Moduli Space of Harmonic Tori in $S^3$

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Let $f : T^2 \rightarrow S^3 = SU(2)$ be a harmonic map.

A harmonic map is a critical point of the energy functional.

Long historical interest in minimal and constant curvature surfaces. A surface is CMC iff its Gauss map is harmonic.

Minimal surfaces = conformal harmonic = CMC with zero mean curvature.

Thought to be quite rare; Hopf Conjecture. Wente (1984) constructed immersed CMC tori.

A classification of such maps is given by spectral data $(\Sigma, \Theta, \tilde{\Theta}, E)$ (Hitchin, Pinkall-Sterling, Bobenko).
Spectral Data ($\Sigma$, $\Theta$, $\tilde{\Theta}$, $E$)

- Spectral curve $\Sigma$ is a real (possibly singular) hyperelliptic curve,

\[ \eta^2 = \prod (\zeta - \alpha_i)(1 - \bar{\alpha}_i \zeta) \]

- $\Theta$, $\tilde{\Theta}$ are differentials with double poles and no residues over $\zeta = 0, \infty$.

- Period conditions: The periods of $\Theta$, $\tilde{\Theta}$ must lie in $2\pi i \mathbb{Z}$.

- Closing conditions: for $\gamma_+$ a path in $\Sigma$ between the two points over $\zeta = 1$, and $\gamma_-$ between the points over $\zeta = -1$ then

\[ \int_{\gamma_+} \Theta, \int_{\gamma_-} \Theta, \int_{\gamma_+} \tilde{\Theta}, \int_{\gamma_-} \tilde{\Theta} \in 2\pi i \mathbb{Z}. \]

- $E$ is a quaternionic line bundle of a certain degree.
CMC Moduli Space (Kilian-Schmidt-Schmitt)

- One can vary the line bundle $E$, so called isospectral deformations.
- CMC non-isospectral deformations. Maps come in one dimensional families.
- $\mathcal{M}_{0}^{CMC}$ is disjoint lines parametrised by $H \in \mathbb{R}$

- Components $\mathcal{M}_{1}^{CMC}$ end in either $\mathcal{M}_{0}^{CMC}$ or bouquet of spheres.
Harmonic Map Example

- \( f(x + iy) = \exp(-4xX) \exp(4yY) \), for

\[
X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & \delta \\ -\delta & 0 \end{pmatrix}, \quad \Im \delta > 0
\]

- This map is periodic. Formula well-defined on any torus \( \mathbb{C}/\Gamma \), where \( \Gamma \) is a sublattice of this periodicity lattice.
- Holding either $x$ or $y$ constant gives circles.
- As $\delta \to \mathbb{R}^\times$, image collapses to a circle.
- As $\delta \to 0, \infty$, the periodicity lattice degenerates.
Constructing Spectral Data

- Up to translations, $f$ is determined by the Lie algebra valued map $f^{-1} df$, the pullback of the Mauer-Cartan form.
- Decompose into its $dz$ and $d\bar{z}$ parts $f^{-1} df = 2(\Phi - \Phi^*)$.
- Use $f$ to pull pack the Levi-Civita connection on $SU(2)$ to get a connection $A$.
- Given a pair $(\Phi, A)$, we can make a family of flat $SL(2, \mathbb{C})$ connections. Let $\zeta \in \mathbb{C}^\times$ be the spectral parameter and define

$$d_\zeta := d_A + \zeta^{-1} \Phi - \zeta \Phi^*$$

Family of connections is

$$d_\zeta = d - \left[ (X - iY) + \zeta^{-1} (X + iY) \right] dz$$
$$- \left[ (X + iY) + \zeta (X - iY) \right] d\bar{z}$$

$$= d - \zeta^{-1} \left[ (X + iY) + \zeta (X - iY) \right] [dz + \zeta d\bar{z}]$$
Holonomy

- Because the connections are flat, we can define holonomy for them.
- Pick a base point and generators for the fundamental group, i.e., take two loops around the torus.
- Parallel translating vectors with $d\zeta$ around one loop gives a linear map on the tangent space at the base point. Call this $H(\zeta)$. Around the other loop call the transformation $\tilde{H}(\zeta)$.

\[
H_\tau(\zeta) = \exp \left\{ \zeta^{-1} \left[ (X + iY) + \zeta(X - iY) \right] \left[ \tau + \zeta \bar{\tau} \right] \right\}
\]
The fundamental group of $T^2$ is abelian, so $H$ and $\tilde{H}$ commute. Therefore they have common eigenspaces.

Define

$$\Sigma = \text{closure } \{ (\zeta, L) \in \mathbb{C}^\times \times \mathbb{C}P^1 \mid L \text{ is an eigenline for } H(\zeta) \}$$

The eigenvalues of $H(\zeta)$ are $\mu(\zeta), \mu(\zeta)^{-1}$. The characteristic polynomial is

$$\mu^2 - (\text{tr } H) \mu + 1 = 0$$

Using the compactness of the torus, one can show that $(\text{tr } H)^2 - 4$ vanishes to odd order only finitely many times. The spectral curve is always finite genus for harmonic maps $T^2 \to \mathbb{S}^3$. 

Spectral curve
From example

$$\Sigma = \left\{ \left( \zeta, \left[ \pm \sqrt{(1 - i\delta)(\zeta - \alpha)} : \sqrt{-\left(1 + i\bar{\delta}\right)(1 - \bar{\alpha}\zeta)} \right] \right) \right\}$$

for

$$\alpha = \frac{1 + i\delta}{-1 + i\bar{\delta}} \quad \Leftrightarrow \quad \delta = i \frac{1 + \alpha}{1 - \alpha}$$

Can write equation for $\Sigma$ as

$$\eta^2 = (\zeta - \alpha)(1 - \bar{\alpha}\zeta)$$
The Differentials

- The differentials come from the eigenvalues $\mu(\zeta), \tilde{\mu}(\zeta)$ of $H(\zeta), \tilde{H}(\zeta)$. These functions have essential singularities.

- However $\log \mu, \log \tilde{\mu}$ are holomorphic on $\mathbb{C}^\times$ and have simple poles above $\zeta = 0, \infty$.

- $d \log \mu$ removes the additive ambiguity of log. Thus we set $\Theta = d \log \mu$ and $\tilde{\Theta} = d \log \tilde{\mu}$.

- In order to recover the eigenvalues, one requires residue free double poles over $\zeta = 0, \infty$ and that the periods of the differentials lie in $2\pi i \mathbb{Z}$. 

The eigenvalues of $H_{\tau}(\zeta)$ are

$$\mu_{\tau}(\zeta, \eta) = \exp \left[ i |1 - i\delta| (\tau + \bar{\tau}\zeta)\eta\zeta^{-1} \right].$$

The corresponding differential is therefore

$$\Theta_{\tau} = i |1 - i\delta| \, d \left[(\tau + \bar{\tau}\zeta)\eta\zeta^{-1}\right].$$

On any given spectral curve, there is a lattice of differentials that may be used in spectral data. Different choices corresponds to coverings of the same image.
Moduli Space $\mathcal{M}_0$

- Every spectral curve in genus zero arises from this class of examples.
- Choice amounts to branch point $\alpha \in D^2$ and choice of pair of differentials from a lattice

$$\mathcal{M}_0 = \bigsqcup D^2$$

- Image degenerates: $\delta \to \mathbb{R}^\times \iff \alpha \to S^1 \setminus \{\pm 1\}$.
- Lattice degenerates: $\delta \to 0, \infty \iff \alpha \to \pm 1$.
- Two dimensional (in contrast to CMC case).
Moduli Space $\mathcal{M}_g$

Theorem

At a point $(\Sigma, \Theta^1, \Theta^2) \in \mathcal{M}_g$ corresponding to a nonconformal harmonic map, if $\Sigma$ is nonsingular, and $\Theta^1$ and $\Theta^2$ vanish simultaneously at most four times on $\Sigma$ and never at a ramification point of $\Sigma$, then $\mathcal{M}_g$ is a two-dimensional manifold in a neighbourhood of this point.

Theorem

At a point $(\Sigma, \Theta^1, \Theta^2) \in \mathcal{M}_g$ corresponding to a conformal harmonic map, if $\Sigma$ is nonsingular, and $\Theta^1$ and $\Theta^2$ never vanish simultaneously on $\Sigma$ then $\mathcal{M}_g$ is a two-dimensional manifold in a neighbourhood of this point.

- Proof uses Whitham deformations.
Genus One

- Spectral curves have two pairs of branch points $\alpha, \beta, \alpha^{-1}, \beta^{-1}$. Let $\mathcal{A}_1 = \{(\alpha, \beta) \in D^2 \times D^2 \mid \alpha \neq \beta\}$.
- Not every spectral curve has differentials that meet all the conditions.
- There is always an exact differential $\Theta^E$ that meets all conditions except closing condition.
- A multiple of $\Theta^E$ meets the closing condition if and only if

$$S(\alpha, \beta) := \frac{|1 - \alpha| |1 - \beta|}{|1 + \alpha||1 + \beta|} \in \mathbb{Q}^+$$
Fix a value of $p \in \mathbb{Q}^+$. Let $A_1(p) = S^{-1}(p)$. It is an open three-ball with a line removed.

Rugby football shaped. Ends are $(\alpha, \beta) = (1, -1), (-1, 1)$. Seams are points with both $\alpha, \beta$ in $S^1$. 
There is a second differential $\Theta^P$ with periods $0$ and $2\pi i$. Every differential that meets period conditions is a combination $\mathbb{R}\Theta^E + \mathbb{Z}\Theta^P$.

Define $T$, up to periods of $\Theta^P$, by

$$2\pi i T := p \int_{\gamma^-} \Theta^P - \int_{\gamma^+} \Theta^P$$

A curve admits spectral data if and only if both $S \in \mathbb{Q}^+$ and $T \in \mathbb{Q}$ (and the latter is well-defined).

The connected components of the space of spectral curves are annuli if $S = 1$ and strips $(0, 1) \times \mathbb{R}$ if $S \neq 1$.

The connected components of the space of spectral data $\mathcal{M}_1$ are all strips $(0, 1) \times \mathbb{R}$.
Method of Proof

- Move to the universal cover of the parameter space
  \[ \pi_p : \tilde{A}_1(p) \to A_1(p). \]

- Define a function \( \tilde{T} \) on \( \tilde{A}_1(p) \) such that \( \tilde{T} = T \circ \pi_p \).

- In the right coordinates, the level sets of \( \tilde{T} \) are graphs over \((0, 1) \times \mathbb{R}\).

- Quotient by deck transformations to recover space of spectral curves.

- Consider how the lattice of differentials change as you change the spectral curve.
Interior Boundary $\mathcal{M}_1$

- $\mathcal{M}_1 \cap A_1(p)$ spirals around the diagonal line $\{\alpha = \beta\} \cap A_1(p)$.
- Just a single point on this diagonal line is reachable along a finite path.
- This limit seems not to be well-defined.
This boundary is where $\alpha$ or $\beta$ tends to $S^1$.

A singular curve with a double point over the unit circle corresponds to genus zero spectral data via normalisation (blow-up).

We can consider $\mathcal{M}_0 \subset \partial \mathcal{M}_1$.

Each face of the football $\overline{A_1(p)}$ is a disc, identified with the space of genus zero spectral curves.

Edges of $\overline{A_1(p)}$ correspond to all branch points on unit circle, ie a map to a circle.
Further questions

- Can we identify geometric properties that parameterise $\mathcal{M}$?
- Is $\mathcal{M}_0 \cup \mathcal{M}_1$ connected? No. What other maps need to be included to make it connected?
- Can one deform a harmonic map to a circle to a harmonic map of any spectral degree?
- How does $\mathcal{M}_g$ sit inside the moduli space of harmonic cylinders? Harmonic planes?
- What deformations lead to topological changes of the image of the harmonic map?