A Joint Adventure in Sasakian and Kähler Geometry

Charles Boyer and Christina Tønnesen-Friedman

Geometry Seminar, University of Bath

March, 2015
Kähler Geometry

Let $N$ be a smooth compact manifold of real dimension $2d_N$.

- If $J$ is a smooth bundle-morphism on the real tangent bundle, $J : TN \rightarrow TN$ such that $J^2 = -Id$ and $\forall X, Y \in TN$

\[ J(\mathcal{L}_X Y) - \mathcal{L}_X JY = J(\mathcal{L}_JX JY - J\mathcal{L}_JX Y), \]

then $(N, J)$ is a complex manifold with complex structure $J$.

- A Riemannian metric $g$ on $(N, J)$ is said to be a Hermitian Riemannian metric if

\[ \forall X, Y \in TN, \quad g(JX, JY) = g(X, Y) \]

- This implies that $\omega(X, Y) := g(JX, Y)$ is a $J$– invariant ($\omega(JX, JY) = \omega(X, Y)$) non-degenerate $2$– form on $N$.
- If $d\omega = 0$, then we say that $(N, J, g, \omega)$ is a Kähler manifold (or Kähler structure) with Kähler form $\omega$ and Kähler metric $g$.
- The second cohomology class $[\omega]$ is called the Kähler class.
- For fixed $J$, the subset in $H^2(N, \mathbb{R})$ consisting of Kähler classes is called the Kähler cone.
Ricci Curvature of Kähler metrics:

Given a Kähler structure \((N, J, g, \omega)\), the Riemannian metric \(g\) defines (via the unique Levi-Civita connection \(\nabla\))

- the **Riemann curvature tensor** \(R : TN \otimes TN \otimes TN \to TN\)
- and the trace thereof, the **Ricci tensor** \(r : TN \otimes TN \to C^\infty(N)\)

This gives us the **Ricci form**, \(\rho(X, Y) = r(JX, Y)\).

The miracle of Kähler geometry is that \(c_1(N, J) = [\rho/2\pi]\).

If \(\rho = \lambda \omega\), where \(\lambda\) is some constant, then we say that \((N, J, g, \omega)\) is **Kähler-Einstein** (or just **KE**).

More generally, if

\[
\rho - \lambda \omega = L_V \omega,
\]

where \(V\) is a holomorphic vector field, then we say that \((N, J, g, \omega)\) is a **Kähler-Ricci soliton** (or just **KRS**).

**KRS** \(\implies c_1(N, J)\) is positive, negative, or null.
Scalar Curvature of Kähler metrics:

Given a Kähler structure \((N, J, g, \omega)\), the Riemannian metric \(g\) defines (via the unique Levi-Civita connection \(\nabla\))

- the **scalar curvature**, \(\text{Scal} \in C^\infty(N)\), where \(\text{Scal}\) is the trace of the map \(X \mapsto \tilde{r}(X)\) where \(\forall X, Y \in TN, g(\tilde{r}(X), Y) = r(X, Y)\).

- If \(\text{Scal}\) is a constant function, we say that \((N, J, g, \omega)\) is a constant scalar curvature Kähler metric (or just **CSC**).

- \(\text{KE} \implies \text{CSC}\) (with \(\lambda = \frac{\text{Scal}}{2d_N}\))

- Not all complex manifolds \((N, J)\) admit CSC Kähler structures.

- There are generalizations of CSC, e.g. **extremal Kähler metrics** as defined by Calabi \((\mathcal{L}_{\nabla g} \text{Scal} J = 0)\).

- Not all complex manifolds \((N, J)\) admit extremal Kähler structures either.
Admissible Kähler manifolds/orbifolds

- Special cases of the more general (admissible) constructions defined by/organized by Apostolov, Calderbank, Gauduchon, and T-F.
- Credit also goes to Calabi, Koiso, Sakane, Simanca, Pedersen, Poon, Hwang, Singer, Guan, LeBrun, and others.
- Let $\omega_N$ be a primitive integral Kähler form of a CSC Kähler metric on $(N, J)$.
- Let $1 \to N$ be the trivial complex line bundle.
- Let $n \in \mathbb{Z} \setminus \{0\}$.
- Let $L_n \to N$ be a holomorphic line bundle with $c_1(L_n) = [n \omega_N]$.
- Consider the total space of a projective bundle $S_n = \mathbb{P}(1 \oplus L_n) \to N$.
- Note that the fiber is $\mathbb{C}\mathbb{P}^1$.
- $S_n$ is called admissible, or an admissible manifold.
Admissible Kähler classes

- Let $D_1 = [1 \oplus 0]$ and $D_2 = [0 \oplus L_n]$ denote the “zero” and “infinity” sections of $S_n \to N$.
- Let $r$ be a real number such that $0 < |r| < 1$, and such that $r n > 0$.
- A Kähler class on $S_n$, $\Omega$, is admissible if (up to scale) $\Omega = \frac{2\pi n[\omega_N]}{r} + 2\pi PD(D_1 + D_2)$.
- In general, the admissible cone is a sub-cone of the Kähler cone.
- In each admissible class we can now construct explicit Kähler metrics $g$ (called admissible Kähler metrics).
- We can generalize this construction to the log pair $(S_n, \Delta)$, where $\Delta$ denotes the branch divisor $\Delta = (1-1/m_1)D_1 + (1-1/m_2)D_2$.
- If $m = \gcd(m_1, m_2)$, then $(S_n, \Delta)$ is a fiber bundle over $N$ with fiber $\mathbb{CP}^1[m_1/m, m_2/m]/\mathbb{Z}_m$.
- $g$ is smooth on $S_n \setminus (D_1 \cup D_2)$ and has orbifold singularities along $D_1$ and $D_2$. 
**Sasakian Geometry:**

**Sasakian geometry:** odd dimensional version of Kählerian geometry and special case of **contact structure**.

A Sasakian structure on a smooth manifold \( M \) of dimension \( 2n + 1 \) is defined by a quadruple \( S = (\xi, \eta, \Phi, g) \) where

- \( \eta \) is **contact 1-form** defining a subbundle (contact bundle) in \( TM \) by \( D = \ker \eta \).
- \( \xi \) is the **Reeb vector field** of \( \eta \) \( [\eta(\xi) = 1 \text{ and } \xi \rfloor d\eta = 0] \)
- \( \Phi \) is an endomorphism field which annihilates \( \xi \) and satisfies \( J = \Phi \big|_D \) is a complex structure on the contact bundle \( (d\eta(J\cdot,J\cdot) = d\eta(\cdot,\cdot)) \)
- \( g := d\eta \circ (\Phi \otimes 1) + \eta \otimes \eta \) is a Riemannian metric
- \( \xi \) is a Killing vector field of \( g \) which generates a one dimensional foliation \( \mathcal{F}_\xi \) of \( M \) whose transverse structure is Kähler.
- (Let \( (g_T, \omega_T) \) denote the transverse Kähler metric)
- \( (dt^2 + t^2g, d(t^2\eta)) \) is Kähler on \( M \times \mathbb{R}^+ \) with complex structure \( I: IY = \Phi Y + \eta(Y)t \frac{\partial}{\partial t} \) for vector fields \( Y \) on \( M \), and \( I(t \frac{\partial}{\partial t}) = -\xi \).
If $\xi$ is **regular**, the transverse Kähler structure lives on a smooth manifold (quotient of regular foliation $\mathcal{F}_\xi$).

If $\xi$ is **quasi-regular**, the transverse Kähler structure has orbifold singularities (quotient of quasi-regular foliation $\mathcal{F}_\xi$).

If not regular or quasi-regular we call it **irregular**... (that’s most of them)

**Transverse Homothety:**

- If $S = (\xi, \eta, \Phi, g)$ is a Sasakian structure, so is $S_a = (a^{-1}\xi, a\eta, \Phi, g_a)$ for every $a \in \mathbb{R}^+$ with $g_a = ag + (a^2 - a)\eta \otimes \eta$.

- So Sasakian structures come in rays.
Deforming the Sasaki structure:

In its contact structure isotopy class:

\[ \eta \rightarrow \eta + d^c \phi, \quad \phi \text{ is basic} \]

This corresponds to a deformation of the transverse Kähler form

\[ \omega_T \rightarrow \omega_T + dd^c \phi \]

in its Kähler class in the regular/quasi-regular case.

“Up to isotopy” means that the Sasaki structure might have to been deformed as above.
In the Sasaki Cone:

- Choose a maximal torus $T^k$, $0 \leq k \leq n + 1$ in the Sasaki automorphism group

\[ \text{Aut}(\mathcal{S}) = \{ \phi \in \text{Diff}(M) \mid \phi^* \eta = \eta, \phi^* J = J, \phi^* \xi = \xi, \phi^* g = g \}. \]

- The unreduced Sasaki cone is

\[ t^+ = \{ \xi' \in t_k \mid \eta(\xi') > 0 \}, \]

where $t^k$ denotes the Lie algebra of $T^k$.

- Each element in $t^+$ determines a new Sasaki structure with the same underlying CR-structure.
Ricci Curvature of Sasaki metrics

- The Ricci tensor of $g$ behaves as follows:
  - $r(X, \xi) = 2n \eta(X)$ for any vector field $X$
  - $r(X, Y) = r_T(X, Y) - 2g(X, Y)$, where $X, Y$ are sections of $\mathcal{D}$ and $r_T$ is the transverse Ricci tensor

- If the transverse Kähler structure is Kähler-Einstein then we say that the Sasaki metric is $\eta$-Einstein.

- $S = (\xi, \eta, \Phi, g)$ is $\eta$-Einstein iff its entire ray is $\eta$-Einstein ("$\eta$-Einstein ray")

- If the transverse Kähler-Einstein structure has positive scalar curvature, then exactly one of the Sasaki structures in the $\eta$-Einstein ray is actually Einstein (Ricci curvature tensor a rescale of the metric tensor). That metric is called Sasaki-Einstein.

- If $S = (\xi, \eta, \Phi, g)$ is Sasaki-Einstein, then we must have that $c_1(\mathcal{D})$ is a torsion class (e.g. it vanishes).
A Sasaki Ricci Soliton (SRS) is a transverse Kähler Ricci soliton, that is, the equation

$$\rho^T - \lambda \omega^T = \mathcal{L}_V \omega^T$$

holds, where $V$ is some transverse holomorphic vector field, and $\lambda$ is some constant.

So if $V$ vanishes, we have an $\eta$-Einstein Sasaki structure.

Our definition allows SRS to come in rays.

We will say that $S = (\xi, \eta, \Phi, g)$ is $\eta$-Einstein / Einstein / SRS whenever it is $\eta$-Einstein / Einstein / SRS up to isotopy.
Scalar Curvature of Sasaki metrics

- The scalar curvature of $g$ behaves as follows
  \[ \text{Scal} = \text{Scal}_T - 2n \]

- $S = (\xi, \eta, \Phi, g)$ has constant scalar curvature (CSC) if and only if the transverse Kähler structure has constant scalar curvature.
- $S = (\xi, \eta, \Phi, g)$ has CSC iff its entire ray has CSC ("CSC ray").
- CSC can be generalized to Sasaki Extremal (Boyer, Galicki, Simanca) such that
- $S = (\xi, \eta, \Phi, g)$ is extremal if and only if the transverse Kähler structure is extremal
- $S = (\xi, \eta, \Phi, g)$ is extremal iff its entire ray is extremal ("extremal ray").
- We will say that $S = (\xi, \eta, \Phi, g)$ is CSC/extremal whenever it is CSC/extremal up to isotopy.
The Join Construction

- The join construction of Sasaki manifolds (Boyer, Galicki, Ornea) is the analogue of Kähler products.
- Given quasi-regular Sasakian manifolds $\pi_i : M_i \to \mathcal{Z}_i$. Let 
  
  $L = \frac{1}{2l_1} \xi_1 - \frac{1}{2l_2} \xi_2$.
- Form $(l_1, l_2)$-join by taking the quotient by the action induced by $L$:
  
  $M_1 \times M_2 \xrightarrow{\pi_L} M_1 \star_{l_1, l_2} M_2 \xrightarrow{\pi} \mathcal{Z}_1 \times \mathcal{Z}_2$

- $M_1 \star_{l_1, l_2} M_2$ is a $S^1$-orbibundle (generalized Boothby-Wang fibration).
- $M_1 \star_{l_1, l_2} M_2$ has a natural quasi-regular Sasakian structure for all relatively prime positive integers $l_1, l_2$. Fixing $l_1, l_2$ fixes the contact orbifold. It is a smooth manifold iff $\gcd(\mu_1 l_2, \mu_2 l_1) = 1$, where $\mu_i$ is the order of the orbifold $\mathcal{Z}_i$. 
Join with a weighted 3-sphere

- Take $\pi_2 : M_2 \rightarrow \mathbb{Z}_2$ to be the $S^1$-orbibundle
  \[
  \pi_2 : S^3_w \rightarrow \mathbb{CP}[w]
  \]
determined by a weighted $S^1$-action on $S^3$ with weights $w = (w_1, w_2)$ such that $w_1 \geq w_2$ are relative prime.

- $S^3_w$ has an extremal Sasakian structure.

- Let $M_1 = M$ be a regular CSC Sasaki manifold whose quotient is a compact CSC Kähler manifold $N$.

- Assume $\gcd(l_2, l_1 w_1 w_2) = 1$ (equivalent with $\gcd(l_2, w_i) = 1$).

\[
\begin{array}{c}
M \times S^3_w \\
\downarrow \pi_{12} \\
\downarrow \\
\downarrow \pi \\
N \times \mathbb{CP}[w]
\end{array}
\]

\[
M \ast_{l_1, l_2} S^3_w =: M_{l_1, l_2, w}
\]
The w-Sasaki cone

- The Lie algebra $\text{aut}(S_{1,2,w})$ of the automorphism group of the join satisfies $\text{aut}(S_{1,2,w}) = \text{aut}(S_1) \oplus \text{aut}(S_w)$, mod $(L_{1,2,w} = \frac{1}{2l_1} \xi_1 - \frac{1}{2l_2} \xi_2)$, where $S_1$ is the Sasakian structure on $M$, and $S_w$ is the Sasakian structure on $S^3_w$.
- The unreduced Sasaki cone $t^+_{1,2,w}$ of the join $M_{1,2,w}$ thus has a 2-dimensional subcone $t^+_w$ is called the w-Sasaki cone.
- $t^+_w$ is inherited from the Sasaki cone on $S^3$.
- Each ray in $t^+_w$ is determined by a choice of $(v_1, v_2) \in \mathbb{R}^+ \times \mathbb{R}^+$.
- The ray is quasi-regular iff $v_2/v_1 \in \mathbb{Q}$.
- $t^+_w$ has a regular ray (given by $(v_1, v_2) = (1, 1)$) iff $l_2$ divides $w_1 - w_2$. 
Motivating Questions

▶ Does $t^+_w$ have a CSC/η-Einstein ray?

▶ What about extremal/Sasaki-Ricci solitons?
Key Proposition (Boyer, T-F)

Let $M_{l_1,l_2,w} = M \star_{l_1,l_2} S^3_w$ be the join as described above. Let $v = (v_1, v_2)$ be a weight vector with relatively prime integer components and let $\xi_v$ be the corresponding Reeb vector field in the Sasaki cone $t^+_w$. Then the quotient of $M_{l_1,l_2,w}$ by the flow of the Reeb vector field $\xi_v$ is $(S_n, \Delta)$ with $n = l_1 \left( \frac{w_1 v_2 - w_2 v_1}{s} \right)$, where $s = \gcd(l_2, w_1 v_2 - w_2 v_1)$, and $\Delta$ is the branch divisor

$$\Delta = (1 - \frac{1}{m_1})D_1 + (1 - \frac{1}{m_2})D_2,$$  \hspace{1cm} (1)

with ramification indices $m_i = v_i \frac{l_2}{s}$.  \hspace{1cm}
The Kähler class on the (quasi-regular) quotient

- is admissible up to scale.
- We can determine exactly which one it is.
- So we can test it for containing admissible KRS, KE, CSC, or extremal metrics.
- Hence we can test if the ray of $\xi_v$ is (admissible and) $\eta$-Einstein/SRS/CSC/extremal (up to isotopy).
- By lifting the admissible construction to the Sasakian level (in a way so it depends smoothly on $(\nu_1, \nu_2)$), we can also handle the irregular rays.
Theorem A (Boyer, T-F)

For each vector \( \mathbf{w} = (w_1, w_2) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \) with relatively prime components satisfying \( w_1 > w_2 \) there exists a Reeb vector field \( \xi_v \) in the 2-dimensional \( \mathbf{w} \)-Sasaki cone on \( M_{l_1,l_2,\mathbf{w}} \) such that the corresponding ray of Sasakian structures \( S_a = (a^{-1}\xi_v, a\eta_v, \Phi, g_a) \) has constant scalar curvature.

Suppose in addition that the scalar curvature of \( N \) is non-negative. Then the \( \mathbf{w} \)-Sasaki cone is exhausted by extremal Sasaki metrics. In particular, if the Kähler structure on \( N \) admits no Hamiltonian vector fields, then the entire Sasaki cone of the join \( M_{l_1,l_2,\mathbf{w}} \) can be represented by extremal Sasaki metrics.

Suppose in addition that the scalar curvature of \( N \) is positive. Then for sufficiently large \( l_2 \) there are at least three CSC rays in the \( \mathbf{w} \)-Sasaki cone of the join \( M_{l_1,l_2,\mathbf{w}} \).
Theorem B (Boyer, T-F)

Suppose $N$ is positive Kähler-Einstein with Fano index $I_N$ and

$$l_1 = \frac{I_N}{\gcd(w_1 + w_2, I_N)}$$

$$l_2 = \frac{w_1 + w_2}{\gcd(w_1 + w_2, I_N)}$$

(ensures that $c_1(D)$ vanishes).

- Then for each vector $w = (w_1, w_2) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ with relatively prime components satisfying $w_1 > w_2$ there exists a Reeb vector field $\xi_v$ in the 2-dimensional $w$-Sasaki cone on $M_{l_1, l_2, w}$ such that the corresponding Sasakian structure $S = (\xi_v, \eta_v, \Phi, g)$ is Sasaki-Einstein.

- Moreover, this ray is the only admissible CSC ray in the $w$-Sasaki cone.

- In addition, for each vector $w = (w_1, w_2) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ with relatively prime components satisfying $w_1 > w_2$ every single ray in the 2-dimensional $w$-Sasaki cone on $M_{l_1, l_2, w}$ admits (up to isotopy) a Sasaki-Ricci soliton.
Remarks

- The Sasaki-Einstein structures were first found by the physicists Guantlett, Martelli, Sparks, Waldram.
- Starting from the join construction allows us to study the topology of the Sasaki manifolds more closely.
- When $N = \mathbb{C}P^1$, $M_{l_1,l_2,w}$ are $S^3$-bundles over $S^2$. These were treated by Boyer and Boyer, Pati, as well as by E. Legendre.
- Our set-up, starting from a join construction, allows for cases where no regular ray in the $w$-Sasaki cone exists. If, however, the given $w$-Sasaki cone does admit a regular ray, then the transverse Kähler structure is a smooth Kähler Ricci soliton and the existence of an SE metric in some ray of the Sasaki cone is predicted by the work of Mabuchi and Nakagawa.
References


- Other papers by **Boyer** et al. For the “join” of Sasaki structures.

- **Boyer and T.-F.** The Sasaki Join, Hamiltonian 2-forms, and Constant Scalar Curvature (to appear in JGA, 2015) and references therein to our previous papers. For the details and proofs behind the statements in this talk.