Discrete line complexes and integrable evolution of minors

by

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[with A.I. Bobenko]

1. The algebraic set-up

For the present purpose, we are concerned with a matrix-valued function

$$\mathcal{M} : \mathbb{Z}^3 \to M_{5,5}(\mathbb{C}),$$

that is, a 5×5 matrix

$$\mathcal{M} = \begin{pmatrix} M^{11} & M^{12} & M^{13} & M^{14} & M^{15} \\ M^{21} & M^{22} & M^{23} & M^{24} & M^{25} \\ M^{31} & M^{32} & M^{33} & M^{34} & M^{35} \\ M^{41} & M^{42} & M^{43} & M^{44} & M^{45} \\ M^{51} & M^{52} & M^{53} & M^{54} & M^{55} \end{pmatrix}$$

as a function of n_1, n_2, n_3. We are interested in the "evolution" of the sub-matrix

$$\hat{\mathcal{M}} = \begin{pmatrix} M^{44} & M^{45} \\ M^{54} & M^{55} \end{pmatrix}$$

which encapsulates the geometry.

The matrix $\mathcal{M}$ is uniquely determined by the fundamental system

$$\boxed{M_l^{ik} = M^{ik} - \frac{M^{il}M^{lk}}{M^{ll}}, \quad l \in \{1,2,3\}\backslash\{i,k\}}$$

and the Cauchy data

$$M^{ik}(S^{ik}), \quad S^{ik} = \{\boldsymbol{n} : n_l = 0,\, l \notin \{i,k\}\}.$$

In particular, $\hat{\mathcal{M}}$ may only be prescribed at one point.

Theorem. Compatible and multi-dimensionally consistent (for the same reason)!

Proof.

$$\left(M_l^{ik}\right)_m = \left(M_m^{ik}\right)_l$$

Consider multi-indices

$$A = (a_1 \cdots a_s), \quad B = (b_1 \cdots b_s)$$

with distinct entries.

Minors of $\mathcal{M} = (M^{ik})_{i,k}$:

$$M^{A,B} = \mathrm{det}(M^{a_\alpha b_\beta})_{\alpha,\beta=1,\ldots,s}, \quad M^{\emptyset,\emptyset} = 1$$

Theorem.

$$M_l^{A,B} = \frac{M^{lA,lB}}{M^{l,l}}, \quad l \notin A \cup B$$

Proof. Laplace expansion.

4. The Jacobi identity

Jacobi's classical identity for determinants:

$$M^{A,B}M^{a\bar{a}A,b\bar{b}B} - M^{aA,bB}M^{\bar{a}A,\bar{b}B} + M^{\bar{a}A,bB}M^{aA,\bar{b}B} = 0$$

Key "observation":

$$\langle \mathsf{W}, \mathsf{W} \rangle = 0,$$

where

$$\mathsf{W} = (M^{A,B}, M^{a\bar{a}A,b\bar{b}B}, M^{aA,bB}, M^{\bar{a}A,\bar{b}B}, M^{\bar{a}A,bB}, M^{aA,\bar{b}B})$$

and the inner product is taken with respect to the block-diagonal metric

$$\mathrm{diag}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right].$$

5. The Plücker quadric

Now, consider all minors of the matrix $\hat{\mathcal{M}}$ and define

$$\boxed{\mathsf{V} = (M^{\emptyset,\emptyset}, M^{45,45}, M^{4,4}, M^{5,5}, M^{5,4}, M^{4,5}).}$$

Then, trivially, $$\langle \mathsf{V}, \mathsf{V} \rangle = 0.$$

Interpretation: Homogeneous coordinates

$$\mathsf{V} : \mathbb{Z}^3 \to \mathbb{C}^{3,3}$$

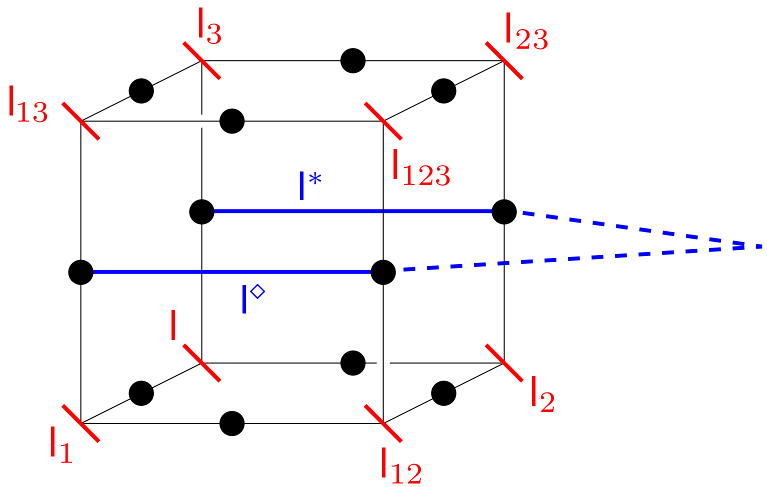
of a lattice of points in a four-dimensional quadric Q^4 embedded in a five-dimensional complex projective space $\mathbb{P}(\mathbb{C}^{3,3})$.

Identification: Q^4 = Plücker quadric and the Plücker correspondence provides a discrete line complex

$$\mathsf{l} : \mathbb{Z}^3 \to \{\text{lines in } \mathbb{CP}^3\},$$

that is, a three-parameter family of lines which are combinatorially attached to the vertices of $\mathbb{Z}^3$.

6. Incidence of lines



Lemma 1. "Neighbouring lines" intersect, that is,

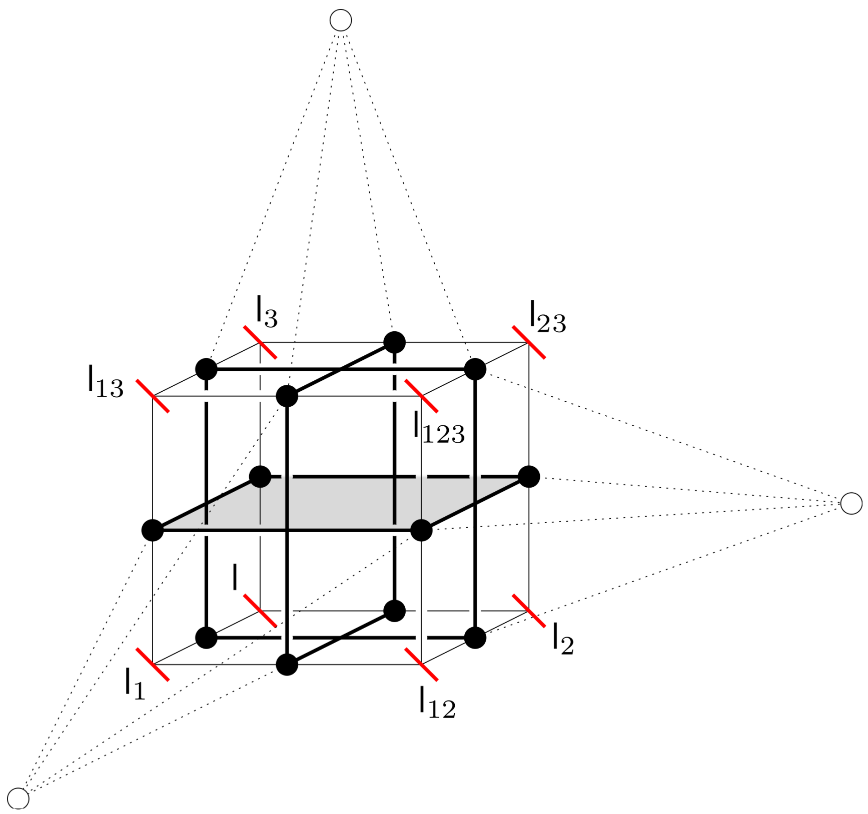
$$\langle V_l, V \rangle = 0.$$

Proof. Jacobi-type identity.

Lemma 2. "Opposite diagonals" intersect, that is,

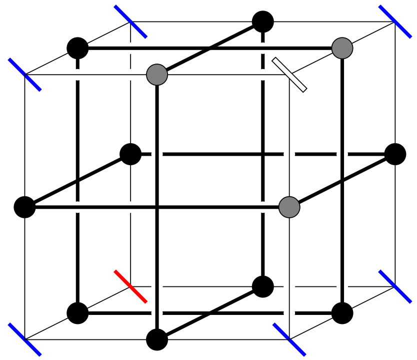
$$\langle V^*, V^\diamond \rangle = 0.$$

7. Fundamental line complexes [cf. Doliwa, Santini & Manas (2000)]



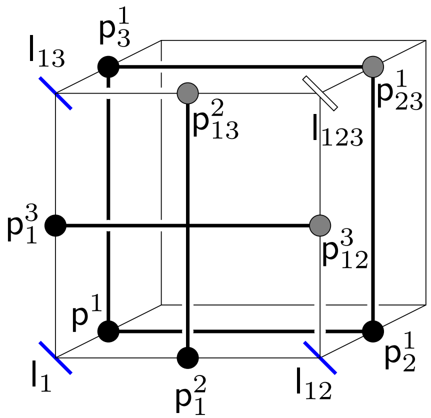
Definition. A line complex $l : \mathbb{Z}^3 \to \{\text{lines in } \mathbb{CP}^3\}$ is termed fundamental if any neighbouring lines l and l_l intersect and the points of intersection enjoy the coplanarity property or, equivalently, the diagonals admit the concurrency property.

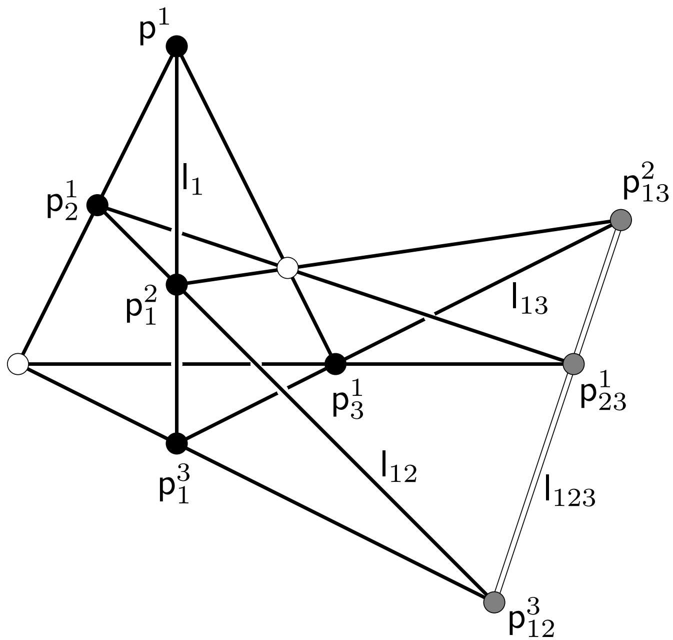
Theorem. Any solution $\mathcal{M}$ of the fundamental system encapsulates a fundamental line complex l via the Plücker correspondence $V \leftrightarrow l$ and, in fact, vice versa!



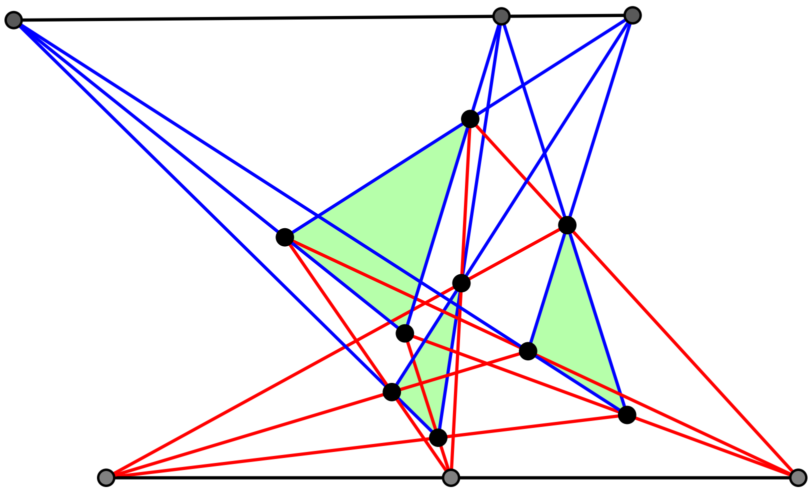
Theorem. For any given hexagon of six lines, the planarity property gives rise to a unique correspondence between the "first" and the "eighth" line.

Proof. Desargues' theorem





Observation. The 8 lines of an elementary cube of a fundamental line complex together with the 12 associated diagonals form a spatial version of the classical point-line configuration $(15_4\, 20_3)$:



[Coxeter, *Projective Geometry* or Baker, *Principles of Geometry* (frontispiece, vol. 1)]

Claim. The lines and diagonals of a fundamental line complex appear on equal footing if one embeds them in a five-dimensional (root) lattice of A type, that is,

$$\mathsf{l} : A_5 \to \{\text{lines in } \mathbb{CP}^3\}.$$

The symmetries of the fundamental system give rise to various admissible reductions:

- $M^{ik} \in \mathbb{R} \quad \to \quad$ real Plücker quadric and line complexes

- $M^{ik} = \bar{M}^{ki}$: Set

$$\tilde{\mathsf{V}} = (M^{\emptyset,\emptyset}, M^{45,45}, M^{4,4}, M^{5,5}, \Re(M^{4,5}), \Im(M^{4,5}))$$

Then, the new inner product is taken with respect to

$$\mathrm{diag}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right]$$

so that

$$\tilde{\mathsf{V}} : \mathbb{Z}^3 \to \mathbb{R}^{4,2}$$

$\to$ 4-dim. Lie quadric $\to$ Lie sphere geometry $\to$ Neighbouring spheres combinatorially attached to vertices have oriented contact.

- $M^{ik} = M^{ki} \in \mathbb{R}$: Set

$$\tilde{\mathsf{V}} = (M^{\emptyset,\emptyset}, M^{45,45}, M^{4,4}, M^{5,5}, M^{4,5})$$

Then, the new inner product is taken with respect to

$$\operatorname{diag}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 2\right]$$

so that

$$\tilde{\mathsf{V}} : \mathbb{Z}^3 \to \mathbb{R}^{3,2}$$

$\to$ 3-dim. Lie quadric $\to$ Lie circle geometry $\to$ Neighbouring circles on the plane combinatorially attached to vertices have oriented contact.

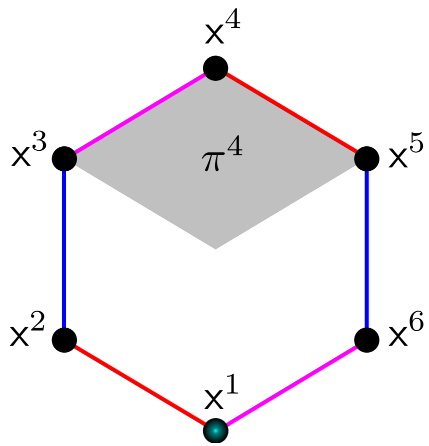
... dCKP equation

Remark. The minors of the symmetric matrix $\mathcal{M}$ may be parametrised in terms of a single function τ $\quad\rightarrow\quad$ discrete CKP equation

$$(\tau\tau_{123} + \tau_1\tau_{23} - \tau_2\tau_{13} - \tau_3\tau_{12})^2 - 4(\tau_{12}\tau_{13} - \tau_1\tau_{123})(\tau_2\tau_3 - \tau\tau_{23}) = 0.$$

The left-hand-side is known to be Cayley's $2\times2\times2$ hyperdeterminant.

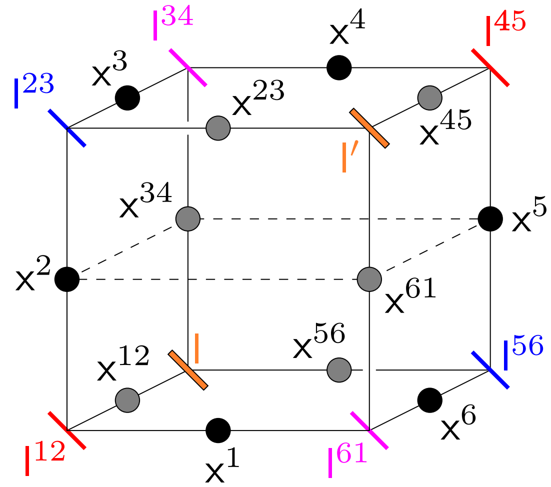
[Kashaev (1996): Star-triangle moves in the Ising model
Schief (2003): Carnot's and Pascal's theorems
Holtz & Sturmfels (2007): Principal minor assignment problem
Kenyon & Pemantle (2014): Dimers and cluster algebras]



Theorem 1. For any hexagon in $\mathbb{CP}^3$ in general position, there exists a unique correlation

$$\kappa : \{\text{points in } \mathbb{CP}^3\} \to \{\text{planes in } \mathbb{CP}^3\}$$

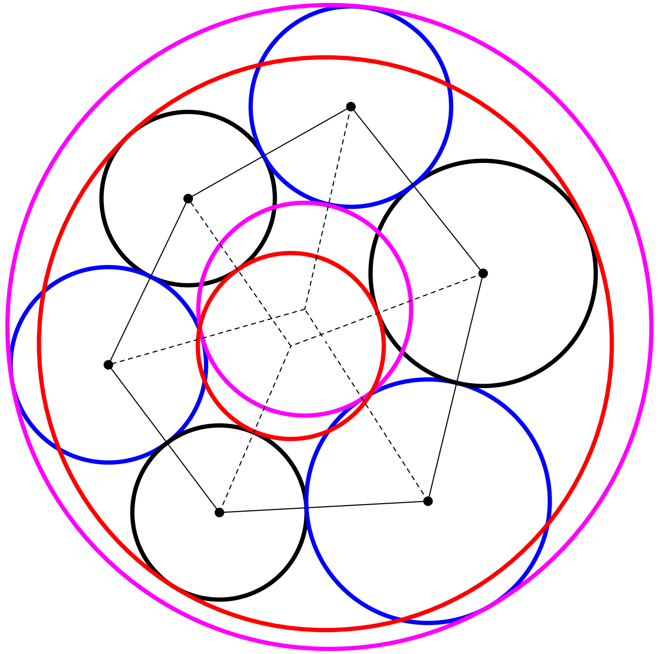
which interchanges "opposite" (extended) edges.



Theorem 2. For any hexagon of six lines, the aforementioned unique correspondence between the "first" and the "eighth" line due to Desargues' theorem coincides with that generated by the above correlation.

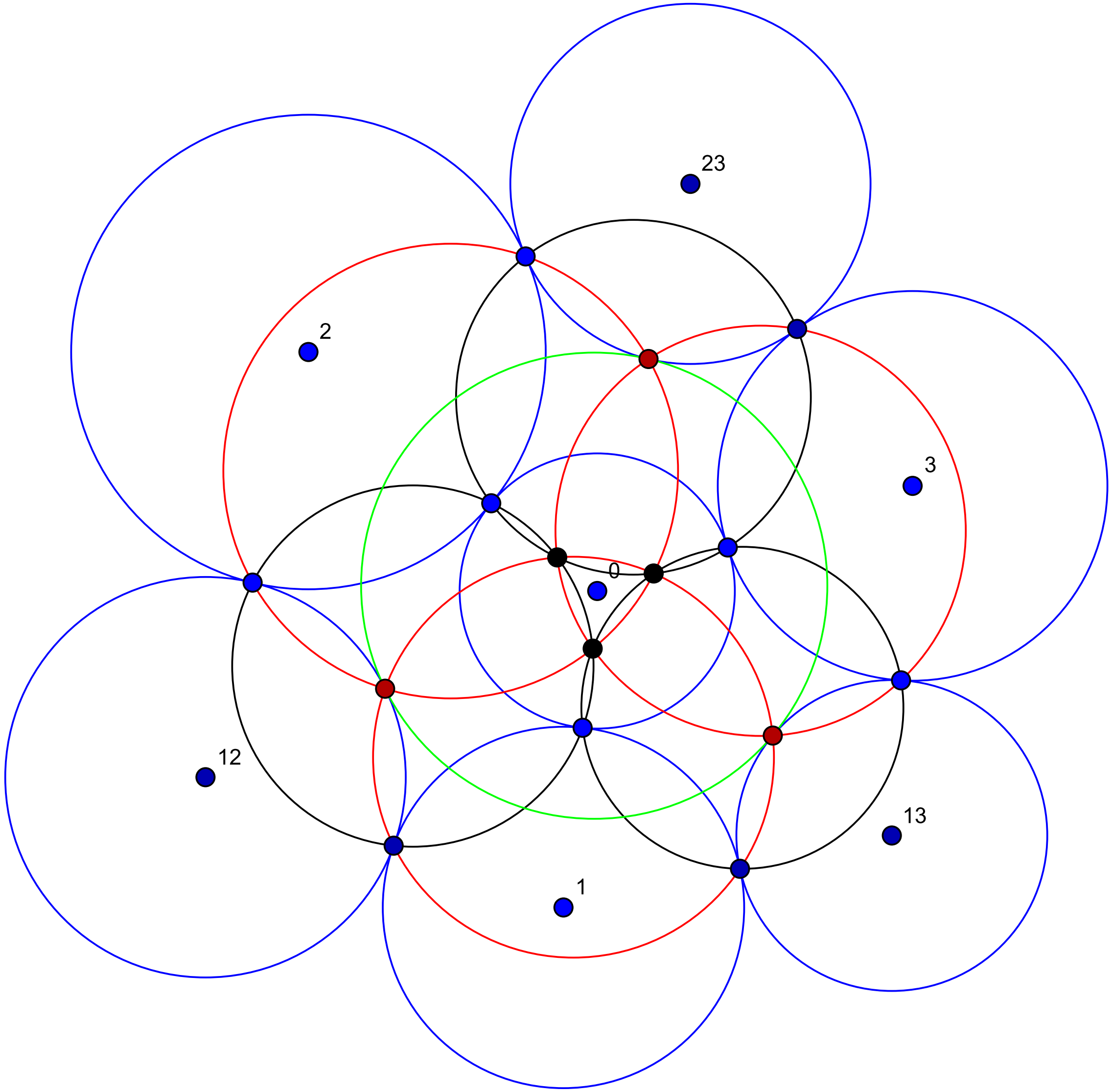
Remark. The correlation "maps" the planarity property to the concurrency property and vice versa!

12. Apollonius circles



Corollary. For any given "hexagon" of six (<u>black</u> and blue) circles which have oriented contact, there exists a unique correspondence between the pairs of (red and purple) Apollonius circles.

13. A canonical eighth circle



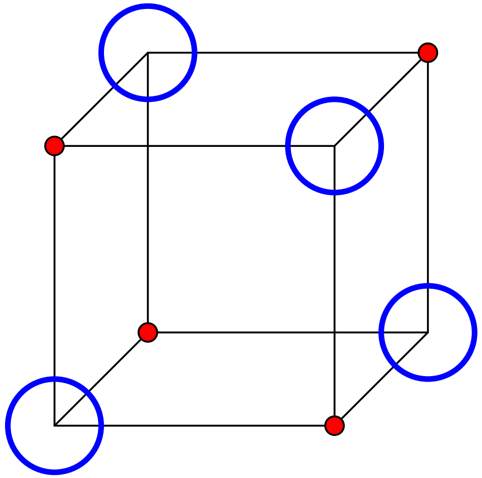
14. Summary

$$\begin{tikzcd}
& \begin{array}{c}\text{Complex Plücker geometry}\\ so(3,3,\mathbb{C}) \cong sl(4,\mathbb{C})\end{array} \arrow[dl, "\bar{\mathcal{M}} = \mathcal{M}"'] \arrow[dr, "\mathcal{M}^{\dagger} = \mathcal{M}"] & \\
\begin{array}{c}\text{Real Plücker geometry}\\ so(3,3) \cong sl(4)\end{array} \arrow[dr, "\mathcal{M}^{T} = \mathcal{M}"'] & & \begin{array}{c}\text{Lie sphere geometry}\\ so(4,2) \cong su(2,2)\end{array} \arrow[dl, "\bar{\mathcal{M}} = \mathcal{M}"] \\
& \begin{array}{c}\text{Lie circle geometry}\\ so(3,2) \cong sp(4)\\ \text{dCKP}\end{array} &
\end{tikzcd}$$

15. "Deeper" reductions

In the spirit of Klein's Erlangen Program, consider the intersection of the Lie quadric with a hyperplane. Depending on the signature of the hyperplane, this identifies

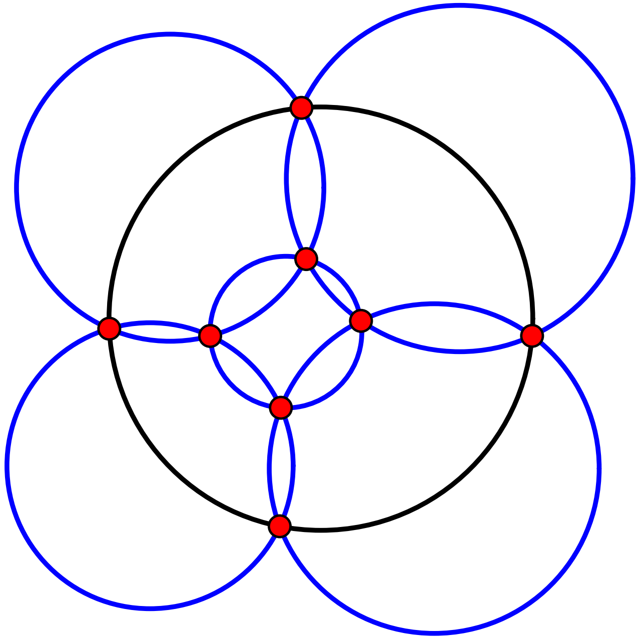
- points $\rightarrow$ Möbius geometry
- lines $\rightarrow$ Laguerre geometry
- "geodesic circles" $\rightarrow$ "hyperbolic" geometry



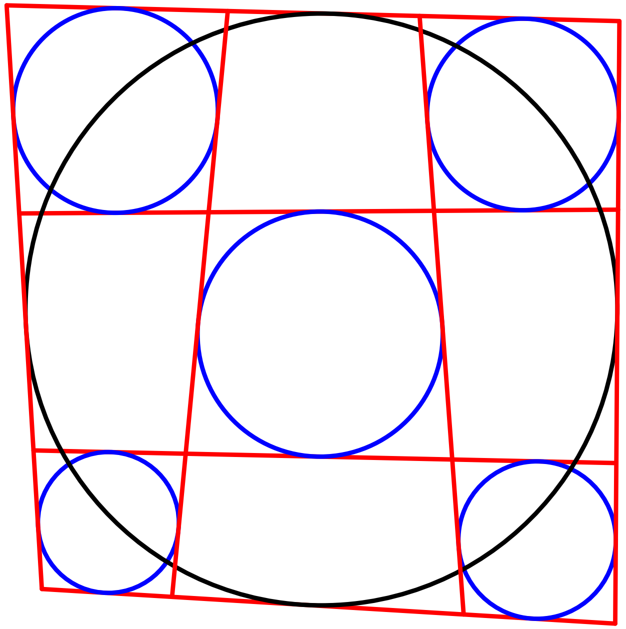
It is then consistent to demand that every second Lie circle be of the above type.

This leads to the consideration of interesting "circle theorems" such as (analogues of) Miquel's theorem and Clifford's chain of circle theorems.

16. Miquel-type theorems



Möbius geometry



Laguerre geometry

[Yaglom, *Complex Numbers in Geometry*]

17. Quaternionic projective geometry ...

... of line complexes leads to configurations in four-dimensional Lie sphere geometry.

Not today ...