On the energy statistic and the cyclic action on invariant tensors

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Introduction

Let $\mathfrak{g}$ be a simple complex Lie algebra and U a simple $\mathfrak{g}$-module.
For each $r \geq 0$, the tensor power $\otimes^r U$ has a natural $\mathfrak{g}$-action and a commuting action of the symmetric group $\mathfrak{S}_r$.
The tensor power $\otimes^r U$ also has a canonical decomposition

$$\otimes^r U \cong \oplus_{\omega \in P_+} \operatorname{Hom}(V(\omega), \otimes^r U) \otimes V(\omega)$$

Each isotypical component has a natural action of $\mathfrak{S}_r$ and this restricts to a natural action of the cyclic group C_r.
The aim of this talk is to give a combinatorial approach to the problem of determining the characters of these representations.
Our main interest is in the case $\omega = 0$ which corresponds to the subspace of invariant tensors.

Cyclic actions

Identify the character ring of the cyclic group of order r with $\mathbb{Z}[q]$ modulo $q^r - 1$

We are interested in any polynomial that reduces to the character modulo $q^r = 1$.

For the *cyclic sieving phenomenon* we have a set X with $c\colon X \to X$ an automorphism of order r.

Example

For an orbit of size r/d the polynomial modulo $q^r = 1$ is

$$\frac{1-q^r}{1-q^d} = 1 + q^d + \cdots + q^{r/d-1}$$

Catalan numbers

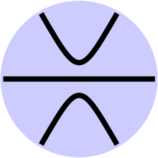
The Catalan numbers and their q-analogue are given by

$$C(n) = \frac{1}{n+1}\binom{2n}{n} \qquad C_q(n) = \frac{1}{[n+1]}\begin{bmatrix} 2n \\ n \end{bmatrix}_q$$

n	1	2	3	4
$n+2$	3	4	5	6
$2n$	2	4	6	8
$C(n)$	1	2	5	14
$C_q(n)$	1	$\frac{[4]}{[2]}$	$\frac{[6][5]}{[3][2]}$	$\frac{[8][7][6]}{[4][3][2]}$

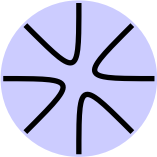
$1+q^2+q^3+q^4+q^6$

$1+q^3$

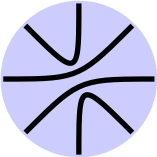


$1+q^2+q^4$

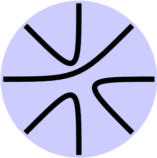
$2 + 2q^2 + q^3 + 3q^4 + q^5 + 2q^6 + q^7 + q^8 + q^9$



$1 + q^4$



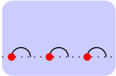
$1 + q^2 + q^4 + q^6$



$1 + q + q^2 + q^3 + q^4 + q^5 + q^6 + q^7$

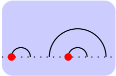
Tableaux

1	3	5
2	4	6



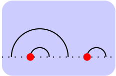
$\{1,3,5\}$
6

1	3	4
2	5	6



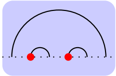
$\{1,4\}$
2

1	2	5
3	4	6



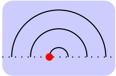
$\{2,5\}$
4

1	2	4
3	5	6



$\{2,4\}$
3

1	2	3
4	5	6



$\{3\}$
0

Promotion = Rotation

Rotation of perfect matchings on $\{1, 2, \ldots, 2n\}$:

- Replace pair $(1, k)$ by pair $(k, 2n + 1)$.
- Decrease all numbers by 1.

Promotion on rectangular $n \times m$ tableaux:

- Remove 1 in top left hand corner.
- Decrease all numbers by 1.
- Slide empty box right and down by moving smaller number into the empty box.
- Repeat until the empty box is the bottom right hand corner.
- Put nm in the empty box.

$\mathrm{SL}(n)$ after Rhoades

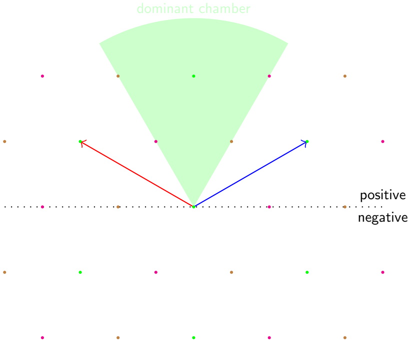
Let $V = \mathbb{C}^n$ be the defining representation of $\mathrm{SL}(n)$. Then a famous result due to Schur is that each isotypical component is an irreducible representation of $\mathfrak{S}_r$.

$$\otimes^r V \cong \bigoplus_{\substack{|\lambda|=r \\ \ell(\lambda)\leq n}} S(\lambda) \otimes V(\lambda)$$

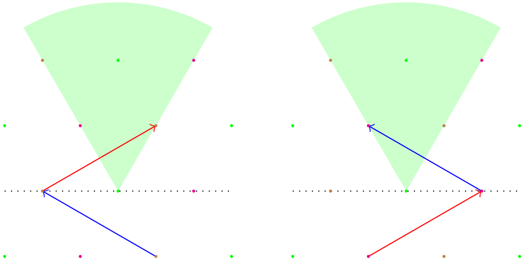
We replace the vector space $S(\lambda)$ by the set of standard tableaux of shape λ. The major index statistic gives the polynomial for the cyclic action.

Invariant tensors are the case λ is a rectangular partition. In this case promotion on standard tableaux corresponds to cyclic action.

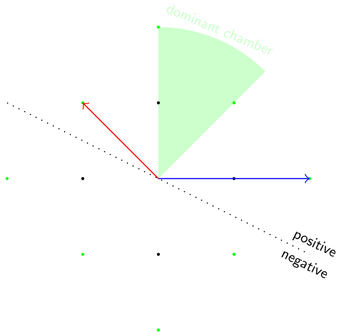
A_2 root system



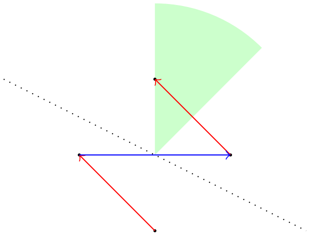
A_2 crystals

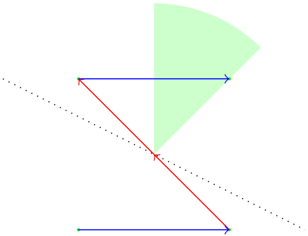


$B_2 = C_2$ root system

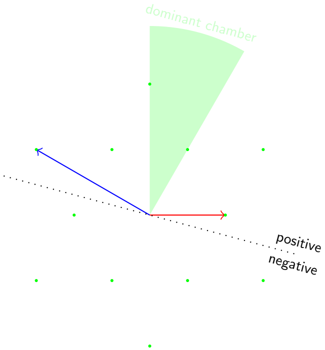


$B_2 = C_2$ crystals

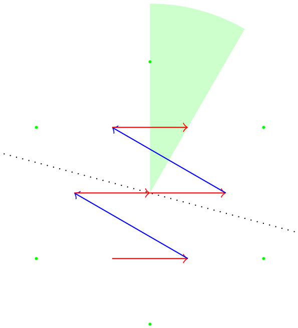


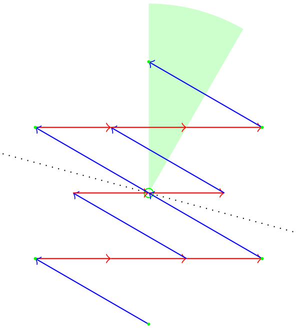


G_2 root system



G_2 crystals





What use are crystals?

- The character can be read off a crystal.
- Branching rules associated to submatrices of Cartan matrix are easy.

There is a tensor product rule for crystals. On the sets of vertices this is Cartesian product.
This gives a vast generalisation of the Robinson-Schensted correspondence.
This gives a vast generalisation of the Littlewood-Richardson rule which replaces each isotypical subspace by a finite set. These are sets of highest weight words.
The set which replaces invariant tensors has an action of the cyclic group. This is a far-reaching generalisation of standard tableaux and promotion on standard rectangular tableaux.

Energy

Let C be a finite crystal. A *local energy* is a function $h\colon C \otimes C \to \mathbb{N}$ which is constant on connected components. The *energy functions* are the functions on words, $H\colon \otimes^r C \to \mathbb{N}$, given by

$$H(x_1, x_2, \dots, x_r) = \sum_{i=1}^{r-1} i.h(x_i, x_{i+1})$$

The character polynomial is then given by

$$P(q) = \sum q^{H(x_1,x_2,\dots,x_r)}$$

where the sum is the set of words which replaces the isotypical component.

Classically irreducible

The irreducible representations for which there is an energy which can be proved to work all have highest weight a multiple of a fundamental weight, so are of the form $V(s\omega_i)$ where $s > 0$ and ω_i is a fundamental weight.

The following are the fundamental representations, $V(\omega_i)$, such that $V(s\omega_i)$ works for all $s > 0$:

- A all fundamental representations
- B the defining representation
- C the extreme representation
- D the defining representation and both half-spin representations
- E the two 26 dimensional representations of E_6 and the 56 dimensional representation of E_7

The following are the fundamental representations, $V(\omega_i)$, such that $V(s\omega_i)$ works for $s = 1$ only:

- B the spin representation
- C the defining representation
- G the 7 dimensional representation
- F the 26 dimensional representation

References

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That's all Folks!