A Numerical Criterion for Lower bounds on K-energy maps of Algebraic manifolds

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Outline

- **Formulation of the problem**: To bound the Mabuchi energy from below on the space of Kähler metrics in a given Kähler class $[\omega]$.

  **Tian’s program ’88 -’97**: In algebraic case should restrict K-energy to “Bergman metrics”.

- **Representation theory**: Toric Morphisms and *Equivariant embeddings*.

- **Discriminants and resultants of projective varieties**: *Hyperdiscriminants* and Cayley-Chow forms.

- **Output**: A *complete description* of the extremal properties of the Mabuchi energy restricted to the space of Bergman metrics.
Formulating the problem

Set up and notation:

- \((X^n, \omega)\) closed Kähler manifold

- \(\mathcal{H}_\omega := \{ \varphi \in C^\infty(X) \mid \omega \varphi > 0 \}\)
  (the space of Kähler metrics in the class \([\omega]\) )

  \[\omega \varphi := \omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \varphi\]

- \(\text{Scal}(\omega)\): = scalar curvature of \(\omega\)

- \(\mu = \frac{1}{V} \int_X \text{Scal}(\omega) \omega^n\)
  (average of the scalar curvature)

\(V=\text{volume}\)
Definition. (Mabuchi 1986)

The **K-energy map** \( \nu_\omega : \mathcal{H}_\omega \to \mathbb{R} \) is given by

\[
\nu_\omega(\varphi) := -\frac{1}{V} \int_0^1 \int_X \dot{\varphi}_t (\text{Scal}(\omega\varphi_t) - \mu) \omega_t^n dt
\]

\( \varphi_t \) is a \( C^1 \) path in \( \mathcal{H}_\omega \) satisfying \( \varphi_0 = 0 \), \( \varphi_1 = \varphi \)

**Observe**: \( \varphi \) is a critical point for \( \nu_\omega \) iff \( \text{Scal}(\omega\varphi) \equiv \mu \) (a constant)

**Basic Theorem** (Bando-Mabuchi, Donaldson, ...., Chen-Tian)

*If there is a \( \psi \in \mathcal{H}_\omega \) with constant scalar curvature then there exists \( C \geq 0 \) such that

\[
\nu_\omega(\varphi) \geq -C \text{ for all } \varphi \in \mathcal{H}_\omega.
\]
Question (*) : 
Given $[\omega]$ how to detect when $\nu_\omega$ is bounded below on $\mathcal{H}_\omega$ ?

N.B. : In general we do not know (!) if there is a constant scalar curvature metric in the class $[\omega]$.

Special Case: Assume that $[\omega]$ is an integral class, i.e. there is an ample divisor $\mathcal{L}$ on $X$ such that

$$[\omega] = c_1(\mathcal{L})$$

We may assume that $X \rightarrow \mathbb{P}^N$ (embedded) and $\omega = \omega_{FS}|_X$
Observe that for $\sigma \in G := SL(N + 1, \mathbb{C})$ there is a $\varphi_\sigma \in C^\infty(\mathbb{P}^N)$ such that

$$\sigma^* \omega_{FS} = \omega_{FS} + \frac{\sqrt{-1}}{2\pi} \partial \overline{\partial} \varphi_\sigma > 0$$

This gives a map

$$G \ni \sigma \longrightarrow \varphi_\sigma \in \mathcal{H}_\omega$$

The space of Bergman Metrics is the image of this map

$$\mathcal{B} := \{ \varphi_\sigma \mid \sigma \in G \} \subset \mathcal{H}_\omega .$$

Tian’s idea: RESTRICT THE K-ENERGY TO $\mathcal{B}$
Question (**) :

Given $X \to \mathbb{P}^N$ how to detect when $\nu \omega$ is bounded below on $\mathcal{B}$?
**Definition.** Let $\Delta(G)$ be the space of *algebraic one parameter subgroups* $\lambda$ of $G$. These are algebraic homomorphisms

$$
\lambda : \mathbb{C}^* \longrightarrow G \quad \lambda_{ij} \in \mathbb{C}[t, t^{-1}].
$$

**Definition.** (The space of *degenerations* in $\mathcal{B}$)

$$
\Delta(\mathcal{B}) := \{ \mathbb{C}^* \xrightarrow{\varphi} \mathcal{B} ; \lambda \in \Delta(G) \}.
$$
Theorem. (Paul 2012)

Assume that for every degeneration \( \lambda \) in \( B \) there is a (finite) constant \( C(\lambda) \) such that

\[
\lim_{\alpha \to 0} \nu_\omega (\varphi_\lambda(\alpha)) \geq C(\lambda).
\]

Then there is a \textit{uniform} constant \( C \) such that for all \( \varphi_\sigma \in B \) we have the lower bound

\[
\nu_\omega (\varphi_\sigma) \geq C.
\]
Equivariant Embeddings of Algebraic Homogeneous Spaces

- $G$ reductive complex linear algebraic group:
  \[ G = GL(N + 1, \mathbb{C}), SL(N + 1, \mathbb{C}), (\mathbb{C}^*)^N, \]
  \[ SO(N, \mathbb{C}), Sp_{2n}(\mathbb{C}). \]

- $H :=$ Zariski closed subgroup.

- $\mathcal{O} := G/H$ associated homogeneous space.
Definition. An embedding of $O$ is an irreducible $G$ variety $X$ together with a $G$-equivariant embedding $i : O \rightarrow X$ such that $i(O)$ is an open dense orbit of $X$. 
Let \((X_1, i_1)\) and \((X_2, i_2)\) be two embeddings of \(\mathcal{O}\).

**Definition.** A **morphism** \(\varphi\) from \((X_1, i_1)\) to \((X_2, i_2)\) is a \(G\) equivariant regular map \(\varphi : X_1 \to X_2\) such that the diagram

\[
\begin{array}{ccc}
O & \xrightarrow{i_1} & X_1 \\
\downarrow & & \downarrow \\
& \varphi \downarrow & \\
& X_2 & \\
\end{array}
\]

commutes.

One says that \((X_1, i_1)\) **dominates** \((X_2, i_2)\).
Assume these embeddings are both projective (hence complete) with very ample linearizations

\[ \mathbb{L}_1 \in \text{Pic}(X_1)^G, \ \mathbb{L}_2 \in \text{Pic}(X_2)^G \]

satisfying

\[ \varphi^*(\mathbb{L}_2) \cong \mathbb{L}_1. \]

Get *injective* map of \( G \) modules

\[ \varphi^* : H^0(X_2, \mathbb{L}_2) \longrightarrow H^0(X_1, \mathbb{L}_1) \]
The adjoint

\[(\varphi^*)^t : H^0(X_1, \mathbb{L}_1)^\vee \longrightarrow H^0(X_2, \mathbb{L}_2)^\vee\]

is \textit{surjective} and gives a rational map:

\[\xymatrix{i_1 & X_1 \ar[r] & \mathbb{P}(H^0(X_1, \mathbb{L}_1)^\vee) \ar[d]^\varphi \ar[d]_{(\varphi^*)^t} & \mathbb{P}(H^0(X_2, \mathbb{L}_2)^\vee) \ar[l]_{i_2} & X_2 \ar[l]}
\]
We abstract this situation:

1. $\mathbb{V}, \mathbb{W}$ finite dimensional rational $G$-modules

2. $v, w$ nonzero vectors in $\mathbb{V}, \mathbb{W}$ respectively

3. Linear span of $G \cdot v$ coincides with $\mathbb{V}$ (same for $w$)

4. $[v]$ corresponding line through $v = \text{point in } \mathbb{P}(\mathbb{V})$

5. $\mathcal{O}_v := G \cdot [v] \subset \mathbb{P}(\mathbb{V})$ (projective orbit)

6. $\overline{\mathcal{O}}_v = \text{Zariski closure in } \mathbb{P}(\mathbb{V})$. 
Definition. \((V; v)\) dominates \((W; w)\) if and only if there exists \(\pi \in Hom(V, W)^G\) such that \(\pi(v) = w\) and the rational map \(\pi : \mathbb{P}(V) \rightarrow \mathbb{P}(W)\) induces a regular finite morphism \(\pi : \overline{G \cdot [v]} \rightarrow \overline{G \cdot [w]}\).
Observe that the map $\pi$ extends to the boundary if and only if

$$(*\,) \quad \overline{G \cdot [v]} \cap \mathbb{P} (\ker \pi) = \emptyset .$$

- $\pi(\mathbb{V}) = \mathbb{W}$
- $\mathbb{V} = \ker(\pi) \oplus \mathbb{W} (G\text{-module splitting})$

Identify $\pi$ with projection onto $\mathbb{W}$

$v = (v_\pi, w) \quad v_\pi \neq 0$

$(*)$ is equivalent to

$$(**\,) \quad \overline{G \cdot [(v_\pi, w)]} \cap \overline{G \cdot [(v_\pi, 0)]} = \emptyset$$

(Zariski closure inside $\mathbb{P}(\ker(\pi) \oplus \mathbb{W})$)
Given \((v, w) \in V \oplus W\) set
\[
O_{vw} := G \cdot [(v, w)] \subset P(V \oplus W)
\]
\[
O_v := G \cdot [(v, 0)] \subset P(V \oplus \{0\})
\]
This motivates:

**Definition.** (Paul 2010) The pair \((v, w)\) is **semistable** if and only if
\[
\overline{O}_{vw} \cap \overline{O}_v = \emptyset
\]
**Example.** Let $V_e$ and $V_d$ be irreducible $SL(2, \mathbb{C})$ modules with highest weights $e, d \in \mathbb{N} \cong$ homogeneous polynomials in two variables. Let $f$ and $g$ in $V_e \setminus \{0\}$ and $W_d \setminus \{0\}$ respectively.

**Claim.** $(f, g)$ is semistable if and only if

\[ e \leq d \text{ and for all } p \in \mathbb{P}^1 \ ord_p(g) - ord_p(f) \leq \frac{d - e}{2}. \]

When $e = 0$ and $f = 1$ conclude that $(1, g)$ is semistable if and only if

\[ ord_p(g) \leq \frac{d}{2} \text{ for all } p \in \mathbb{P}^1. \]
Toric Morphisms

If the pair \((v, w)\) is semistable then we certainly have that

\[
T \cdot [(v, w)] \cap T \cdot [(v, 0)] = \emptyset
\]

for all maximal algebraic tori \(T \leq G\). Therefore there exists a morphism of \textit{projective} toric varieties.

\[
\begin{array}{c}
T \\ \downarrow \pi \\
T \cdot [(v, w)] & \rightarrow & \mathbb{P}(V \oplus W) \\
\downarrow \pi & & \downarrow \pi \\
T \cdot [(0, w)] & \leftarrow & \mathbb{P}(W)
\end{array}
\]

We expect that the existence of such a morphism is completely dictated by the \textit{weight polyhedra} \(\mathcal{N}(v)\) and \(\mathcal{N}(w)\).
Theorem. (Paul 2012)
The following statements are equivalent.

1. $(v, w)$ is semistable. Recall that this means
   \[ \overline{O}_{vw} \cap \overline{O}_v = \emptyset \]

2. $\mathcal{N}(v) \subset \mathcal{N}(w)$ for all maximal tori $H \leq G$.
   We say that $(v, w)$ is numerically semistable.

3. For every maximal algebraic torus $H \leq G$ and $\chi \in \mathcal{A}_H(v)$ there exists an integer $d > 0$
   and a relative invariant $f \in \mathbb{C}_d[\mathbb{V} \oplus \mathbb{W}]_d^H$ such that
   \[ f(v, w) \neq 0 \text{ and } f|_{\mathbb{V}} \equiv 0. \]
Corollary A. If $\overline{O}_{vw} \cap \overline{O}_v \neq \emptyset$ then there exists an alg. 1psg $\lambda \in \Delta(G)$ such that

$$\lim_{\alpha \to 0} \lambda(\alpha) \cdot [(v, w)] \in \overline{O}_v.$$
Equip $\mathbb{V}$ and $\mathbb{W}$ with Hermitian norms. The \textit{energy of the pair} $(v, w)$ is the function on $G$ defined by

$$G \ni \sigma \rightarrow p_{vw}(\sigma) := \log ||\sigma \cdot w||^2 - \log ||\sigma \cdot v||^2.$$ 

**Corollary B.**

$$\inf_{\sigma \in G} p_{vw}(\sigma) = -\infty$$

if and only if there is a degeneration $\lambda \in \Delta(G)$ such that

$$\lim_{\alpha \to 0} p_{vw}(\lambda(\alpha)) = -\infty.$$
<table>
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<tr>
<th><strong>Hilbert-Mumford Semistability</strong></th>
<th><strong>Semistable Pairs</strong></th>
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<tbody>
<tr>
<td>For all $H \leq G$ $\exists d \in \mathbb{Z}<em>{&gt;0}$ and $f \in \mathbb{C}</em>{\leq d}[W]^H$ such that $f(w) \neq 0$ and $f(0) = 0$</td>
<td>For all $H \leq G$ and $\chi \in \mathcal{A}<em>H(v)$ $\exists d \in \mathbb{Z}</em>{&gt;0}$ and $f \in \mathbb{C}<em>d[V \oplus W]^H</em>{d\chi}$ such that $f(v, w) \neq 0$ and $f</td>
</tr>
<tr>
<td>$0 \notin G \cdot w$</td>
<td>$\mathcal{O}_{vw} \cap \mathcal{O}_v = \emptyset$</td>
</tr>
<tr>
<td>$w_\lambda(w) \leq 0$ for all 1psg’s $\lambda$ of $G$</td>
<td>$w_\lambda(w) - w_\lambda(v) \leq 0$ for all 1psg’s $\lambda$ of $G$</td>
</tr>
<tr>
<td>$0 \in \mathcal{N}(w)$ all $H \leq G$</td>
<td>$\mathcal{N}(v) \subset \mathcal{N}(w)$ all $H \leq G$</td>
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<tr>
<td>$\exists C \geq 0$ such that $\log</td>
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To summarize, the context for the study of **SEMISTABLE PAIRS** is

1. A reductive linear algebraic group $G$.


3. A pair of (non-zero) vectors $(v, w) \in V \oplus W$. 
Resultants and Discriminants

Let $X$ be a smooth linearly normal variety

$$X \longrightarrow \mathbb{P}^N$$

Consider two polynomials:

$R_X := X$-resultant

$\Delta_{X \times \mathbb{P}^{n-1}} := X$-hyperdiscriminant

Let’s normalize the degrees of these polynomials

$$X \rightarrow R = R(X) := R_X^{\deg(\Delta_{X \times \mathbb{P}^{n-1}})}$$

$$X \rightarrow \Delta = \Delta(X) := \Delta_{X \times \mathbb{P}^{n-1}}^{\deg(R_X)}$$
It is known that

\[ R(X) \in E_{\lambda} \setminus \{0\} , \ (n + 1)\lambda \cdot = (r, r, \ldots, r, 0, \ldots, 0) . \]

\[ \Delta(X) \in E_{\mu} \setminus \{0\} , \ n\mu \cdot = (r, r, \ldots, r, 0, \ldots, 0) . \]

\[ r = \deg(R(X)) = \deg(\Delta(X)) . \]

\( E_{\lambda} \) and \( E_{\mu} \) are irreducible \( G \) modules.

The associations \( X \longrightarrow R(X) , \ X \longrightarrow \Delta(X) \)
are \( G \) equivariant:

\[ R(\sigma \cdot X) = \sigma \cdot R(X) \]

\[ \Delta(\sigma \cdot X) = \sigma \cdot \Delta(X) . \]
K-Energy maps and Semistable Pairs

Let $P$ be a numerical polynomial

\[ P(T) = c_n \binom{T}{n} + c_{n-1} \binom{T}{n-1} + O(T^{n-2}) \quad c_n \in \mathbb{Z}_{>0} . \]

Consider the Hilbert scheme

\[ \mathcal{H}^P_{\mathbb{P}^N} := \{ \text{all (smooth) } X \subset \mathbb{P}^N \text{ with Hilbert polynomial } P \} . \]

Recall the $G$-equivariant morphisms

\[ R, \Delta : \mathcal{H}^P_{\mathbb{P}^N} \longrightarrow \mathbb{P}(E_{\lambda \bullet}), \mathbb{P}(E_{\mu \bullet}) . \]
**Theorem** (Paul 2012)

There is a constant $M$ depending only on $c_n$, $c_{n-1}$ and the Fubini Study metric such that for all $[X] \in \mathcal{H}_{\mathbb{PN}}^P$ and all $\sigma \in G$ we have

$$\left| \nu \omega_{FS} |_{X}(\varphi_{\sigma}) - p_{R(X)\Delta(X)}(\sigma) \right| \leq M.$$