1} = 0$); local minimum
- Attractor points $\sim$ arithmetic varieties

  e.g. $X = K3 \times T^2$ with $\hat{\gamma} = p \oplus q \in H^3(X;\mathbb{Z}) \simeq H^2(K3;\mathbb{Z}) \otimes H^1(T^2;\mathbb{Z}) \simeq H^2(K3;\mathbb{Z})^2$, then

  Attractor point is $\quad T^2 = \mathbb{E}_\tau; \quad K3 = Y_{Q_{p,q}}$

  where Y is the Shioda-Inose K3 associated to quadratic form $Q_{p,q}$ and $\tau = \frac{p\cdot q + i\sqrt{D_{p,q}}}{p^2}$, $D_{p,q} := (p\cdot q)^2 - p^2 q^2$

Summary


K3 Surfaces
Exceptional: Max Picard #=20
Extremal Elliptic: 325
EssE: 112
33 genus 0
30 Eta–Products
Finite index Subgroups
Schreier–Cayley Graphs
Modular Group
Type II / K3 x T2
(CHL orbifold of Het/T6)
Attractors/Arithmetic K3
1/2–BPS states
Multiplicative Functions
Moonshine
Dessins d'Enfants