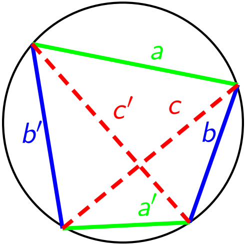
Quantum cluster algebras from geometry

Marta Mazzocco

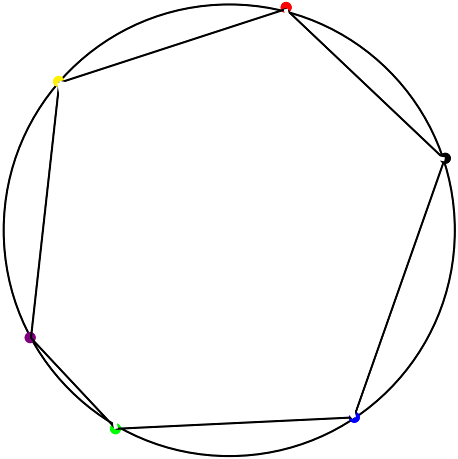
Based on Chekhov-M.M. arXiv:1509.07044 and Chekhov-M.M.-Rubtsov arXiv:1511.03851

Ptolemy Relation

$$aa' + bb' = cc'$$



Ptolemy Relation



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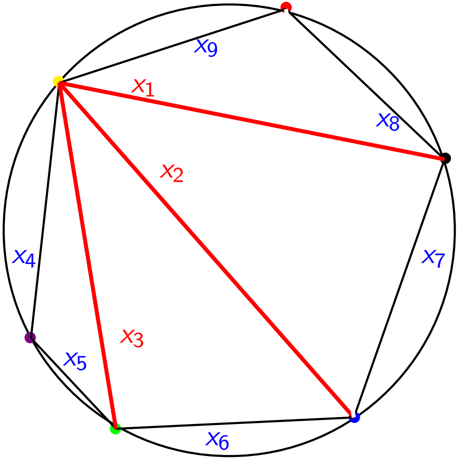

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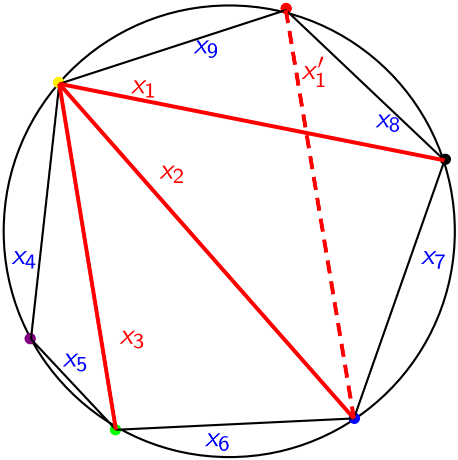
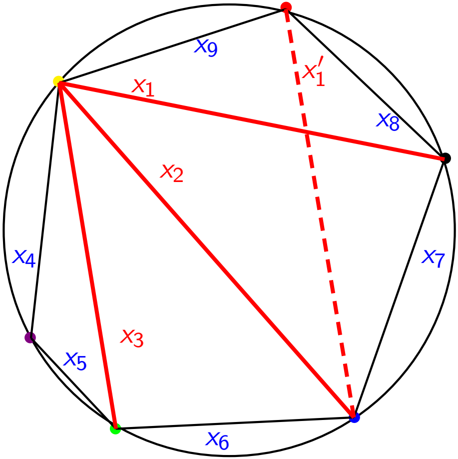


Ptolemy Relation

$$x_1 x_1' = x_9 x_7 + x_8 x_2$$



Ptolemy Relation

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) \to (x_1', x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9)$$

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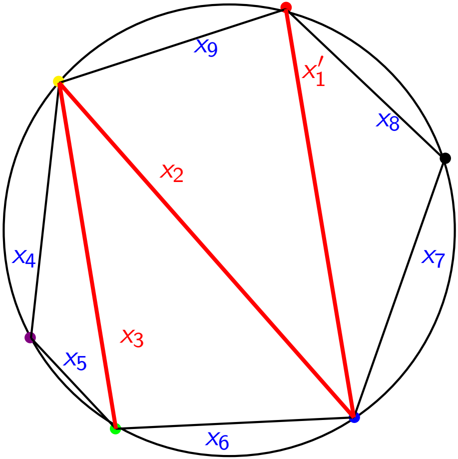

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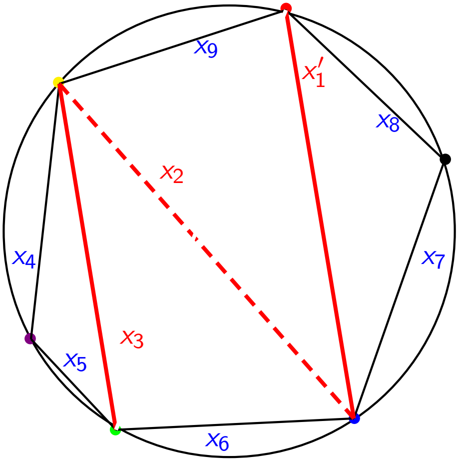

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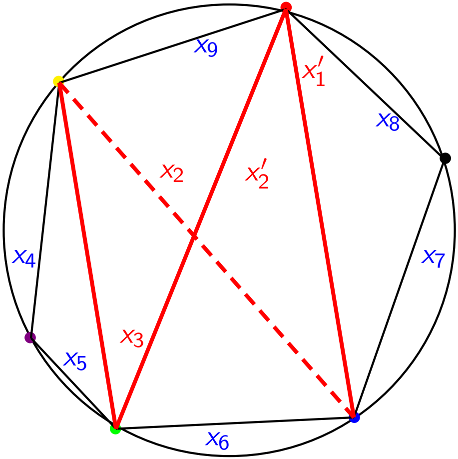


Cluster algebra

- We call a set of n numbers $(x_1, \dots, x_n)$ a cluster of rank n.
- A seed consists of a cluster and an exchange matrix B, i.e. a skew–symmetrisable matrix with integer entries.
- A mutation is a transformation $\mu_i : (x_1, x_2, \dots, x_n) \to (x'_1, x'_2, \dots, x'_n)$, $\mu_i : B \to B'$ where

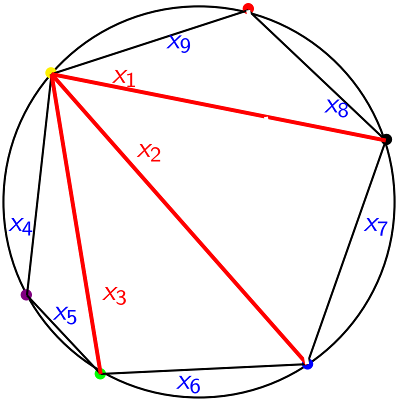
$$x_i x'_i = \prod_{j:b_{ij}>0} x_j^{b_{ij}} + \prod_{j:b_{ij}<0} x_j^{-b_{ij}}, \quad x'_j = x_j \, \forall j \neq i.$$

Definition

A cluster algebra of rank n is a set of all seeds $(x_1, \dots, x_n, B)$ related to one another by sequences of mutations $\mu_1, \dots, \mu_k$. The cluster variables $x_1, \dots, x_k$ are called exchangeable, while $x_{k+1}, \dots, x_n$ are called frozen. [Fomin-Zelevnsky 2002].

Example

Cluster algebra of rank 9 with 3 exchangeable variables x_1, x_2, x_3 and 6 frozen ones $x_4, \ldots, x_9$.



Outline

Are all cluster algebras of geometric origin?

- Introduce bordered cusps
- Geodesics length functions on a Riemann surface with bordered cusps form a cluster algebra.

All Berenstein-Zelevinsky cluster algebras are geometric

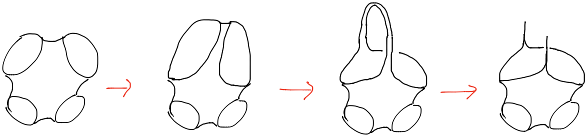
Teichmüller space

For Riemann surfaces with holes:

$$Hom\left(\pi_1(\Sigma), \mathbb{P}SL_2(\mathbb{R})\right)/GL_2(\mathbb{R}).$$

Idea:

- Teichmüller theory for a Riemann surfaces with holes is well understood.
- Take confluences of holes to obtain cusps.



- Develop bordered cusped Teichmüller theory asymptotically.

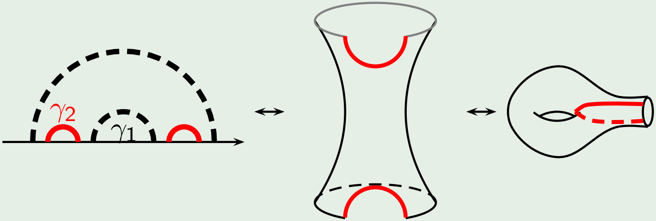
This will provide cluster algebra of geometric type

Poincaré uniformsation

$$\Sigma = \mathbb{H}/\Delta,$$

where Δ is a *Fuchsian group*, i.e. a discrete sub-group of $\mathbb{P}SL_2(\mathbb{R})$.

Examples



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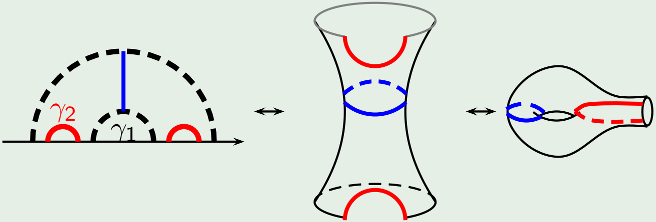


Theorem

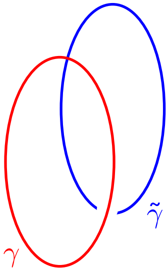
Elements in $\pi_1(\Sigma_{g,s})$ are in 1-1 correspondence with conjugacy classes of closed geodesics.

Coordinates: geodesic lengths

Theorem

The geodesic length functions form an algebra with multiplication:

$$G_\gamma G_{\tilde{\gamma}} = G_{\gamma\tilde{\gamma}} + G_{\gamma\tilde{\gamma}^{-1}}.$$



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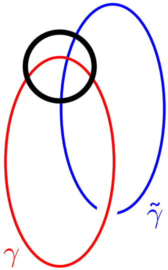

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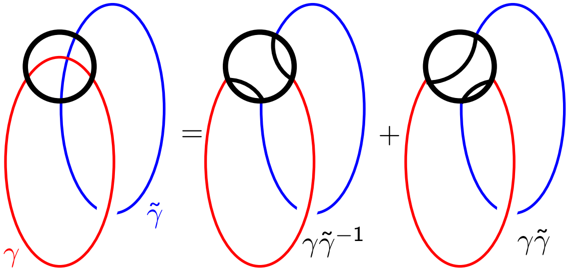
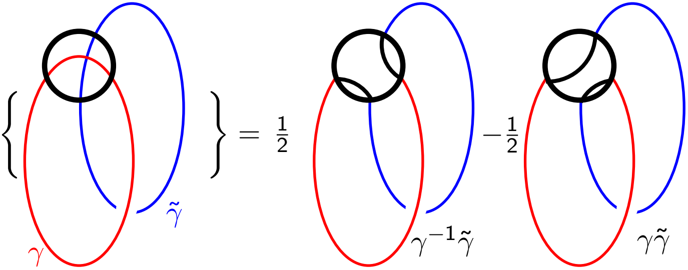


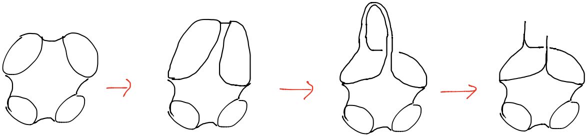
Poisson structure

$$\{G_\gamma, G_{\tilde\gamma}\} = \frac{1}{2} G_{\gamma\tilde\gamma} - \frac{1}{2} G_{\gamma\tilde\gamma^{-1}}.$$



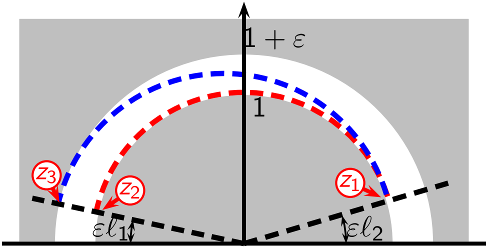
Two types of chewing-gum moves

- **Connected result:**



- **Disconnected result:**

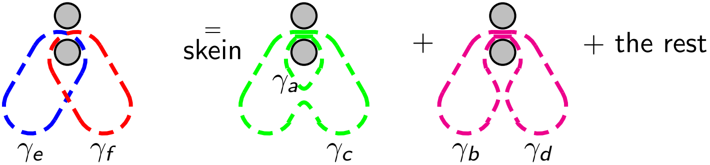
Chewing gum



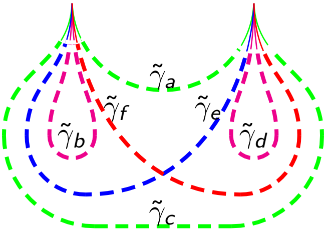
- $\left(\sinh \frac{d_{\mathbb{H}}(z_1,z_2)}{2}\right)^2 = \frac{|z_1-z_2|^2}{4\Im z_1 \Im z_2}$
- $e^{d_{\mathbb{H}}(z_1,z_2)} \sim \frac{1}{l_1 l_2 \epsilon^2} + \frac{(l_1+l_2)^2}{l_1 l_2} + \mathcal{O}(\epsilon)$,
- $e^{d_{\mathbb{H}}(z_1,z_3)} \sim e^{d_{\mathbb{H}}(z_1,z_2)} + \frac{1}{l_1 l_2} + \mathcal{O}(\epsilon)$.

$\Rightarrow$ Rescale all geodesic lengths by e^{ϵ} and take the limit $\epsilon \to 0$.

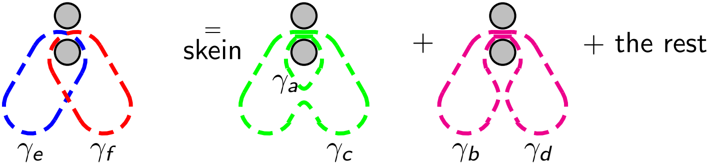
[Chekhov-M.M. arXiv:1509.07044]

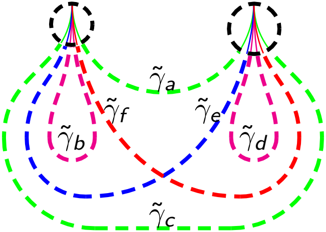


γ_e
γ_f
$\stackrel{=}{\text{skein}}$
γ_a
γ_c
$+$
γ_b
γ_d
$+$ the rest



$\tilde{\gamma}_a$
$\tilde{\gamma}_f$
$\tilde{\gamma}_e$
$\tilde{\gamma}_b$
$\tilde{\gamma}_d$
$\tilde{\gamma}_c$

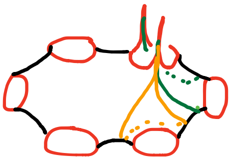




$$G_{\tilde{\gamma}_e} G_{\tilde{\gamma}_f} = G_{\tilde{\gamma}_a} G_{\tilde{\gamma}_c} + G_{\tilde{\gamma}_b} G_{\tilde{\gamma}_d}$$

Poisson bracket

- Introduce cusped laminations



- Compute Poisson brackets between arcs in the cusped lamination.

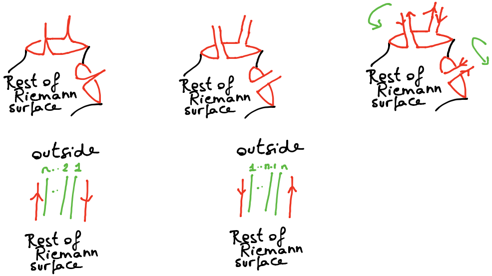
Theorem

Given a Riemann surface of any genus, any number of holes and at least one cusp on a boundary, there always exists a complete cusped lamination [Chekhov-M.M. ArXiv:1509.07044].

Poisson structure

Theorem

The Poisson algebra of the λ-lengths of a complete cusped lamination is a cluster algebra [Chekhov-M.M. ArXiv:1509.07044].

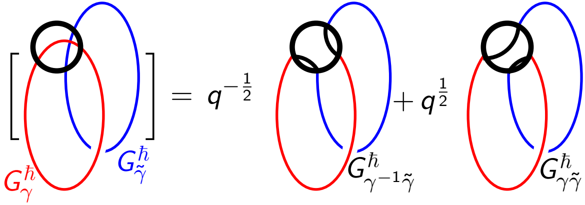


$$\{g_{s_i,t_j}, g_{p_r,q_l}\} = g_{s_i,t_j} g_{p_r,q_l} \mathcal{I}_{s_i,t_j,p_r,q_l}$$

$$\mathcal{I}_{s_i,t_j,p_r,q_l} = \frac{\epsilon_{i-r}\delta_{s,p} + \epsilon_{j-r}\delta_{t,p} + \epsilon_{i-l}\delta_{s,q} + \epsilon_{j-l}\delta_{t,q}}{4}$$

Quantisation

For standard geodesic lengths $G_\gamma \to G_\gamma^\hbar$ [Chekhov-Fock '99]:



$$[G_\gamma^\hbar, G_{\tilde\gamma}^\hbar] = q^{-\frac{1}{2}} G_{\gamma^{-1}\tilde\gamma}^\hbar + q^{\frac{1}{2}} G_{\gamma\tilde\gamma}^\hbar$$

For arcs $g_{s_i,t_j} \to g_{s_i,t_j}^\hbar$:

$$q^{\mathcal{I}_{s_i,t_j,p_r,q_l}} g_{s_i,t_j}^\hbar g_{p_r,q_l}^\hbar = g_{p_r,q_l}^\hbar g_{s_i,t_j}^\hbar q^{\mathcal{I}_{p_r,q_l,s_i,t_j}}$$

This identifies the geometric basis of the quantum cluster algebras introduced by Berenstein and Zelevinsky.

Decorated character variety

What is the *character variety* of a Riemann surface with cusps on its boundary?

For Riemann surfaces with holes:

$$Hom\left(\pi_1(\Sigma), \mathbb{P}SL_2(\mathbb{C})\right)/GL_2(\mathbb{C}).$$

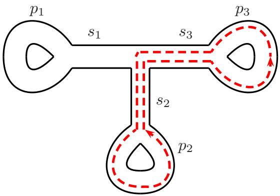
For Riemann surfaces with bordered cusps:

Decorated character variety [Chekhov-M.M.-Rubtsov arXiv:1511.03851]

- Replace $\pi_1(\Sigma)$ with the groupoid of all paths γ_{ij} from cusp i to cusp j modulo homotopy.
- Replace tr by two characters: tr and tr_K.

Shear coordinates in the Teichmüller space

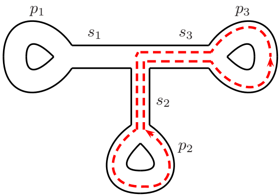
Fatgraph:



Decompose each hyperbolic element in Right, Left and Edge matrices Fock, Thurston

$$R := \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad L := \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix},$$

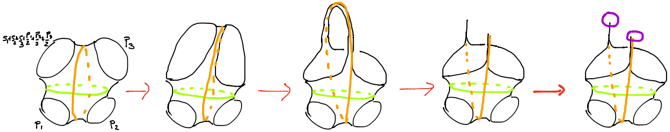
$$X_y := \begin{pmatrix} 0 & -\exp\left(\frac{y}{2}\right) \\ \exp\left(-\frac{y}{2}\right) & 0 \end{pmatrix}.$$



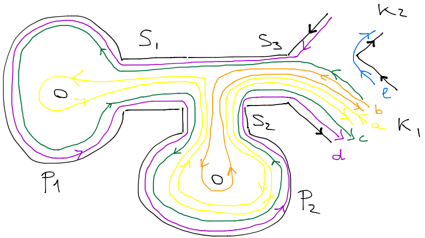
The three geodesic lengths: $x_i = \mathrm{Tr}(\gamma_{jk})$

$$x_1 = e^{s_2+s_3} + e^{-s_2-s_3} + e^{-s_2+s_3} + (e^{\frac{p_2}{2}} + e^{-\frac{p_2}{2}})e^{s_3} + (e^{\frac{p_3}{2}} + e^{-\frac{p_3}{2}})e^{-s_2}$$
$$x_2 = e^{s_3+s_1} + e^{-s_3-s_1} + e^{-s_3+s_1} + (e^{\frac{p_3}{2}} + e^{-\frac{p_3}{2}})e^{s_1} + (e^{\frac{p_1}{2}} + e^{-\frac{p_1}{2}})e^{-s_3}$$
$$x_3 = e^{s_1+s_2} + e^{-s_1-s_2} + e^{-s_1+s_2} + (e^{\frac{p_1}{2}} + e^{-\frac{p_1}{2}})e^{s_2} + (e^{\frac{p_2}{2}} + e^{-\frac{p_2}{2}})e^{-s_1}$$

$$\{x_1, x_2\} = 2x_3 + \omega_3, \quad \{x_2, x_3\} = 2x_1 + \omega_1, \quad \{x_3, x_1\} = 2x_2 + \omega_2.$$



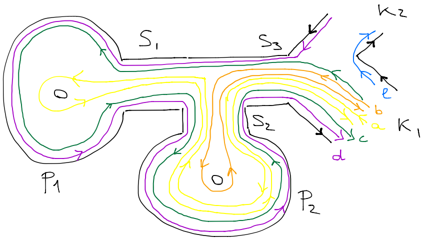
PV



$$\gamma_b = X(k_1)RX(s_3)RX(s_2)RX(p_2)RX(s_2)LX(s_3)LX(k_1)$$

BUT its length is $b = \mathrm{tr}_K(\gamma_b) = \mathrm{tr}(bK)$, $K = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$

PV



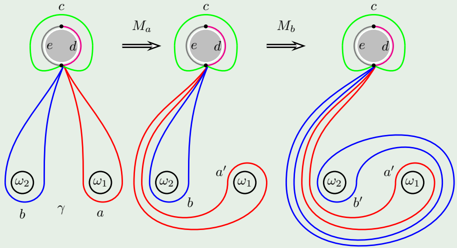
$$\{g_{s_i,t_j}, g_{p_r,q_l}\} = g_{s_i,t_j} g_{p_r,q_l} \frac{\epsilon_{i-r}\delta_{s,p}+\epsilon_{j-r}\delta_{t,p}+\epsilon_{i-l}\delta_{s,q}+\epsilon_{j-l}\delta_{t,q}}{4}$$

$$\begin{aligned}\{b, d\} &= \{g_{1_3,1_4}, g_{2_1,1_8}\}\\ &= g_{1_3,1_4} g_{2_1,1_8} \frac{\epsilon_{3-1}\delta_{1,2} + \epsilon_{4-1}\delta_{1,2} + \epsilon_{3-8}\delta_{1,1} + \epsilon_{4-8}\delta_{1,1}}{4}\\ &= -bd\frac{1}{2}\end{aligned}$$

Mutations

Example

Riemann sphere with three holes, and two cusps on one of the holes. Frozen variables: c, d, e. Exchangeable variables: a, b.



$a = g_{1_5,1_6}$, $b = g_{1_3,1_4}$, $d = g_{1_8,2_2}$, $\{a, b\} = ab$, $\{a, d\} = -\frac{ad}{2}$.
Sub-algebra of functions that commute with the frozen variables

Chekhov-M.M.-Rubtsov arXiv:1511.03851:

$$\{x_1, x_2\} = 2x_3 + \omega_3, \quad \{x_2, x_3\} = 2x_1 + \omega_1, \quad \{x_3, x_1\} = 2x_2 + \omega_2.$$

Conclusion

- A Riemann surface of genus g, n holes and k cusps on the boundary admits a complete cusped lamination of $6g-6+2n+2k$ arcs which triangulate it.
- Any other cusped lamination is obtained by the cluster algebra mutations.
- By quantisation: quantum cluster algebra of geometric type.
- New notion of decorated character variety

Many thanks for your attention!!!