Integrability, Geometry and Deformations

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based on joint works with
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We will consider two geometries staying behind many integrable dispersionless PDEs in 3D and 4D.

Weyl structure on M^3 is the pair $([g],\mathbb{D})$ consisting of a conformal structure and a linear connection preserving it (we allow any signature, but prefer Lorentzian g). Then the condition on $\mathbb{D}$ writes via 1-form ω

$$\mathbb{D}g=\omega\otimes g.$$

A choice of ω is equivalent to a choice of $\mathbb{D}$.

Indeed, denoting by ∇ the Levi-Civita connection, we have $\mathbb{D}=\nabla+\rho(\omega)$, $2\rho(\omega)(X,Y)=\omega(X)Y+\omega(Y)X-g(X,Y)\omega^\sharp$. In coordinates $\mathbb{D}_i v^j=\nabla_i v^j+\gamma^j_{ik}v^k$, where $\gamma^j_{ik}=\frac12(\omega_k\delta^j_i+\omega_i\delta^j_k-\omega^j g_{ik})$ (raising is done by g).

Under the change $g\mapsto\kappa g$ the form changes so: $\omega\mapsto\omega+d\log\kappa$. We encode Weyl structures as pairs (g,ω) mod the above gauge.

For the general linear connection $\mathbb{D}$, its Ricci tensor $\mathrm{Ric}_{\mathbb{D}}$ needs not be symmetric. Its skew-symmetric part $\mathrm{Ric}_{\mathbb{D}}^{\mathrm{alt}}$ is proportional to $d\omega$. The symmetric part $\mathrm{Ric}_{\mathbb{D}}^{\mathrm{sym}}$ leads to Einstein-Weyl equation

$$\mathrm{Ric}_{\mathbb{D}}^{\mathrm{sym}} = \Lambda\, g.$$

Here Λ is a function on M. The pair $([g], \mathbb{D})$ is called an Einstein-Weyl structure if the above equation is satisfied (these yields 5 PDEs of the 2nd order on the 5 entries of the conformal structure and 3 of the covector). Notice that Einstein-Weyl is an invariant property of conformal (not metric) structure.

In particular, for $\omega = 0$ the connection $\mathbb{D}$ is Levi-Civita, and the above is just the Einstein equation. Thus Einstein-Weyl structures are rich generalizations of the Einstein structures. For instance, in 3D all Einstein manifolds are the spaces of constant curvature. But $S^1 \times S^2 = (\mathbb{R}^3 \setminus 0)/\mathbb{Z}$ has a flat Einstein-Weyl structure.

Self-duality equation

Given a metric g in 4D, its Weyl curvature W (as (3,1)-tensor) is an invariant of the conformal structure $[g]$. The Hodge operator acts on the space $(S^2\Lambda^2TM)_0$ of Weyl tensors and it is an involution for Riemannian or neutral signature, whence the split $W = W_+ + W_-$ into self-dual and anti-self-dual parts. The structure $[g]$ is called self-dual, resp. anti-self-dual, if $W_- = 0$, resp. $W_+ = 0$ (these are 5 PDEs of the 2nd order on the 9 entries of $[g]$).

Both EW and SD (or ASD) equations are integrable, as well as some of their reductions, e.g. anti-self dual Einstein (=heavenly) equations. Many other PDEs can be obtained as (~~symmetry~~) reductions of the two equations, thus allowing to think of them as master-equations in 3D and 4D respectively.

EW structures in 3D are reductions from 4D of: (1) hypercomplex manifolds with triholomorphic symmetry; (2) ASD manifolds with conformal symmetry. Hypercomplex geometry gives rise to integrable systems as well, but will not be discussed.

Cartan related EW structures to the geometry of 3rd order ODEs w.r.t. point transformations

$$y''' = F(x, y, y', y'').$$

Denoting $p = y', q = y''$ we have the following (relative) differential invariants ($\mathcal{D} = \partial_x + p\partial_y + q\partial_p + F\partial_q$ is the total derivative):

$$W = \tfrac{1}{6}\mathcal{D}^2 F_q - \tfrac{1}{3}F_q\mathcal{D}F_q - \tfrac{1}{2}\mathcal{D}F_p + \tfrac{2}{27}F_q^3 + \tfrac{1}{3}F_qF_p + F_y$$

$$C = (\tfrac{1}{3}\mathcal{D}F_q - \tfrac{1}{9}F_q^2 - F_p)F_{qq} + \tfrac{2}{3}F_qF_{qp} - 2F_{qy} + F_{pp} + 2W_q$$

(Wünschmann and Cartan invariants).

Provided $W = 0$, $C = 0$ the solution space $\mathcal{S} \simeq \mathbb{R}^3(y, p, q)$ of the ODE carries EW geometry with the conformal structure

$$g = 2\,dy\,dq - \tfrac{2}{3}F_q\,dy\,dp + (\tfrac{1}{3}\mathcal{D}F_q - \tfrac{2}{9}F_q^2 - F_p)\,dy^2 - dp^2$$

and the Weyl potential

$$\omega = \tfrac{2}{3}(F_{qp} - \mathcal{D}F_{qq})\,dy + \tfrac{2}{3}F_{qq}dp.$$

EW and PDEs

It was also known that examples of EW structures can come from solutions of integrable PDEs.

(1) The metric $g = 4dxdt - dy^2 + 4udt^2$ and the covector $\omega = -4u_x dt$ form EW structure provided $u = u(t, x, y)$ satisfies the dKP equation (Dunajski, Mason, Tod)

$$u_{xt} - (uu_x)_x - u_{yy} = 0$$

(2) The metric $g = dx^2 + dy^2 - e^{-u}dt^2$ and the covector $\omega = u_t dt - u_x dx - u_y dy$ form EW structure provided $u = u(t, x, y)$ satisfies the Boyer-Finley equation (Ward, LeBrun)

$$u_{xx} + u_{yy} = (e^u)_{tt}$$

(3) Calderbank found Einstein-Weyl structures from solutions of the gauge field equations with the gauge group $\mathrm{SDiff}(2)$ modelled on Riccati spaces in the class of PDEs related to the generalized Nahm equation.

Dispersionless PDEs

Consider the quasi-linear system of PDEs

$$A(\mathbf{u})\mathbf{u}_x + B(\mathbf{u})\mathbf{u}_y + C(\mathbf{u})\mathbf{u}_t = 0, \qquad (\dagger)$$

where $\mathbf{u} = (u_1, \dots, u_m)^t$ is an m-component vector and A, B, C are $l \times m$ matrices. We assume the system involutive, with the general solution depending on 2 functions of 1 variable.

Systems of type $(\dagger)$ will be referred to as 3D dispersionless PDEs. Typically, they arise as dispersionless limits of integrable soliton equations: The canonical example is the KP equation:

$$u_t - u\,u_x + \epsilon^2 u_{xxx} - w_y = 0, \quad w_x = u_y,$$

which assumes the form $(\dagger)$ in the limit $\epsilon \to 0$.

Notice that $(\dagger)$ is translation invariant, which is the standard requirement for dispersionless PDEs (another approach: scaling limit in independent vars).

Integrability by the method of hydrodynamic reductions

As proposed by Ferapontov and Khusnutdinova, the method of hydrodynamic reductions consists of seeking N-phase solutions

$$\mathbf{u} = \mathbf{u}(R^1, \dots, R^N).$$

The phases (Riemann invariants) $R^i(x, y, t)$ are required to satisfy a pair of commuting equations

$$R^i_y = \mu^i(R) R^i_x, \quad R^i_t = \lambda^i(R) R^i_x,$$

Compatibility of this system writes (commutativity conditions):

$$\frac{\partial_j \mu^i}{\mu^j - \mu^i} = \frac{\partial_j \lambda^i}{\lambda^j - \lambda^i}.$$

Definition

A quasiliner system is called integrable if, for any N, it possesses infinitely many N-component reductions parametrized by N arbitrary functions of 1 variable ($N = 3$ is sufficient).

Example of dKP

Let's rewrite the dKP equation $(u_t - uu_x)_x = u_{yy}$ in the first order (hydrodynamic) form:

$$u_t - uu_x = w_y, \quad u_y = w_x.$$

N-phase solutions are obtained by $u = u(R^1, \dots, R^N)$, $w = w(R^1, \dots, R^N)$, where

$$R^i_y = \mu^i(R) R^i_x, \quad R^i_t = \lambda^i(R) R^i_x.$$

Then

$$\partial_i w = \mu^i \partial_i u, \quad \lambda^i = u + (\mu^i)^2.$$

Functions $u(R)$, $\mu^i(R)$ satisfy the Gibbons-Tsarev equations:

$$\partial_j \mu^i = \frac{\partial_j u}{\mu^j - \mu^i}, \quad \partial_i \partial_j u = \frac{2 \partial_i u \partial_j u}{(\mu^j - \mu^i)^2}.$$

This system is involutive and its solutions depend on N functions of 1 variable.

Formal linearization

Given a PDE

$$F(x^i, u, u_{x^i}, u_{x^i x^j}, \dots) = 0,$$

its formal linearization ℓ_F results upon setting $u \to u + \epsilon v$, and keeping terms of the order ϵ. This leads to a linear PDE for v,

$$\ell_F(v) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} F(u + \epsilon v) = 0,$$

In coordinates we have:

$$\ell_F = F_u + F_{u_{x^i}} \mathcal{D}_{x^i} + F_{u_{x^i x^j}} \mathcal{D}_{x^i} \mathcal{D}_{x^j} + \dots .$$

Example: Linearization of the dKP equation, $u_{xt} - (uu_x)_x - u_{yy} = 0$, reads as $v_{xt} - (uv)_{xx} - v_{yy} = 0$. In the latter linear PDE u is the background solution.

Types I-IV of PDEs studied:

I. Equations possessing the 'central quadric ansatz':

$$(a(u))_{xx}+(b(u))_{yy}+(c(u))_{tt}+2(p(u))_{xy}+2(q(u))_{xt}+2(r(u))_{yt}=0.$$

Equivalence group: $\mathrm{GL}(3)\times\mathrm{Diff}(\mathbb{R}):\mathbb{R}^3(x,y,t)\times\mathbb{R}^1(u)\circlearrowleft$.

II. Quasilinear wave equations:

$$f_{11}u_{xx}+f_{22}u_{yy}+f_{33}u_{tt}+2f_{12}u_{xy}+2f_{13}u_{xt}+2f_{23}u_{yt}=0,$$

$f_{ij}=f_{ij}(u_x,u_y,u_t)$. Equivalence group: $\mathrm{GL}(4):\mathbb{R}^4(x,y,t,u)\circlearrowleft$.

III. Hirota-type equations:

$$F(u_{xx},u_{xy},u_{yy},u_{xt},u_{yt},u_{tt})=0.$$

Equivalence group: $\mathrm{Sp}(6):T^*\mathbb{R}^3(x,y,t,u_x,u_y,u_t)\circlearrowleft$.

IV. Two-component systems of hydrodynamic type:

$$\mathbf{u}_t=A(\mathbf{u})\mathbf{u}_x+B(\mathbf{u})\mathbf{u}_y,\quad \mathbf{u}=(u_1,u_2)^T.$$

Equiv. group $\mathrm{GL}(3)\times\mathrm{Diff}(\mathbb{R}^2):\mathbb{R}^3(x,y,t)\times\mathbb{R}^2(u_1,u_2)\circlearrowleft$.

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Canonical conformal structure

For the equations of the considered type the linearized equation

$$\ell_F(v) = g^{ij} v_{ij} + f^i v_i + c\,v = 0$$

is the second order PDE linear in v. The matrix of higher derivatives represents a symmetric bi-vector $g^{ij} = g^{ij}(u)$ (depending on the 2-jet $j^2 u$ of the solution u) defined up to multiplication by a function.
Thus, provided this matrix is non-degenerate, its inverse $(g_{ij}) = (g^{ij})^{-1}$ determines a canonical conformal metric structure

$$g = g_{ij}\,dx^i\,dx^j,$$

depending on a finite jet of the solution (this encodes the symbol of the equation = dispersion relation). We say that there is a canonical conformal structure on every solution.

A remarkable formula for the Weyl potential

Given a conformal structure $g = g_{ij}(u)dx^i dx^j$ let us introduce the covector $\omega = \omega_s dx^s$ by the universal formula

$$\omega_s = 2g_{sj}\mathcal{D}_{x^k}(g^{jk}) + \mathcal{D}_{x^s}(\ln\det g_{ij}).$$

To interpret this formula, note that the covector ω is given by the identity

$$g^{ij}v_{ij} = \nabla^i\nabla_i v - \tfrac{1}{2}\omega^i\nabla_i v,$$

where ∇ is the Levi-Civita connection. Equivalently, the contracted Christoffel symbols $\Gamma_i = g_{il}g^{jk}\Gamma^l_{jk} = \frac{1}{2}g^{jk}(\partial_j g_{ik} + \partial_k g_{ij} - \partial_i g_{jk})$ equal to

$$\Gamma_i = -g_{ij}\partial_k g^{jk} - \tfrac{1}{2}\partial_i \log|\det(g_{jk})|,$$

and so (in 3D only!) we relate $\omega_i = -2\Gamma_i$.

Due to dispersionless setup, the formula for ω is not contact invariant, but it is invariant w.r.t. equivalence transformations.

Theorem (E. Ferapontov & BK)

A second order PDE is linearizable (by a transformation from the natural equivalence group) if and only if the conformal structure g is conformally flat on every solution (has vanishing Cotton tensor).

Theorem (E. Ferapontov & BK)

A second order PDE is integrable by the method of hydrodynamic reductions if and only if, on every solution, the conformal structure g satisfies the Einstein-Weyl equations, with the covector $\omega = \omega_s dx^s$ given by the universal formula.

According to a theorem of E. Cartan, the triple $(\mathbb{D}, g, \omega)$ is EW iff there exists a two-parameter family of g-null surfaces that are totally geodesic with respect to $\mathbb{D}$. For our classes of integrable PDEs, these totally geodesic null surfaces are provided by the corresponding dispersionless Lax pair.

Lax pairs

Integrability of the equations of types I-IV above is equivalent to existence of a dispersionless Lax pair

$$S_t = f(S_x, u_x, u_y, u_t), \quad S_y = g(S_x, u_x, u_y, u_t). \qquad (\flat)$$

This means that the compatibility condition $S_{ty} = S_{yt}$ is equivalent to the considered PDE. Lax pairs of this form arise in dispersionless limits of solitonic Lax pairs (Zakharov).

Differentiate $(\flat)$ by x and set $S_x = \lambda$, $u_x = a$, $u_y = b$, $u_t = c$:

$$\lambda_t = f_\lambda \lambda_x + f_a a_x + f_b b_x + f_c c_x, \ \lambda_y = g_\lambda \lambda_x + g_a a_x + g_b b_x + g_c c_x. \ (\sharp)$$

The vector fields in the extended space $\mathbb{R}^4(x, y, t, \lambda)$

$$X = \frac{\partial}{\partial t} - f_\lambda \frac{\partial}{\partial x} + (f_a a_x + f_b b_x + f_c c_x)\frac{\partial}{\partial \lambda},$$
$$Y = \frac{\partial}{\partial y} - g_\lambda \frac{\partial}{\partial x} + (g_a a_x + g_b b_x + g_c c_x)\frac{\partial}{\partial \lambda},$$

commute iff the compatibility $\lambda_{ty} = \lambda_{yt}$ of $(\sharp)$ holds.

Consider the cotangent bundle $Z^6 = T^*\mathbb{R}^3(x, y, t, S_x, S_y, S_t)$ of the soluton $u = u(t, x, y)$, equipped with the symplectic form

$$\omega = dS_x \wedge dx + dS_y \wedge dy + dS_t \wedge dt.$$

Equations (♭) specify a submanifold $M^4 \subset Z^6$ parametrized by x, y, t, λ. The compatibility of (♭) means this submanifold is coisotropic and we have:

$$\mathrm{Ker}(\Omega|_{M^4}) = \langle X, Y \rangle.$$

This distribution is integrable and is tangent to the hypersurface $\lambda = \lambda(x, y, t)$ in M^4.

The two-parameter family of integral leaves of the distribution $\langle X, Y \rangle$ projects to the space $\mathbb{R}^3(x, y, z)$ to a 2-parameter family of null totally geodesic surfaces of the Weyl connection $\mathbb{D}$.

Example 1: dKP. $u_{xt} - (uu_x)_x - u_{yy} = 0$.
The corresponding EW structure is as follows:

$$g = 4dxdt - dy^2 + 4udt^2, \ \omega = -4u_x dt.$$

The dispersionless Lax pair is given by vector fields

$$X = \partial_y - \lambda\partial_x + u_x\partial_\lambda, \quad Y = \partial_t - (\lambda^2 + u)\partial_x + (u_x\lambda + u_y)\partial_\lambda,$$

Example 2: The system of Ferapontov, Khusnutdinova and Tsarev.
The Euler-Lagrange equation

$$u_x u_{yt} + u_y u_{xt} + u_t u_{xy} = 0.$$

for the density $u_x u_y u_t$ is integrable by the method of hydrodynamic reductions. Its EW structure is given by:

$$g = (u_x dx + u_y dy + u_t dt)^2 - 2u_x^2 dx^2 - 2u_y^2 dy^2 - 2u_t^2 dt^2,$$
$$\omega = -4\frac{u_x u_{yt}}{u_y u_t}\, dx - 4\frac{u_y u_{tx}}{u_t u_x}\, dy - 4\frac{u_t u_{xy}}{u_x u_y}\, dt.$$

Example 3: Integrable system of Pavlov.

$$u_{tt} = \frac{u_{xy}}{u_{xt}} + \frac{1}{6}\eta(u_{xx})u_{xt}^2.$$

Integrability condition: Chazy equation $\eta''' + 2\eta\eta'' = 3(\eta')^2$.

Conformal structure is:

$$g = 4u_{xt}dxdy - \left(\frac{2}{3}\eta' u_{xt}^4 + s^2\right) dy^2 + 2sdydt - dt^2,$$

where $s = \frac{1}{3}\eta u_{xt}^2 - \frac{u_{xy}}{u_{xt}}$.

Covector ω equals:

$$\Big[(\tfrac{2}{3}\, u_{tx}\eta + 4u_{xy}u_{tx}^{-2})u_{ttx} + (\tfrac{2}{9}\, u_{tx}^2\eta^2 + \tfrac{8}{3}\, u_{tx}^2\eta' - u_{xy}^2 u_{tx}^{-4} - \tfrac{1}{3}\, u_{xy}u_{tx}^{-1}\eta)u_{txx}$$
$$+(\tfrac{1}{9}\, u_{tx}^3\eta\,\eta' + \tfrac{2}{3}\, u_{tx}^3\eta'' - \tfrac{1}{3}\, u_{xy}\eta')u_{xxx} + (u_{xy}u_{tx}^{-3} - \tfrac{1}{3}\,\eta)u_{xxy} - 2\, u_{tx}^{-1}u_{txy}\Big]\, dy$$
$$- \left[(u_{xy}u_{tx}^{-3} + \tfrac{2}{3}\eta)u_{txx} + \tfrac{1}{3}\eta'\, u_{tx}u_{xxx} - u_{t,x}^{-2}u_{xxy} - 2u_{tx}^{-1}u_{ttx}\right] dt.$$

This structure is EW iff η solves the Chazy equation.

Explicit form of EW system

According to a theorem of Hitchin, the system of EW equations is integrable. We will write its PDEs in a proper gauge.

Theorem (M. Dunajski, E. Ferapontov & BK)

Any Lorentzian EinsteinWeyl structure is locally of the form

$$g = -(dy - v_x dt)^2 + 4(dx - (u - v_y)dt)dt,$$
$$\omega = -v_{xx}dy + (4u_x - 2v_{xy} + v_x v_{xx})dt,$$

where the functions u, v *on* M^3 *satisfy*

$$P(u) = -u_x^2,\ P(v) = 0; \quad P = \partial_x\partial_t - \partial_y^2 + (u - v_y)\partial_x^2 + v_x\partial_x\partial_y.$$

The above coupled system of second-order PDEs, known as the Manakov-Santini system, has the Lax pair

$$L_1 = \partial_y - (\lambda + v_x)\partial_x - u_x\partial_\lambda,$$
$$L_2 = \partial_t - (\lambda^2 + v_x\lambda - u + v_y)\partial_x - (u_x\lambda + u_y)\partial_\lambda.$$

Integrability in 4D and self-duality

In 4D PDEs of Monge-Ampère type are linearizable iff the corresponding conformal structure is flat on every solution.

Integrable equations of Monge-Ampère type in 4D have the following normal forms (Doubrov-Ferapontov):

- $u_{tt} - u_{xx} - u_{yy} - u_{zz} = 0$ (linear wave equation)
- $u_{xz} + u_{yt} + u_{xx}u_{yy} - u_{xy}^2 = 0$ (second heavenly equation)
- $u_{xz} = u_{xy}u_{tt} - u_{xt}u_{yt}$ (modified heavenly equation)
- $u_{xz}u_{yt} - u_{xt}u_{yz} = 1$ (first heavenly equation)
- $u_{xx} + u_{yy} + u_{xz}u_{yt} - u_{xt}u_{yz} = 0$ (Husain equation)
- $u_{xy}u_{zt} - \beta u_{xz}u_{yt} + (\beta - 1)u_{xt}u_{yz} = 0$ (general heavenly).

Their conformal structures are self-dual on every solution.

Criterion: A 2nd order dispersionless PDE in 4D is integrable iff the corresponding conformal structure is SD/ASD on every solution.

Explicit form of SD/ASD equations

According to a theorem of Penrose, the system of (A)SD equations is integrable. We will write its PDEs in a proper gauge.

Theorem (M. Dunajski, E. Ferapontov & BK)

Any ASD conformal structure of signature (2,2) has local form

$$g = dx\,dw + dy\,dz + u_y dw^2 - (u_x + v_y)\,dz\,dw + v_x dz^2,$$

where the functions u, v on M^4 satisfy

$$\partial_x Q(u) - \partial_y Q(v) = 0,$$
$$(\partial_w - u_y\partial_x + v_y\partial_y)Q(v) + (\partial_z + u_x\partial_x - v_x\partial_y)Q(u) = 0,$$
$$Q = \partial_x\partial_w + \partial_y\partial_z - u_y\partial_x^2 + (u_x + v_y)\partial_x\partial_y - v_x\partial_y^2.$$

The above coupled system of third-order PDEs has the Lax pair

$$L_1 = \partial_w - u_y\partial_x + (\lambda + v_y)\partial_y + Q(u)\partial_\lambda,$$
$$L_2 = \partial_z + (\lambda + u_x)\partial_x - v_x\partial_y - Q(v)\partial_\lambda.$$

Application: integrable symmetric deformation - example

Theorem (BK & O.Morozov)

Every integrable Monge-Ampère equations of Hirota type in 4D has 4 copies of Lie algebra $\mathrm{SDiff}(2)$ in its symmetry algebra, realizing all 5 Petrov types: N, D, III, II, I and O (for linear equations).

Let us split off one of the copies of $\mathrm{SDiff}(2)$ and investigate invariant equations with respect to the smaller symmetry.

Theorem (BK & O.Morozov)

Integrable deformations of the above equations are the following:

- N *deformation rigid*
- D $u_{xt}u_{yz} - u_{xz}u_{yt} = \partial_z Q(z,t)\, u_t - \partial_t Q(z,t)\, u_z + b(z,t)$,
- III $u_{yt} - u_{xt}u_{zz} + u_{xz}u_{zt} = Q(t,u_t)\, u_{zt}$,
- II $u_{xy}u_{zt} - u_{xz}u_{yt} = Q(t,u_t)\, u_{xt}$,
- I $u_{xy}u_{zt} - u_{xz}u_{yt} = Q(t,u_t)\,(u_{xy}u_{zt} - u_{xt}u_{yz})$.